


HETEROCLINIC BIFURCATION THEORY AND RIEMANN PROBLEMS

Stephen Schecter 

Abstract

The role of heteroclinic bifurcation theory in solving Riemann problems for systems of two conservation laws is presented. This theory is used to study shock waves with viscous profiles, which correspond to heteroclinic orbits of vector fields.

1. Introduction

Solutions of Riemann problems contain shock waves, and shock waves with viscous profiles correspond to heteroclinic orbits of associated vector fields. Thus heteroclinic bifurcation theory should be a useful tool in the study of Riemann problems. Of course, it is not a tool that can do the whole job. In the first place, solutions of Riemann problems also contain rarefaction waves. In the second place, information relevant to shock waves can be lost in passing from the partial differential equation to a heteroclinic bifurcation diagram. Nevertheless, heteroclinic bifurcation theory provides important information about the solution of Riemann problems.

In this paper, I propose to describe, with technicalities generally omitted, the heteroclinic bifurcation theory approach to Riemann problems for systems of two conservation laws. More details can be found in a series of papers by myself and Michael Shearer [19–23]. I will also try to make some observations on the virtues and limitations of this approach.

The paper is organized as follows. In Section 2 the relation of shock waves to heteroclinic bifurcation diagrams is described. In particular, it is pointed

*Research supported by National Science Foundation grant DMS 9002803

out that unstable bifurcation diagrams can occur, and that the theory of singularities of maps can help analyze them. In Section 3 the singularity theory approach to heteroclinic bifurcation diagrams is presented. In Section 4 a particular unstable heteroclinic bifurcation diagram is discussed from the point of view of singularity theory, and a proof is sketched that this diagram occurs in a concrete system of conservation laws. In Section 5 various issues that arise from the earlier sections are discussed more fully.

2. Shock Waves and Heteroclinic Bifurcation Diagrams

A system of two conservation laws in one space-dimension is a partial differential equation of the form

$$U_t + F(U)_x = 0 \quad (2.1)$$

where $U \in \mathbb{R}^2$ and $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. A shock solution with speed s of (2.1) is a discontinuous function

$$U = \begin{cases} U_-, & x < st \\ U_+, & x > st \end{cases} \quad (2.2)$$

that satisfies the Rankine-Hugoniot condition

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0. \quad (2.3)$$

The shock (2.2) has a viscous profile if the equation

$$U_t + F(U)_x = \epsilon U_{xx} \quad (2.4)$$

has a traveling wave solution

$$U\left(\frac{x-st}{\epsilon}\right) \quad (2.5)$$

with

$$\lim_{\xi \rightarrow \pm\infty} U(\xi) = U_{\pm}, \quad \lim_{\xi \rightarrow \pm\infty} U'(\xi) = 0. \quad (2.6)$$

If we substitute (2.5) into (2.4), integrate once, and use the left-hand boundary conditions from (2.6), we find that a shock solution of (2.1) with left state U_- , speed s , and viscous profile corresponds to a heteroclinic orbit of

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \quad (2.7)$$

from U_- to a second equilibrium U_+ .

It is thus natural to regard (2.7), with U_- fixed, as defining a one-parameter heteroclinic bifurcation problem; the parameter is s . Notice that for each s , U_- is an equilibrium of (2.7). One wishes to identify all pairs (s, U_+) such that the Rankine-Hugoniot condition (2.3) holds (i.e., U_+ is a second equilibrium of (2.7)), and an orbit of (2.7) goes from U_- to U_+ .

Let us emphasize that the discussion is restricted to systems of two conservation laws and to the simple viscosity U_{xx} . The restriction on the viscosity is largely for convenience. Some work relevant to the extension of the discussion to systems of n conservation laws is in [1].

