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THE CAUCHY PROBLEM FOR A CLASS OF 2×2 NONSTRICTLY HYPERBOLIC SYSTEMS

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Abstract

In this paper we solve the Cauchy problem for the systems $\partial_t z - \partial_x \bar{z}^\gamma = 0$, where $z = u + iv \in C$ and $\gamma \leq 1 < 2$. These systems are nonstrictly hyperbolic possessing an isolated umbilic point at $z = 0$. We use the vanishing viscosity method with the help of theory of compensated compactness. Uniform bounds in L^∞ for the solutions of the viscous systems are not available, but such bounds can be found in L^2. This makes necessary the use of the generalized Young measures and improvements in known techniques of the compensated compactness theory applied to conservation laws.

1.Introduction

In what follows we will be interested in solving the Cauchy problem for the class of 2×2 nonstrictly hyperbolic systems given by:

$$\partial_t z - \partial_x \bar{z}^\gamma = 0, \tag{1.1}$$

where $z = u + iv \in C$, $t > 0$, $x \in \mathbb{R}$ and

$$1 < \gamma < 2. \tag{1.2}$$

For the initial condition we set

$$z(x, 0) = z_0(x). \tag{1.3}$$

Our main result states that there exists a global weak solution to (1.1)-(1.2) , provided the initial data satisfy:

(i) $$z_0 \in L^2 \cap L^\infty(\mathbb{R}; \mathbb{R}^2)$$

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(ii) z_0 takes its values in some closed wedge C of the plane, with vertex at the origin and angle $\frac{\pi}{\gamma+1}$ radians. The Riemann problem for (1.1) with $\gamma = 2$ is solved in [14]. This is a particular case ($a = -1,\ b = 0$) of the systems

$$\begin{cases} \partial_t u + \partial_x(au^2 + 2buv + v^2) = 0 \\ \partial_t v + \partial_x(bu^2 + 2uv) = 0. \end{cases} \tag{1.4}$$

that give canonical forms of the quadratic systems which posses an isolated umbilic point at the origin (see [17]). As it is shown in [17] they can be classified in four cases according to the range in which vary the parameters a and b. In each case, the wave curves and the solutions to the Riemann problems have the same structure from a qualitative point of view. The solution of Riemann problems for systems included in the general form (1.3) was the subject of a number of works (see, e.g., [9], [14], [19]). Up to now, the only result proving the existence of a global weak solution to the Cauchy problem for a system of the type (1.3) is due to P. Kan [13], for the case $a = 3,\ b = 0$.

As we will see, systems (1.1), with $1 < \gamma < 2$, form a beautiful example of the far-reaching power of the techniques developed in connection with the application of the compensated compactness theory to the nonlinear hyperbolic systems. Besides the fact of being nonstrictly hyperbolic, these systems present as an additional complexity the lack of available bounded invariant regions. This is due to the fact that the level curves of the Riemann invariants, or, briefly, the rarefaction curves, have their concavities turned to the origin and so they do not bound any finite convex region. Because of this we are led to consider uniform bounds in L^2, instead of L^∞, and to use the generalized Young measures which were introduced in [18] and later extended in [8]. The fact that these measures may have unbounded support, imposing certain cares with the growth of the functions to be integrated, is an important aspect to be considered in our procedure for reducting these measures to point masses, which constitutes a slight improvement in a technique originally due to D. Serre [21] (cf. §5).

To get a sequence of approximate solutions to (1.1) we use the vanishing viscosity method, that is, we consider solutions of the systems formed by the addition to the right-hand member of (1.1) of the viscous term $\varepsilon\frac{\partial^2 z}{\partial x^2}$.

The initial value problem for the viscous systems presents two interesting features. The first is due to the fact that the flow function $\overline{z}^{\gamma}$, with $1 < \gamma < 2$, is not even twice differentiable at the origin. The curious thing here is that the very fact that the convex part of the plane delimited by any rarefaction curve, not passing through the origin, is that not containing the origin, plays now a decisive positive role to surmount this difficulty. It permits us to define extensions of the function $\overline{z}^{\gamma}$ out of invariant regions, which are C^{∞} in all the plane (see §3). The second is due to the already mentioned lack of a priori L^{∞} bounds for the solutions of the viscous systems. This demands a careful verification on the way in which the constants, appearing in the process of extension of local solutions to global ones, depend on the viscous parameter ε. More precisely, we have to show that these constants can be choosen independent of ε (cf. §3).

This work is divided in five sections. After this introduction which constitutes section 1, we state in section 2 the basic facts about the system (1.1) and an important result which gives a direct way to get Riemann invariants for systems of a more general type including (1.1). In section 3 we solve the Cauchy problem for the viscous systems. In section 4 we show how to construct entropies for system (1.1) of certain types, which will be useful in section 5 where we expose the process of reduction of the Young measures, proving that they are in fact Dirac measures, that is, their masses are concentrated in only one point. At the end of section 5 we then state our main result as theorem 5.7.

2. Riemann invariants

We initiate our study of the systems (1.1) by stating some preliminary facts about these systems such as what are their eigenvalues and the corresponding eigenvectors. We will end this basic discussion with a result which gives a direct way of obtaining Riemann invariants for a class of systems including (1.1). For these purposes the following expedient is very convenient. Take the conjugate in both sides of (1.1), set $\overline{z} = w$, and thus form the following 2×2 systems of complex equations in the dependent variables z, w:

$$\begin{cases} \partial_t z - \partial_x w^\gamma = 0, \\ \partial_t w - \partial_x z^\gamma = 0. \end{cases} \tag{2.1}$$

The Jacobian matrix of (2.1) is

$$A(z,w) = \begin{pmatrix} 0 & -\gamma w^{\gamma-1} \\ -\gamma z^{\gamma-1} & 0 \end{pmatrix}. \tag{2.2}$$

Its eigenvalues are then:

$$\lambda_1 = -\gamma(zw)^{\frac{\gamma-1}{2}}, \qquad \lambda_2 = \gamma(zw)^{\frac{\gamma-1}{2}}. \tag{2.3}$$

The corresponding eigenvectors are easily seen to be:

$$r_1 = (w^{\frac{\gamma-1}{2}}, z^{\frac{\gamma-1}{2}}), \qquad r_2 = (w^{\frac{\gamma-1}{2}}, -z^{\frac{\gamma-1}{2}}). \tag{2.4}$$