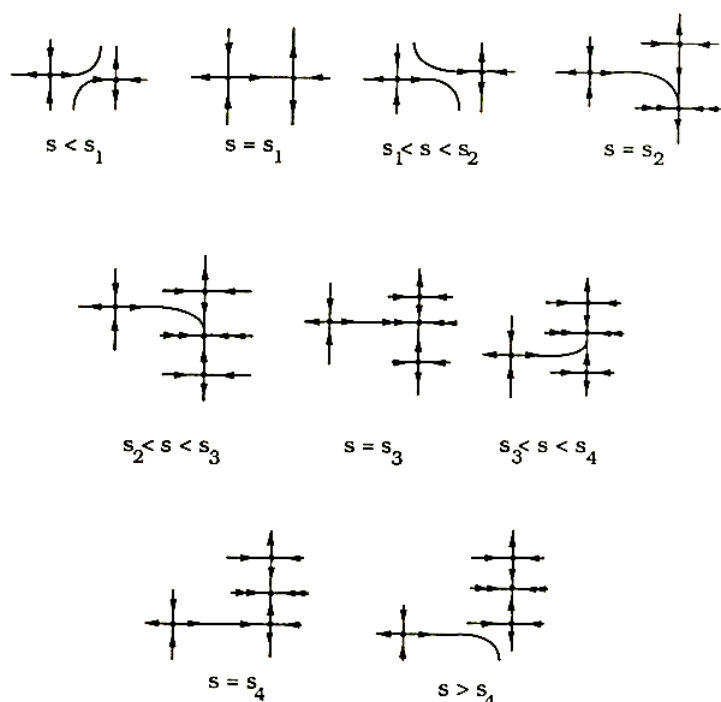


Figure 1. The left-hand equilibrium is U_- .

In (2.7), suppose that for a fixed U_- , as s varies in some interval, the phase

portrait undergoes the changes portrayed in Figure 1. Then for $s = s_1$ and for $s_2 \leq s \leq s_4$, there are shocks with viscous profile with left state U_- . Notice that saddle-to-saddle heteroclinic orbits occur at $s = s_1$ and $s = s_4$, and a saddle-node bifurcation occurs at $s = s_2$. The bifurcation at $s = s_3$, where the unstable manifold of U_- meets the strong stable manifold of a node, has no real importance.

The sequence of phase portraits shown in Figure 1 can be encoded in the heteroclinic bifurcation diagram of Figure 2. (Stability of the equilibria has not been represented in order to simplify the picture.) The solid curve represents the right hand equilibria; the dashed curve represents the position of the incoming unstable manifold of the saddle at U_- .

The following assumptions about Figure 1 are consistent with what has been said so far:

1. The dashed and solid curves are regular curves that are transverse to the vertical except at $s = s_2$, where the solid curve has a quadratic intersection with the vertical. This assumption expresses the fact that all equilibria are hyperbolic except at $s = s_2$, where one undergoes a saddle-node bifurcation.
2. The dashed curve is transverse to the solid curve. Thus at $s = s_1$ and $s = s_4$, the saddle-to-saddle connection breaks in a nondegenerate manner as s increases.
3. The dashed and solid curves do not meet at $s = s_2$. This means that at the moment of bifurcation of a right-hand equilibrium, the incoming unstable manifold from U_- does not meet the stable manifold of the bifurcating equilibrium.

Under these assumptions, the bifurcation diagram of Figure 2 is stable to perturbation. Thus, if we change U_- slightly, we will have qualitatively the same bifurcation diagram over the s -interval shown.

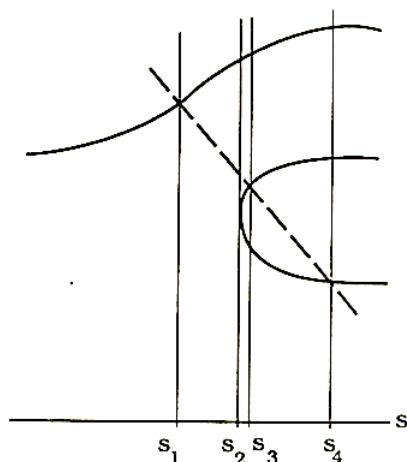


Figure 2. Bifurcation diagram for Figure 1.

When we speak of how a bifurcation diagram changes as U_- varies, we are regarding s as a “distinguished” parameter and U_- as a vector of “unfolding” parameters, in the terminology of [2]. Since $U_- \in \mathbb{R}^2$, the family of vector fields (2.7) has two unfolding parameters; additional unfolding parameters may be present in the formula for F . Because of the presence of unfolding parameters, one expects to observe, for certain U_- , bifurcation diagrams that are not stable to perturbation. It should be useful to be able to recognize the unstable diagrams that frequently occur, to know what other diagrams occur for nearby values of the unfolding parameters, and to understand how the space of unfolding parameters is divided into regions in which different bifurcation diagrams occur.

In particular, I have studied bifurcation diagrams in which, at the s -value at which the right-hand equilibrium bifurcates, the unstable manifold of U_- meets the stable manifold of the bifurcating equilibrium. In other words, at least the third condition above is violated. The motivation for beginning with this type

of instability is that it comes up in the study of hyperbolic problems with an umbilic point (i.e., in (2.1), $DF(U)$ has distinct real eigenvalues except at a point $U = U_0$ where $DF(U_0)$ is a multiple of the identity.)