Now, setting $w = \overline{z}$, we get

$$\lambda_1 = -\gamma|z|^{\gamma-1}, \qquad \lambda_2 = \gamma|z|^{\gamma-1}, \tag{2.5}$$

$$r_1 = (z^{-\frac{\gamma-1}{2}}, z^{\frac{\gamma-1}{2}}), \qquad r_2 = (z^{-\frac{\gamma-1}{2}}, -z^{\frac{\gamma-1}{2}}). \tag{2.6}$$

It is easily to check that (2.5)–(2.6) really give eigenvalues and eigenvectors to the original systems (1.1). We remark that, since (1.1) is symmetric, r_1 and r_2 given by (2.6) are also left–eigenvectors. A quick look on the expressions giving r_1, r_2 in (2.4) can provides us with candidates to Riemann invariants for (1.1) which then are easily comproved to be actually Riemann invariants for (1.1). We have in fact the more general result below.

Proposition 2.1. *Let us consider a* 2×2 *system*

$$\partial_t z - \partial_x \overline{f(z)} = 0, \tag{2.7}$$

where f is any holomorphic function. If $G = w_1 + iw_2$ is a primitive to $(f')^{1/2}$ then (w_1, w_2) is a pair of Riemann invariants for (2.7), i.e.

$$\nabla w_k dF = (-1)^k |f\,'| \nabla w_k \tag{2.8}$$

$k = 1, 2$, where $F \stackrel{\text{def}}{=} -\overline{f}$ is the map from $dom.(f) \subset \mathbb{R}^2$ to $\mathbb{R}^2$ whose coordinates are $(-Ref, Imf)$.

Proof: In this proof we are using the following formula which is valid for complex numbers $z_1 = a + ib$ and $z_2 = c + id$:

$$(a,b)\begin{pmatrix} -c & d \\ d & c \end{pmatrix} = -\overline{z_1 z_2}. \tag{2.9}$$

We are using the Cauchy–Riemann equation also.

We have that

$$(f\,')^{1/2} = G\,' = \partial_u w_1 - i\partial_v w_1 = \partial_v w_2 + i\partial_u w_2, \tag{2.10}$$

so,

$$\nabla w_1 = \overline{(f\,')}^{1/2} \text{ and } \nabla \mathrm{w}_2 = \mathrm{i}\overline{(\mathrm{f}\,')}^{1/2}. \tag{2.11}$$

By the other side,

$$df = \begin{pmatrix} -\partial_u Ref & \partial_u Imf \\ \partial_u Imf & \partial_u Ref \end{pmatrix}. \tag{2.12}$$

Then

$$\begin{aligned} \nabla w_1 df &= -(f\,')^{1/2}(\partial_u Ref - i\partial_u Imf) \\ &= -(f\,')^{1/2}\overline{f\,'} \\ &= -|f\,'|\overline{(f\,')}^{1/2} \\ &= -|f\,'|\nabla w_1. \end{aligned} \tag{2.13}$$

Analogously,

$$\nabla w_2 dF = |f\,'|\nabla w_2. \tag{2.14}$$

By noting that $\pm|f\,'|$ are the eigenvectors of df, we have finished the proof. ∎

Note that (2.8) says that ∇w_k is the left eigenvector associated to the eigenvalue $(-1)^k|f\,'|$ of dF. By the proposition (2.1) we have, in polar coordinates $(u = rcos\theta, v = rsen\theta)$, that

$$w_1^0 \stackrel{\text{def}}{=} r^a cosa\theta \tag{2.15}$$

and

$$w_2^0 \stackrel{\text{def}}{=} r^a sena\theta, \tag{2.16}$$

where

$$a \stackrel{\text{def}}{=} \frac{\gamma+1}{2}, \tag{2.17}$$

form a pair of Riemann invariants for (1.1). Then

$$w_1 \stackrel{\text{def}}{=} -r^{2a} cos^2 a\theta = -(w_1^0)^2 \tag{2.18}$$

and

$$w_2 \stackrel{\text{def}}{=} r^{2a} sen^2 a\theta = (w_2^0)^2 \tag{2.19}$$

form also such a pair. This last is more convenient for our work.

These Riemann invariants will provide us with invariant regions and will furnish a suitable system of coordinates for the construction of entropies.

By the theorem of Chueh—Conley—Smoller ([2]) we have that the convex regions limited by the rarefaction curves, i.e., the level curves of w_1 and w_2, are positively invariant regions for

$$\partial_t z^\varepsilon - \partial_x(\overline{z^\varepsilon}^\gamma) = \varepsilon \partial_{xx} z^\varepsilon, \ \varepsilon > 0, \tag{2.20}$$

3. The viscuos system

Let C be some wedge

$$C \stackrel{\text{def}}{=} \{z \in \mathbb{R}^2 : k\frac{\pi}{\gamma+1} \leq arg(z) \leq (k+1)\frac{\pi}{\gamma+1}\}, \tag{3.1}$$

where $k \in \mathbb{Z}$.

Let

$$Q \stackrel{\text{def}}{=} exp(i(\frac{k+1}{2})\frac{\pi}{\gamma+1}) \tag{3.2}$$

(or any vector pointing to the interior of C).

Consider the Cauchy problem

$$\begin{cases} \partial_t z^\varepsilon - \partial_x(\overline{z^\varepsilon})^\gamma = \varepsilon \partial_{xx} z^\varepsilon, & \varepsilon > 0 \\ z^\varepsilon(x,0) = z_0^\varepsilon(x), & x \in \mathbb{R}, \end{cases} \tag{3.3}$$

where

$$z_0^\varepsilon \stackrel{\text{def}}{=} z_0 + \varepsilon Q. \tag{3.4}$$

Note that, since the image of z_0 is contained in the wedge C, the image of z_0^ε defined by (3.4) is also contained in C and can be bounded away from the origin by some rarefaction curve, for instance the level curve $w_2 \equiv w_2(\varepsilon Q)$. So, z_0^ε will take values in the positively invariant region:

$$S \stackrel{\text{def}}{=} \{U \in \mathbb{R}^2 : w_2(U) \geq w_2(\varepsilon Q)\}. \tag{3.5}$$

To avoid the singularity of the second derivatives of the map

$$f(z) \stackrel{\text{def}}{=} -\overline{z}^\gamma \tag{3.6}$$

at the origin, in order to obtain a global solution to (3.1), we consider $F \in C^\infty$ satisfying:

$$F(z) = f(z) \ \forall z \in S \tag{3.7}$$

and

$$F'(z) = o(|z|),\ F''(z) = O(1) \text{ and } \mathrm{F'''(z) = O(1)} \quad \text{when } |z| \to \infty. \tag{3.8}$$

Following [11] and [12] we obtain a local solution to

$$\begin{cases} \partial_t z^\varepsilon + \partial_x F(z^\varepsilon) = \varepsilon z^\varepsilon_{xx} \\ z^\varepsilon(x,0) = z_0^\varepsilon(x), \quad x \in \mathbb{R} \end{cases} \tag{3.9}$$

and, accordingly [11] and [12], this local solution can be extended globally if we have that

$$\|z_0\|_{L^2} < c(\varepsilon, r) \tag{3.10}$$

for some constant $c(\varepsilon, r)$ which can depend on ε and $r \geq \|z_0\|_{L^\infty}$. Using the above growth conditions (3.8) we calculate for our case this constant explicitly and we show that,

$$\lim_{r\to\infty} c(\varepsilon, r) = \infty, \tag{3.11}$$

for each fixed ε, so the condition (3.10) can be omitted and we obtain a global solution z^ε for (3.9) (see [16]). Moreover, since S is invariant for (2.20) we have that this solution z^ε belongs to S everywhere, so z^ε is in fact a global solution to (3.3).

Since (1.1) is symmetric we have, as it is well known ([7],[13]), that z^ε satisfies

$$\|z^\varepsilon(t) - \varepsilon Q\|_{L^2(\mathbb{R})} \leq \|z_0\|_{L^2} \tag{3.12}$$

$\forall t > 0, \forall \varepsilon > 0$, and

$$2\varepsilon \|\partial_x z^\varepsilon\|_{L^2(\mathbb{R}^2_+)} \leq \|z_0\|_{L^2} \tag{3.13}$$

$\forall \varepsilon > 0$.

4. Entropies

Consider a 2×2 system of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \tag{4.1}$$

$U = (u, v) \in \mathbb{R}^2$.

Definition 4.1. A function $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called an *entropy* for (4.1) if there exists another function $q : \mathbb{R}^2 \longrightarrow \mathbb{R}$, which is called *entropy flux*, such that

$$\nabla \eta(U) dF(U) = \nabla q(U) \tag{4.2}$$

$\forall U \in$dom.(F).

Note that (4.2) is equivalent to

$$\partial_t \eta(U) + \partial_x q(U) = 0 \tag{4.3}$$

for every classical solution U of (4.1).

In terms of Riemann invariants, it is well known that a pair (ϕ, ψ) is an entropy–entropy flux pair for (4.1) if

$$\nabla \psi = (\lambda_1 \frac{\partial \phi}{\partial w_1}, \lambda_2 \frac{\partial \phi}{\partial w_2}). \tag{4.4}$$

Eliminating ψ in (4.4) we obtain that ϕ satisfies

$$\frac{\partial^2 \phi}{\partial w_1 \partial w_2} + \frac{1}{\lambda_2 - \lambda_1} \{ \frac{\partial \lambda_2}{\partial w_1} \frac{\partial \phi}{\partial w_2} - \frac{\partial \lambda_1}{\partial w_2} \frac{\partial \phi}{\partial w_1} \} = 0. \tag{4.5}$$

Now taking the pair of Riemann invariants in (2.18)-(2.19) we have

$$w_2 - w_1 = r^{2a} = r^{\gamma+1} \tag{4.6}$$

and

$$\lambda_1 = -\lambda_2 = -\gamma r^{\gamma-1} = -\gamma(w_2 - w_1)^{\frac{\gamma-1}{\gamma+1}} \tag{4.7}$$

so, the equation (4.5) becomes

$$\frac{\partial^2\phi}{\partial w_1 \partial w_2} - \frac{\alpha}{w_2 - w_1}\left(\frac{\partial\phi}{\partial w_2} - \frac{\partial\phi}{\partial w_1}\right) = 0 \tag{4.8}$$

with

$$\alpha \overset{\text{def}}{=} \frac{\gamma - 1}{2(\gamma+1)}. \tag{4.9}$$

Note that $0 < \alpha < 1/6$.

The equation (4.8) is the *Euler (—Poisson—Darboux) equation* ([3], [5], [25]). We will only consider the Euler equation (4.11) in the quadrant $w_1 \leq 0 \leq w_2$ where our Riemann invariants take their values. We consider the Goursat problem for (4.11) which consists in solving it submitted to the conditions

$$\begin{cases} \phi(w_1, w_2^*) = \theta_1(w_1), & w_1 \leq 0, \\ \phi(w_1^*, w_2) = \theta_2(w_2), & w_2 \geq 0, \end{cases} \tag{4.10}$$

where $\theta_1,\ \theta_2$ are given smooth functions, $w_1^* \leq 0 \leq w_2^*$ are fixed constants and we impose the compatibility conditions $\theta(w_1^*) = \theta_2(w_2^*)$.

The solution for (4.8)–(4.10), obtained using the Riemann's method, is given by

$$\begin{aligned} \phi(w_1, w_2) &= \theta_1(w_1) G(w_1^*, w_2^*, w_1, w_2) \\ &+ \int_{w_1^*}^{w_1} G(t, w_2^*, w_1, w_2)[\theta_1{}'(t) - \alpha(w_2^* - t)^{-1}\theta_1(t)]dt \\ &+ \int_{w_2^*}^{w_2} G(w_1^*, \tau, w_1, w_2)[\theta_2{}'(\tau) + \alpha(\tau - w_1^*)^{-1}\theta_2(\tau)]d\tau \end{aligned} \tag{4.11}$$

([25], p.24) where G is the *Riemann function*, which in our case is

$$G(x_1, x_2, x_3, x_4) = \left(\frac{x_2 - x_1}{x_4 - x_3}\right)^{\alpha} H(\sigma) \tag{4.12}$$

where

$$\sigma = \sigma(x_1, x_2, x_3, x_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_2 - x_1)(x_4 - x_3)} \tag{4.13}$$

and

$$H(\sigma) \stackrel{\text{def}}{=} F(1-\alpha, \alpha; 1; \sigma) \tag{4.14}$$

is the hypergeometric function (see [25], p.20 and properties of hypergeometric functions).