For this type of instability at least, the questions I have posed can be answered using the theory of singularities of maps. This theory grew out of Whitney's work on maps from \mathbf{R}^2 to itself, where the stable singularities are folds and cusps [28]. Mather, in a classic series of papers [10–15], gave the theory its modern form. A recent exposition is [9]. More directly relevant to the present situation is the work of Golubitsky and Schaeffer [2], who adapted singularity theory to the study of one-parameter problems, and the work of Vegter [27], who showed how to use singularity theory to study heteroclinic connections to semihyperbolic equilibria in the plane (equilibria with one zero eigenvalue). The Golubitsky-Schaeffer and Vegter theories can be combined to yield a theory of heteroclinic bifurcation problems with a distinguished parameter. This theory is described in the following section.

3. Singularity Theory and Heteroclinic Bifurcation Diagrams

Let us consider a C^∞ vector field in \mathbf{R}^2 with one parameter,

$$\dot{y} = f(y, \lambda), \quad y \in \mathbf{R}^2, \quad \lambda \in \mathbf{R}. \quad (3.1)$$

We assume that for $\lambda = 0$,

- (1) there is a hyperbolic saddle at p_0 ;
- (2) there is an equilibrium at q_0 with one zero eigenvalue and one negative eigenvalue;
- (3) the unstable manifold of p_0 meets the stable manifold of q_0 in an orbit Γ .

See Figure 3.

There are a number of consequences of these assumptions.

1. For each λ near 0 there is a hyperbolic saddle $p(\lambda)$ of (3.1); the map $p: \mathbf{R} \rightarrow \mathbf{R}^2$ is C^∞ .

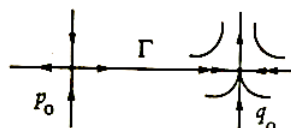


Figure 3.

2. Choose C^∞ coordinates $y = (u, v)$ such that $q_0 = (0, 0)$ and such that the u -axis is not the eigendirection for the eigenvalue 0 of $D_1 f(q_0, 0)$. In these coordinates (3.1) becomes

$$\begin{aligned}\dot{u} &= f_1(u, v, \lambda), \\ \dot{v} &= f_2(u, v, \lambda).\end{aligned}\tag{3.2}$$

The extended center manifold of $\dot{y} = f(y, \lambda)$ at $(q_0, 0)$ then has the form

$$u = \psi(v, \lambda).$$

See Figure 4. The function ψ is C^p , where p can be made arbitrarily large by restricting the neighborhood of $(0, 0)$ on which it is defined, but is not in general C^∞ . The series expansion of ψ is computable [3].

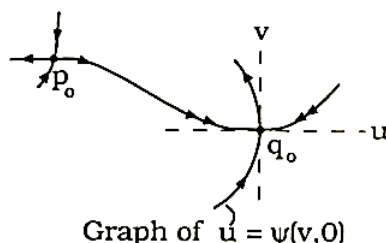


Figure 4. Phase portrait of $\dot{u} = f_1(u, v, 0)$, $\dot{v} = f_2(u, v, 0)$.

3. The flow of (3.2) on the extended center manifold is

$$\dot{v} = g_2(v, \lambda) := f_2(\psi(v, \lambda), v, \lambda),$$

where g_2 is C^p .

4. For fixed λ near $\lambda = 0$, equilibria of $\dot{y} = f(y, \lambda)$ near $y = q_0$ lie in the invariant curve $u = \psi(v, \lambda)$. Thus they can be found by solving the equation

$$g_2(v, \lambda) = 0. \quad (3.3)$$

We always have $g_2(0, 0) = \frac{\partial g_2}{\partial v}(0, 0) = 0$. If $\frac{\partial g_2}{\partial \lambda}(0, 0) \neq 0$, the equation (3.3) can be solved for λ , so solutions of (3.3) are parameterized by v . In the case of saddle-node bifurcation, for example, if we have

$$\dot{v} = g_2(v, \lambda) = -\lambda + v^2 + \dots,$$

then

$$\lambda(v) = v^2 + \dots;$$

see Figure 5. The equilibria of (3.2) are then $(q(v), \lambda(v))$, $\lambda(v)$ given by (3.3),

$$q(v) = (\psi(v, \lambda(v)), v).$$

If $\frac{\partial g_2}{\partial \lambda}(0, 0) = 0$, one needs additional parameters in order to be able to parameterize the equilibria. An example is discussed in Section 4.