The constants w_1^* and w_2^* will be called *limits* of the entropy.

We observe that the Euler equation (4.8) is invariant under the transformation $(w_1 - w_k^*, w_2 - w_k^*)$, $k = 1, 2$, so we can restrict our attention to the case where $w_1^* = 0$ or $w_2^* = 0$.

We recall that we have the following integral representation for H:

$$H(\sigma) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{-\alpha}(1-\sigma s)^{\alpha-1} ds, \tag{4.15}$$

since $0 < \alpha < 1$.

From this we have that H is analytic on the interval $(-\infty, 1)$, $H(\sigma) > 0 \ \forall \ \sigma \in (-\infty, 1)$ and H is bounded on $(-\infty, \eta] \ \forall \ \eta < 1$.

For our purposes, following [21] and [13], we will consider four types of special entropies, solutions of (4.11)–(4.13), namely *east, west, south* and *north.*

East type: It is defined by choosing $w_1^* < 0, \theta_2 \equiv 0$, $\theta_1(w_1) = 0$ if $w_1 \leq w_1^*$ and $\theta(w_1) = 0$ if $-\delta \leq w_1 \leq 0$ for a given $0 < \delta < -w_1^*$. By (4.11)–(4.13) we have that an east type entropy ϕ is given by

$$\phi(w_1, w_2) = (w_2 - w_1)^{-\alpha} \int_{w_1^*}^{w_1} H(\sigma)\theta'(t)dt \tag{4.16}$$

if the limit $w_2^* = 0$, where

$$\theta(t) \stackrel{\text{def}}{=} (-t)^{\alpha}\theta_1(t), \tag{4.17}$$

H satisfies (4.15) and α and σ are defined in (4.9) and (4.13), respectively.

From (4.16) we immediately have that the support of ϕ is contained in $\{w_1 \geq w_1^*\}$ and we see that the terms contributing to singularities are $(w_2 - w_1)^{-\alpha}$ at the umbilic point $w_1 = w_2 = 0$ and the Hypergeometric function when $\sigma \longrightarrow 1$.

In (4.16) we have

$$\sigma = \frac{(w_1 - t)w_2}{(w_1 - w_2)t} \tag{4.18}$$

so

$$\sigma = 1 \Leftrightarrow w_1 = 0 \tag{4.19}$$

Then the singularity of ϕ is concentrated on the axis $w_1 = 0$, away from which ϕ is smooth and bounded.

The others types of entropies are defined as follows:

West: $w_1^* < 0, \theta_2 \equiv 0$ and $\theta_1(w_1) = 0$ if $w_1 \geq w_1^*$;

South: $w_2^* > 0, \theta_1 \equiv 0, \theta_2(w_2) = 0$ if $w_2 \geq w_2*$ e $\theta_2(w_2) = 0$ if $0 \leq w_2 \leq \delta$ for a given $0 < \delta < w_2^*$.

North: $w_2^* > 0, \theta_1 \equiv 0$ and $\theta_2(w_2) = 0$ if $w_2 \geq w_2^*$.

All these entropies have similar integral representation to (4.16) and the suitable vanishing properties: *east* is supported to the right of the line $w_1 = w_1^*$; *west* is supported to the left of the line $w_1 = w_1^*$; *south* is supported below the line $w_2 = w_2^*$ and; *north* is supported above the line $w_2 = w_2^*$.

To control the singularity of the east entropies on the axis $w_1 = 0$, we make use of the two following lemmas due to P. Kan [13]. The proof of them can be found in [13] or also in [16].

Lemma 4.3. *Given an east entropy* ϕ *, consider the operator*

$$T(\ .\ ) \stackrel{\text{def}}{=} \int_{w_1^*}^{-\delta} (\ .\ )\theta\,'(t)dt. \tag{4.20}$$

Suppose that for some $n \in \mathbb{N}, n \geq 1$,

$$T((-t)^l) = 0 \tag{4.21}$$

$\forall\, l \in \{1, 2, ..., n-1\}$. *Then in some small box* $0 \leq -w_1, w_2 < \varepsilon < \delta$ *we have that*

$$\phi = O(r^{((2n-1)\gamma+2n+1)/2})$$

$$\phi_{w_1} = O(r^{((2n-3)\gamma+2n-1)/2})$$

$$\phi_{w_2} = O(r^{((2n-3)\gamma+2n-1)/2})$$

$$\phi_{w_1 w_2} = O(r^{((2n-5)\gamma+2n-3)/2})$$

$$\phi_{w_1 w_1} = O(r^{((2n-5)\gamma+2n-3)/2})$$

$$\phi_{w_2 w_2} = O(r^{((2n-5)\gamma+2n-3)/2}),$$

where $r = \|(u,v)\| = (w_2 - w_1)^{\frac{1}{\gamma+1}}$.

Lemma 4.4. *Let* $\eta(u,v) = \phi(w_1, w_2)$ *an east entropy on the* (u,v) *space. Suppose that (4.21) happens for some* $n \geq 3$. *Then* $\eta, d\eta, d^2\eta$ *are bounded.*

We have similar results for the south entropies. The west and north types are regulars , since they vanish on the singular axis, and so they and their derivatives will be bounded if we assume that $\frac{d}{dt}[(-t)^\alpha \theta_k(t)]$ has compact support.

Another integral representation for our special entropies which will be used in the next section is obtained from (4.16), through integration by parts. In fact we have

$$\phi(w_1, w_2) = I(w_1, w_2)\theta(w_1) + \int_{w_1^*}^{w_1} J(t, w_1, w_2)\theta_1(t)dt, \tag{4.22}$$

where

$$I(w_1, w_2) \stackrel{\text{def}}{=} \left(\frac{w_1}{w_1 - w_2}\right)^\alpha \tag{4.23}$$

$$J(t, w_1, w_2) \stackrel{\text{def}}{=} \frac{-w_1 w_2 (-t)^\alpha - 2}{(w_2 - w_1)^\alpha + 1} H'(\sigma). \tag{4.24}$$

for an east or west entropy with flux

$$\psi(w_1, w_2) = K(w_1, w_2)\theta_1(w_1) + \int_{w_1^*}^{w_1} L(t, w_1, w_2)\theta_1(t)dt, \tag{4.25}$$

where

$$K(w_1, w_2) \stackrel{\text{def}}{=} \lambda_1(w_1, w_2) I(w_1, w_2) \tag{4.26}$$

and

$$L(t, w_1, w_2) \stackrel{\text{def}}{=} \lambda_1(t, w_2) J(t, t, w_2) - \frac{\partial \lambda_1}{\partial w_1}(t, w_2) I(t, w_2)$$
$$+ \int_t^{w_1} \lambda_1(s, w_2) \frac{\partial J}{\partial w_1}(t, s, w_2) ds. \tag{4.27}$$

We have similar integral representation for the south and north entropies.

5. Generalized Young Measure, Compensated Compactness and Reduction of Measure.

Associated to the family $\{z^\varepsilon\}_{\varepsilon>0}$ of solutions to (3.3) satisfying (3.12) we have the generalized Young measure

$$\{\nu_{(x,t)}\}\,,\ \text{a.e. } (x,t) \in \mathrm{R}^2_+ \tag{5.1}$$

([18] and [8]). This means that for a.e. $(x,t) \in \mathbb{R}^2_+$, $\nu_{(x,t)}$ is a probability measure and we have the following representation for limits of nonlinear composite functions of $\{z^\varepsilon\}$ when $\varepsilon \to 0$, in the sense of the distributions (taking subsequence if necessary):

$$g(z^\varepsilon) \rightharpoonup \langle \nu_\cdot,\ g\rangle, \tag{5.2}$$

for every $g \in C(\mathbb{R}^2)$ such that

$$g(\lambda) = o(|\lambda|^2) \tag{5.3}$$

when $|\lambda| \longrightarrow \infty$.

The notation in (5.2) stands for

$$\langle \nu_{(x,t)},\ g\rangle \overset{\text{def}}{=} \int_{\mathbb{R}^2} g(\lambda)d\nu_{(x,t)}(\lambda), \tag{5.4}$$

a.e. $(x,t) \in \mathbb{R}^2_+$.

Taking in (5.2) $g = \text{id.}$ and $g(z) = \overline{z}^{\,\gamma}$ we have

$$z^\varepsilon \rightharpoonup z \overset{\text{def}}{=} \langle \nu_\cdot,\ id.\rangle \in L^2_{loc} \tag{5.5}$$

and

$$(\overline{z^\varepsilon}\,)^\gamma \rightharpoonup \langle \nu_\cdot,\ \overline{\lambda}^\gamma\rangle. \tag{5.6}$$

Then to show that z is a (weak) solution to (1.1) what we need is to proof that

$$\nu \overset{\text{def}}{=} \nu_{(x,t)} \tag{5.7}$$

is a Dirac measure for a.e. $(x,t) \in \mathbb{R}^2+$.

The growth condition (5.3) at infinity is a sharp one. It is not difficulty to find a function that grows exact as $|\lambda|^2$ at infinity such that (5.2) does not occur (see [18]).

To reduce the Young measures to Dirac measures we use a slight improvement of the method of Denis Serre ([21]). This method consists in showing that

$$supp\ \nu \cap \{w_1 = a\} = \emptyset \quad \forall\, a \in (w_1^-, w_1^+), \tag{5.8}$$

and

$$supp\ \nu \cap \{w_2 = b\} = \emptyset \quad \forall\, b \in (w_2^-, w_2^+), \tag{5.9}$$

where w_k^- and w_k^+ are the inf. and sup., respectively, of the projection of $supp\ \nu$ on the axis $w_l = 0$, $l, k = 1, 2$. This means that the rectangle R whose vertices are $(w_1^-, w_2^-), (w_1^+, w_2^-), (w_1^+, w_2^+)$, and (w_1^-, w_2^+) is the minimal rectangle in (w_1, w_2) space containing the support of ν. The cases $w_1^- = -\infty$ or $w_2^+ = +\infty$ are also allowed.

After the proof of (5.8) and (5.9) we will have that there exist points A_l, $l = 1, ..., 4$ belonging to the quadrant $w_1 \le 0 \le w_2$ and constants $\beta_l \ge 0$, $l = 1, ..., 4$ such that $\sum_{l=1}^4 \beta_l = 1$ and

$$\nu = \sum_{l=1}^{4} \beta_l \delta_{A_l}. \tag{5.10}$$

By the theorem 6.1 in [21], (5.10) implies that there exists at least one $\beta_l = 0$. Having $supp\ \nu$ concentrated in three points, we will show then how to reduce it to a unique point in the (w_1, w_2) space, using the entropies in section 4. If $\nu_{(x,t)}$ is a Dirac measure in (w_1, w_2) space for a.e. $(x, t) \in \mathbb{R}^2_+$, then this is also true in the (u, v) space and

$$\nu_{(x,t)} = \delta_{z(x,t)}\ , \quad \text{a.e. } (\mathrm{x}, \mathrm{t}) \in \mathrm{R}^2_+, \tag{5.11}$$

for z given in (5.5), because the wedge C, where the initial data takes its values, is a positively invariant region for the viscous approximations of (1.1) and the map

$$(u, v) \longmapsto (w_1, w_2) \tag{5.12}$$

is 1–1 on this region.

We will assume that R contains the umbilic point $w_1 = w_2 = 0$, i.e., $w_1^+ = w_2^- = 0$. The other case is similar and less complicated. We will show (5.8) by using the east and west entropies described in the section 4. Analogously one can show (5.9) by using the south and north entropies.

The basic fact used in the theory of Compensated Compacteness that permits to reduce the Young measures to Dirac measures is the *commutation relation*:

$$\langle \nu, \phi\tilde{\psi} - \tilde{\phi}\psi \rangle = \langle \nu, \phi \rangle \langle \nu, \tilde{\psi} \rangle - \langle \nu, \tilde{\phi} \rangle \langle \nu, \psi \rangle, \tag{5.13}$$

i.e.,

$$\left\langle \nu, \begin{vmatrix} \phi & \psi \\ \tilde{\phi} & \tilde{\psi} \end{vmatrix} \right\rangle = \begin{vmatrix} \langle \nu, \phi \rangle & \langle \nu, \psi \rangle \\ \langle \nu, \tilde{\phi} \rangle & \langle \nu, \tilde{\psi} \rangle \end{vmatrix} \tag{5.14}$$

valid for any entropy-entropy flux pairs $(\phi, \psi), (\tilde{\phi}, \tilde{\psi})$ satisfying the lemma 4.4, since we have (3.13).