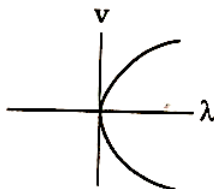


Figure 5. Equilibria of $\dot{v} = -\lambda + v^2 + \dots$.

5. The equilibrium $(q(v), \lambda(v))$ of (3.2) has a unique invariant manifold $W(v)$ near the stable manifold of $(q_0, 0)$. If $q(v)$ is a saddle, $W(v)$ is its stable manifold; if $q(v)$ is a node, $W(v)$ is its strong stable manifold. See Figure 6.

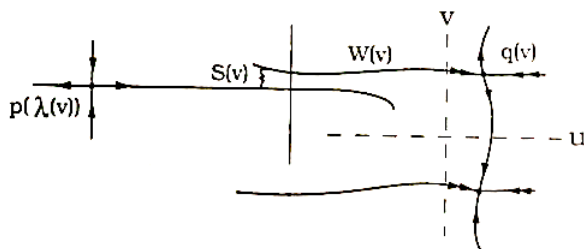


Figure 6. Phase portrait of $\dot{y} = f(y, \lambda(v))$ showing invariant manifolds and the separation function.

6. The signed separation $S(v)$ between the unstable manifold of $p(\lambda(v))$ and $W(v)$ can be measured at a convenient cross-section to the flow. See Figure 6. The function S is C^p . Its first derivatives can be computed as "Melnikov integrals", except that a boundary term must be added to the integral since q_0 is not hyperbolic [19].

Melnikov [16] derived the integral formula now named for him in his study of the separation between the stable and unstable manifolds of a hyperbolic saddle in a one-degree-of-freedom Hamiltonian system subject to small time-periodic forcing. Holmes [4,3] extended and popularized Melnikov's work. At an IMPA conference it is perhaps appropriate to note that at about the same time as Melnikov, Sotomayor, in his IMPA thesis [26], derived the same integral formula. He was studying the splitting of heteroclinic orbits joining hyperbolic saddles of planar autonomous systems.

Let us define

$$g_1(v, \lambda) := S(v).$$

Then the map $(g_1, g_2)(v, \lambda)$ from \mathbf{R}^2 to \mathbf{R}^2 , which we shall call the *bifurcation mapping*, encodes the behavior of $\dot{y} = f(y, \lambda)$ near Γ . In fact:

1. Equilibria of $\dot{y} = f(y, \lambda)$ near $(q_0, 0)$ correspond to solutions of $g_2 = 0$, as we have seen. The solution corresponds to a saddle if $\frac{\partial g_2}{\partial v} > 0$, a node if $\frac{\partial g_2}{\partial v} < 0$, and a semihyperbolic equilibrium if $\frac{\partial g_2}{\partial v} = 0$.

2. Connections from $p(\lambda)$ to a hyperbolic saddle near q_0 correspond to solutions of $g_1 = g_2 = 0$ at which $\frac{\partial g_2}{\partial v} > 0$.

3. Connections from $p(\lambda)$ to a node near q_0 along the strong stable manifold of the latter correspond to solutions of $g_1 = g_2 = 0$ at which $\frac{\partial g_2}{\partial v} < 0$.

Bifurcation diagrams like Figure 2 show the curve $g_1 = 0$ dashed and the curve $g_2 = 0$ solid.

In order to analyze the mapping (g_1, g_2) it may be convenient to make coordinate changes to simplify it. One wants to allow a large enough class of coordinate changes to make simplification possible, but not such a large class that important information may be lost. These two objectives can be in conflict, and that appears to be the case here. Motivated by [2] and [27], we proceed as follows. We shall say that a bifurcation mapping (g_1, g_2) is C^* \mathcal{U} -equivalent to (h_1, h_2) if there are real-valued C^* functions $A(v, \lambda)$, $B(v, \lambda)$, $C(v, \lambda)$, $V(v, \lambda)$, $\Lambda(\lambda)$, defined on neighborhoods of the origin, with $A > 0$, $C > 0$, $V(0, 0) = 0$, $\frac{\partial V}{\partial v} > 0$, $\Lambda(0) = 0$, $\Lambda' > 0$, such that

$$\begin{pmatrix} g_1(v, \lambda) \\ g_2(v, \lambda) \end{pmatrix} = \begin{pmatrix} A(v, \lambda) & B(v, \lambda) \\ 0 & C(v, \lambda) \end{pmatrix} \begin{pmatrix} h_1(V(v, \lambda), \Lambda(\lambda)) \\ h_2(V(v, \lambda), \Lambda(\lambda)) \end{pmatrix}. \quad (3.4)$$