With a slight adaptation of an argument in [21], using the integral representations (4.22)–(4.27), we have ([16])

Lemma 5.1. *Given $-\infty < a \in [w_1^-, 0]$, let $0 < \delta < a$. For $0 < \varepsilon \ll 1$, let (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$ be an east and west entropy–entropy flux pair, with limits $a + \varepsilon$ and $a - \varepsilon$, respectively, where the Goursat datas satisfy $\theta_1(t) \stackrel{\text{def}}{=} \phi(t, 0) > 0, \widetilde{\theta_1}(t) \stackrel{\text{def}}{=} \tilde{\phi}(t, 0) < 0$ if $t \in (a - \varepsilon, a + \varepsilon)$ and $\theta_1(t) = 0$ if $t \in (-\delta, 0)$. Under these conditions, on the strip $a - \varepsilon < w_1 < a + \varepsilon, w_2 \geq 0$, we have*

$$\phi\tilde{\psi} - \tilde{\phi}\psi = \frac{\partial \lambda_1}{\partial w_1}(a, w_2) I(a, w_2)^2 \rho^\varepsilon(w_1) + \rho^\varepsilon(w_1)(w_2 - w_1)^\alpha O(\varepsilon), \tag{5.15}$$

where

$$I(t, w_2) \stackrel{\text{def}}{=} \left(\frac{t}{t - w_2}\right)^\alpha \tag{5.16}$$

and

$$\rho^\varepsilon(w_1) \stackrel{\text{def}}{=} \theta_1(w_1) \int_{a+\varepsilon}^{w_1} \widetilde{\theta_1}(t) dt - \widetilde{\theta_1}(w_1) \int_{a-\varepsilon}^{w_1} \theta_1(t) dt. \tag{5.17}$$

Now let us fix $a \in [w_1^-, 0]$ with $a < -\delta < 0$.

Lemma 5.2. *Let* (ϕ, ψ) *and* $(\tilde{\phi}, \tilde{\psi})$ *as in the lemma 5.1. If*

$$\langle \nu, \phi\tilde{\psi} - \tilde{\phi}\psi\rangle = 0 \quad \forall 0 < \varepsilon \ll 1, \tag{5.18}$$

then (5.8) *holds.*

Proof. The support of $\phi\tilde{\psi} - \tilde{\phi}\psi$ is contained in the strip $a-\varepsilon < w_1 < a+\varepsilon, w_2 \geq 0$,. Let χ_ε denote the characteristic function of this strip. Then

$$\begin{aligned} \phi\tilde{\psi} - \tilde{\phi}\psi &= (\phi\tilde{\psi} - \tilde{\phi}\psi)\chi_\varepsilon \\ &= \left(\tfrac{\partial \lambda_1}{\partial w_1}(a, w_2) I(a, w_2)^2 \rho^\varepsilon(w_1)\right)\chi_\varepsilon \\ &+ \left(\rho^\varepsilon(w_1)(w_2 - w_1)^\alpha\right)\chi_\varepsilon O(\varepsilon) \end{aligned} \tag{5.19}$$

by (5.15). If we have (5.18) then (5.19) implies that

$$0 = \langle \nu,\ h(w_2)\rho^\varepsilon(w_1)(w_2 - a)^\alpha \chi_\varepsilon\rangle + \langle \nu,\ \rho^\varepsilon(w_1)(w_2 - a)^\alpha \chi_\varepsilon\rangle O(\varepsilon) \tag{5.20}$$

where

$$h(w_2) \stackrel{\text{def}}{=} \frac{\partial \lambda_1}{\partial w_1}(a, w_2) I(a, w_2)^2 (w_2 - a)^{-\alpha} = \frac{2\alpha\gamma(-a)^{2\alpha}}{(w_2 - a)^{1+\alpha}}. \tag{5.21}$$

Suppose that (5.8) does not occur. Then $\langle \nu,\ \rho^\varepsilon(w_1)(w_2 - a)^\alpha \chi_\varepsilon\rangle > 0\ \forall \varepsilon > 0$, so we have a well defined probability measure μ^ε on the half line $w_1 = a, w_2 \geq 0$, given by:

$$\langle \mu^\varepsilon,\ \zeta\rangle \stackrel{\text{def}}{=} \frac{\langle \nu,\ \zeta\rho^\varepsilon(w_1)(w_2 - a)^\alpha \chi_\varepsilon\rangle}{\langle \nu, \rho^\varepsilon(w_1)(w_2 - a)^\alpha \chi_\varepsilon\rangle}. \tag{5.22}$$

We observe that

$$(w_2 - a)^\alpha = O(r^{\frac{\gamma-1}{2}}) \tag{5.23}$$

when $r = \|(u, v)\| \to \infty$ and $\frac{\gamma-1}{2} < 2$, so

$$\langle \nu,\ (w_2 - a)^\alpha\rangle < \infty. \tag{5.24}$$

From (5.22) we have that $\mu^\varepsilon \rightharpoonup \mu$, where μ is a certain probability measure on $w_1 = a, w_2 \geq 0$, which we call the *trace* of ν. Then, by (5.20) we get

$$\langle \mu,\ h\rangle = 0. \tag{5.25}$$

This is a contradition since $h > 0$. ∎

So, in order to prove (5.8) it suffices to show that (5.18) is true. We do this by means of the following three lemmas:

Lemma 5.3. *For all $\tilde{w}_1 \in (w_1^-, 0)$ there exists a west entropy $\tilde{\phi}$ bounded with limit $w_1^* = \tilde{w}_1$ such that*

$$\langle \nu, \ \phi \rangle \neq 0. \tag{5.26}$$

Lemma 5.4. *Let $w_1^* \in (w_1^-, 0)$. There exists a constant c such that*

$$\langle \nu, \ \psi \rangle = c \langle \nu, \ \phi \rangle \tag{5.27}$$

for all (ϕ, ψ) east entropy–entropy flux pair with limit w_1^*.

Lemma 5.5. *For every east entropy–entropy flux pair (ϕ, ψ) with limit $w_1^* \in (w_1^-, 0)$, we have*

$$\langle \nu, \ \phi \rangle = \langle \nu, \ \psi \rangle = 0. \tag{5.28}$$

The lemma 5.5 and the *commutation relation* imply (5.18). The lemmas 5.3 and 5.4 are used to prove the lemma 5.5.

Proof of the lemma 5.3. Consider

$$\phi(w_1, w_2) = (w_2 - w_1)^{-\alpha} \int_{\tilde{w}_1}^{w_1} H(\sigma)\theta\,'(t)dt \tag{5.29}$$

a west entropy with limit $w_1^* = \tilde{w}_1$, where $\theta(t) \stackrel{\text{def}}{=} (-t)^\alpha \theta_1(t)$ for some smooth function θ_1 vanishing in $(\tilde{w}_1, 0)$ and H satisfies (4.16). By choosing $\theta_1 \not\equiv 0$ in $(w_1^-, \tilde{w}_1)$ such that θ results to be a monotone function having compact support, we get $\phi \not\equiv 0$ with constant sign and bounded. Then $\langle \nu, \ \phi \rangle \neq 0$ as required. ∎

Proof of the lemma 5.4. Take $w_1 * \in (w_1^-, 0)$. By lemma 5.3 we have a west entropy–entropy flux pair $(\tilde{\phi}, \tilde{\psi})$ with limit $\tilde{w}_1 \leq w_1^*$ such that $\langle \nu, \tilde{\phi} \rangle \neq 0$. Since that $supp \nu \cap supp \tilde{\phi} = \emptyset$, we get the result with

$$c \stackrel{\text{def}}{=} \frac{\langle \nu, \tilde{\psi} \rangle}{\langle \nu, \tilde{\phi} \rangle} \tag{5.30}$$

by applying the *commutation relation* to the pairs (ϕ, ψ), $(\tilde{\phi}, \tilde{\psi})$. ∎

Remark 5.6. It is clear from the above proof that the constant c in the lemma 5.4 can be taken independently of the limit w_1^* varying in any interval of the type $(\eta, 0)$ with $w_1^- < \eta < 0$. In the case w_1^- finite, if there exists an east entropy–entropy flux pair $(\hat{\phi}, \hat{\psi})$ with limit $w_1^* = \hat{w}_1$ larger than w_1^- such that $\langle \nu, \hat{\phi} \rangle \neq 0$ then we can extend this independence of c on w_1^* up to w_1^- by a limiting argument. Besides, in this case the relation (5.27) is also valid for west entropy–entropy flux pair with the universal constant $c \stackrel{\text{def}}{=} \langle \nu, \hat{\psi} \rangle / \langle \nu, \hat{\phi} \rangle$.

We note that the existence of an east entropy $\hat{\phi}$ such that $\langle \nu, \hat{\phi} \rangle \neq 0$ is possible only if *supp.*ν it is not contained in the union of the lines $\{w_1 = w_1^-\}$ and $\{w_1 = 0\}$, since east entropies vanish on these lines.

Proof of the lemma 5.5. Suppose that there exists an east entropy–entropy flux pair $(\hat{\phi}, \hat{\psi})$ with limit $w_1^* = \hat{w}_1$ such that $w_1^- < \hat{w}_1 < 0$ and $\langle \nu, \hat{\phi} \rangle \neq 0$.

By the definition of w_1^-, there exists $-\infty < a \in [w_1^-, \hat{w}_1)$ such that $\nu(\{a \leq w_1 < a + \varepsilon\}) > 0 \quad \forall 0 < \varepsilon \ll 1$. By the lemma 5.4 and remark 5.6, there exists a constant c such that

$$\langle \nu, \ \psi \rangle = c \langle \nu, \ \phi \rangle \tag{5.31}$$

for every east entropy–entropy flux pair (ϕ, ψ) with limit $w_1^* = a$ and

$$\langle \nu, \ \tilde{\psi} \rangle = c \, \langle \nu, \ \tilde{\phi} \rangle \tag{5.32}$$

for every west entropy–entropy flux pair $(\tilde{\phi}, \tilde{\psi})$ with limit $w_1^* = a + \varepsilon < \hat{w}_1$.

Then the *commutation relation* implies that

$$\langle \nu, \ \phi \tilde{\psi} - \tilde{\phi} \psi \, \rangle = 0. \tag{5.33}$$

for these pairs. So we get a contradicttion with lemma 5.2. ∎

We have shown (5.10). By theorem 6.1 in [21] at least one $\beta_l = 0$. Let us consider the Figure 5.1 below.

The *commutation relation* now is

$$\sum_{l=1}^{4}(\beta_l - \beta_l^2)(\phi_l\tilde{\psi}_l - \tilde{\phi}_l\psi_l) = \sum_{\substack{l,k=1 \\ l\neq k}}^{4} \beta_l\beta_k(\phi_l\tilde{\psi}_k - \tilde{\phi}_l\psi_k) \tag{5.34}$$

where, if f is a function, f_l denote $f(A_l)$.

Suppose $\beta_4 = 0$. From (5.34), we have

$$(\beta_3 - \beta_3^2)(\phi_3\tilde{\psi}_3 - \tilde{\phi}_3\psi_3) = 0 \tag{5.35}$$

$\forall(\phi,\psi)$, $(\tilde{\phi},\tilde{\psi})$ west entropy–entropy flux pairs with limits $w_1^* = \frac{a_2}{2}$ and $w_2^* = 0$. Note that

$$\begin{aligned}\psi_3 = \int_{A_1}^{A_3} \frac{\partial\psi}{\partial w_1} dw_1 &= \int_{A_1}^{A_3} \lambda_1 \frac{\partial\phi}{\partial w_1} dw_1 \\ &= \lambda_1(A_3)\phi_3 - \lambda_1(A_1)\phi_1 - \int_{A_1}^{A_3} \frac{\partial\lambda_1}{\partial w_1}\phi dw_1 \\ &= \lambda(A_3)\phi_3 - \int_{A_1}^{A_3} \frac{\partial\lambda_1}{\partial w_1}\phi dw_1.\end{aligned} \tag{5.36}$$

Then

$$\phi_3\tilde{\psi}_3 - \tilde{\phi}_3\psi_3 = \int_{A_1}^{A_3} (\tilde{\phi}_3\phi - \phi_3\tilde{\phi}) \frac{\partial\lambda_1}{\partial w_1} dw_1. \tag{5.37}$$