Note that, because of the role of λ as a parameter, the change of coordinates in λ can only depend on λ , but the change of coordinates in v can depend on both v and λ . The conditions $\frac{\partial V}{\partial v} > 0$ and $\Lambda' > 0$ preserve the orientations of v -space and λ -space. The required 0 in the matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ means that the coordinate change $(v, \lambda) \mapsto (V(v, \lambda), \Lambda(\lambda))$ takes zeros of g_2 to zeros of h_2 , and takes solutions of $g_1 = g_2 = 0$ to solutions of $h_1 = h_2 = 0$. The conditions $A > 0$ and $C > 0$ preserve the signs of the derivatives of g_1 and g_2 at these points.

The zero set of g_1 is not, however, taken to that of h_1 . In fact, from the definition of the separation function, g_1 should really be thought of as defined on the zero set of g_2 , i.e., as equivalent to g_1 plus any function times g_2 . This is why we allow $B(v, \lambda)$ to be nonzero.

These coordinate changes generally allow simplification of (g_1, g_2) to a polynomial normal form (h_1, h_2) . Unfortunately at least one key piece of information

for conservation laws has been lost: the Hugoniot locus, which is the projection of

$$\{(u, v, \lambda) : g_2(v, \lambda) = 0 \text{ and } u = \psi(v, \lambda)\}$$

to the uv -plane, cannot be recovered from the normal form (h_1, h_2) , unless we also keep track of the coordinate changes $(V(v, \lambda), \Lambda(\lambda))$ used to produce the normal form. The reason is that the coordinate change $V(v, \lambda)$ is not a function of v only. Another difficulty is discussed in Section 5.

A shorthand notation for (3.4) is

$$g(v, \lambda) = S(v, \lambda)h(V(v, \lambda), \Lambda(\lambda)),$$

where g and h map \mathbf{R}^2 to \mathbf{R}^2 and S maps \mathbf{R}^2 into \mathcal{U} , the space of upper triangular 2×2 matrices with positive diagonal entries. A k -parameter unfolding of g is a map $G : \mathbf{R}^2 \times \mathbf{R}^k \rightarrow \mathbf{R}^2$ such that $G(v, \lambda, 0) = g(v, \lambda)$. Let $G(v, \lambda, \beta)$ and $H(v, \lambda, \alpha)$ be two unfoldings of g , possibly with different numbers of parameters. We say G C^s -factors through H if there are C^s maps $S(v, \lambda, \beta)$ into \mathcal{U} , $V(v, \lambda, \beta)$ into \mathbf{R} , $\Lambda(\lambda, \beta)$ into \mathbf{R} , and $A(\beta)$ into α -space, with $S(v, \lambda, 0) = I$, $V(v, \lambda, 0) = v$, $\Lambda(\lambda, 0) = \lambda$, $A(0) = 0$, such that

$$G(v, \lambda, \beta) = S(v, \lambda, \beta)H(V(v, \lambda, \beta), \Lambda(\lambda, \beta), A(\beta)).$$

H is (s, p) -universal if every C^p unfolding G of g C^s -factors through H , and the number of parameters is minimal. If g has a universal unfolding with k parameters, then g is said to be of codimension k ; if g does not have a universal unfolding, then g is of infinite codimension. (If g is C^q and $1 \leq s \leq p \leq q$, it turns out that the codimension of g is independent of s and p as long as the difference $p - s$ is big enough.) If g is of codimension k , then bifurcation mappings equivalent to g should occur in typical problems with k unfolding parameters. A universal unfolding of g then exhibits all the perturbations of g , or any germ equivalent to it, that can occur, modulo the equivalence relation (3.4). Moreover, consider the division of the parameter space for a universal unfolding into subsets representing equivalent bifurcation mappings. Via the

mapping $\beta \mapsto A(\beta)$, this division can be pulled back to a division for an arbitrary unfolding of g .

The apparatus of singularity theory, which I shall not describe here, can produce, for a polynomial normal form h of finite codimension, criteria for recognizing maps g equivalent to h ; a simple polynomial universal unfolding of h ; and criteria for recognizing when an unfolding of a map g equivalent to h is a universal unfolding of g [19,20]. Universal unfoldings of g and h are equivalent (under an extension of the equivalence relation (3.4) to mappings with unfolding parameters), so the polynomial universal unfolding of h can be studied as a prototype.