Since ϕ and $\tilde{\phi}$ are arbitrary on the segment A_1A_3, from (5.34) it follows that

$$\beta_3 = 0 \quad \text{or} \quad \beta_3 = 1. \tag{5.38}$$

Suppose $\beta_3 = 0$. From (5.34), we have

$$(\beta_4 - \beta_4^2)(\phi_4\tilde{\psi}_4 - \widetilde{\phi_4}\psi_4) = 0, \tag{5.39}$$

$\forall(\phi,\psi), (\tilde{\phi},\tilde{\psi})$ west entropy–entropy flux pairs with limits $w_1^* = \frac{a_2}{2}$ and $w_2^* = b_2$. Then

$$\beta_4 = 0 \quad \text{or} \quad \beta_4 = 1. \tag{5.40}$$

If $\beta_2 = 0$, analogously we have $\beta_4 = 0$ or $\beta_4 = 1$ by using $(\phi,\psi), (\tilde{\phi},\tilde{\psi})$ north entropy–entropy flux pairs with limits $w_1^* = a_2$ and $w_2^* = \frac{b_2}{2}$.

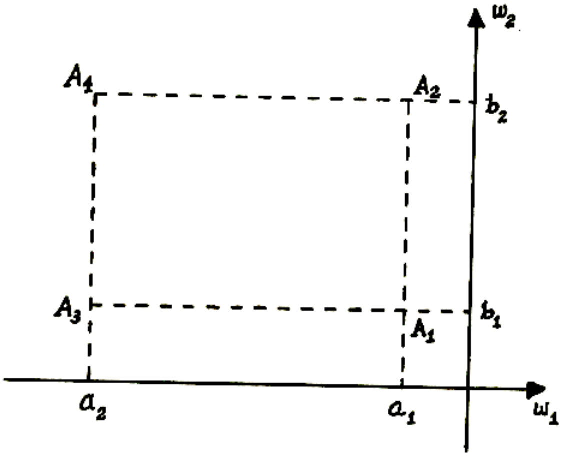

Figure 5.1

Now suppose $\beta_1 = 0$. Here the arguments above do not apply with east or south entropies because these vanish when $w_1 = 0$ or $w_2 = 0$. But we can combine west and north type to obtain, from (5.34),

$$(\beta_4 - \beta_4^2)(\phi_4\tilde{\psi}_4 - \tilde{\phi}_4\psi_4) = 0, \tag{5.41}$$

$\forall(\phi, \psi)$ west with limit $w_1^* = \frac{a_2}{2}$ and $\forall(\tilde{\phi}, \tilde{\psi})$ north with limit $w_2^* = \frac{b_2}{2}$ such that

$$\phi_3 = \psi_3 = \tilde{\phi}_2 = \tilde{\psi}_2 = 0. \tag{5.42}$$

Further, it is not difficult to prove that the functions $J(\cdot, a_2, b_2)$, $L(\cdot, a_2, b_2)$ and $\frac{\partial \lambda_1}{\partial w_1}(\cdot, a_2, b_1)$ in the integral representations (4.22)–(4.27) are linearly independents, so we can choose

$$\phi_4 = \int_{\frac{a_2}{2}}^{a_2} J(t, a_2, b_2)\phi(t, b_1)dt = 1, \tag{5.43}$$

and

$$\psi_4 = \int_{\frac{a_2}{2}}^{a_2} L(t, a_2, b_2)\phi(t, b_1)dt = 1. \tag{5.44}$$

Analogously, we can choose

$$\tilde{\phi}_4 = -\tilde{\psi}_4 = 1. \tag{5.45}$$

So

$$\beta_4 = 0 \quad \text{or} \quad \beta_4 = 1. \tag{5.46}$$

In all cases, we conclude that ν is the sum of at most two delta functions, i.e.,

$$\nu = \rho_1\delta_{B_1} + \rho_2\delta_{B_2}, \tag{5.47}$$

where $\rho_1 + \rho_2 = 1$, $\rho_1, \rho_2 \geq 0$, $B_1 = (c_1, d_1)$ and $B_2 = (c_2, d_2)$ with $c_2 \leq c_1 \leq d_1 \leq d_2$ and $B_1 \neq B_2$.

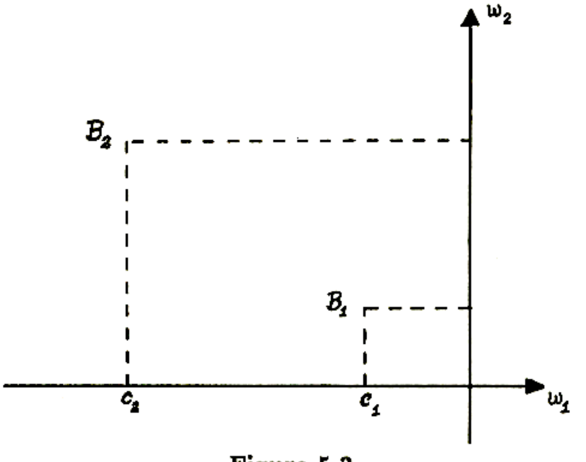

Figure 5.2

Now, if $c_2 < c_1$, taking west entropy–entropy flux pairs with limits $c_1 < w_1^* < c_2$ and $w_2^* = d_1$, (5.34) implies that

$$(\rho_2 - \rho_2^2)(\phi_2\bar{\psi}_2 - \bar{\phi}_2\psi_2) = 0, \tag{5.48}$$

and so $\rho_2 = 0$ or $\rho_2 = 1$, i.e., ν is a Dirac measure. If $c_2 = c_1$, since $B_1 \neq B_2$, we have $d_1 < d_2$, so, taking north entropy–entropy flux pairs with limits $d_1 <$

$w_2^* < d_2$ and $w_1^* = c_1$, we conclude also that ν is a Dirac measure. This was our objective. We have then proved the following:

Theorem 5.7. *Let $\gamma \in (1,2)$ and $z_0: I\!R \longrightarrow I\!R^2$ such that $z_0 \in L^2 \cap L^\infty$ and $k\frac{\pi}{\gamma+1} \leq arg(z_0(x)) \leq (k+1)\frac{\pi}{\gamma+1}$ $\forall x \in I\!R$, for some $k \in Z$. Then the family $\{z^\epsilon\}_{\epsilon>0}$ of solutions of (3.3) contains a subsequence that converges weakly as $\epsilon \to 0+$ to a solution of (1.1).*

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