The recognition criteria for normal forms and universal unfoldings generally have geometric interpretations as transversality conditions. This will be illustrated in the following section.

4. An Example

In order to indicate the power of these ideas, let us consider the two normal forms

$$\begin{aligned} h_1(v, \lambda) &= -v + \delta\lambda, \\ h_2(v, \lambda) &= v^3 - v\lambda, \end{aligned} \quad (4.1)$$

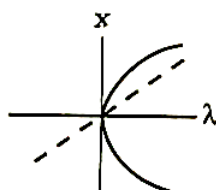
with $\delta = \pm 1$. These normal forms represent an equilibrium undergoing a pitchfork bifurcation at the moment when the unstable manifold of a distant saddle meets its stable manifold; see Figure 7. If $1 \leq s \leq p$ and $p-s$ is large enough, a C^p bifurcation mapping $(g_1(v, \lambda), g_2(v, \lambda))$ is C^s \mathcal{U} -equivalent to (4.1) provided at $(0, 0)$,

$$g_1 = g_2 = g_{2v} = g_{2\lambda} = g_{2v\lambda} = g_{2vv} = 0; \quad (4.2)$$

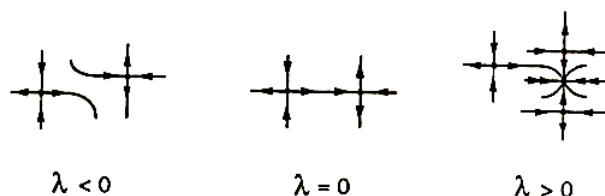
$$g_{1v} < 0, \quad g_{2v\lambda} < 0, \quad g_{2vvv} > 0, \quad \text{sgn}(g_{1v}g_{2\lambda\lambda} - 2g_{1\lambda}g_{2v\lambda}) = \delta. \quad (4.3)$$

(This and other results in this section are from [21].) The criteria involving only g_2 are recognition criteria for a pitchfork bifurcation; see [2]. The condition $g_{1v}g_{2\lambda\lambda} - 2g_{1\lambda}g_{2v\lambda} \neq 0$ says that the curve $g_1 = 0$ is transverse to the curve

of "trivial equilibria" for the pitchfork bifurcation. The sign of the inequality indicates how the curves cross.



(a) Bifurcation Diagram



(b) Phase Portraits

Figure 7. $h(v, \lambda) = (-v + \lambda, v^3 - v\lambda)$. For $\lambda = 0$, the right hand equilibrium is semihyperbolic, with cubic behavior on its vertical center manifold.

A universal unfolding of (h_1, h_2) is

$$\begin{aligned} H_1(v, \lambda, \alpha_1, \alpha_2, \alpha_3) &= -v + \delta\lambda + \alpha_1, \\ H_2(v, \lambda, \alpha_1, \alpha_2, \alpha_3) &= v^3 - v\lambda + \alpha_2 v^2 + \alpha_3. \end{aligned} \quad (4.4)$$

Several transition varieties in α -space can be identified, which correspond to unstable bifurcation diagrams. In general these transition varieties are given by the simultaneous vanishing of three functions.

1. The *equilibrium bifurcation variety* \mathcal{B} is the set of α for which there is a point (v, λ) where $H_2 = H_{2v} = H_{2\lambda} = 0$.

2. The *hysteresis variety* \mathcal{H} is the set of α for which there is a point (v, λ) where $H_2 = H_{2v} = H_{2vv} = 0$.

3. The *equilibrium/heteroclinic bifurcation variety* \mathcal{E} is the set of α for which there is a point (v, λ) where $H_1 = H_2 = H_{2v} = 0$.

4. The *variety of simultaneous equilibrium and heteroclinic bifurcations* \mathcal{F} is the closure of the set of α for which there is a point (v_1, λ) where $H_1 = H_2 = 0$ and a distinct point (v_2, λ) (same λ) where $H_2 = H_{2v} = 0$.

5. The *degenerate heteroclinic bifurcation variety* \mathcal{N} is the set of α for which there is a point (v, λ) where $H_1 = 0$ and $H_2 = 0$ are tangent, i.e., where $H_1 = H_2 = H_{1v}H_{2\lambda} - H_{1\lambda}H_{2v} = 0$.

Figures 8 and 9 show the intersection of these varieties with a plane $\alpha_1 = \text{constant} < 0$, and the stable bifurcation diagrams that occur in the various connected components of the complement of the union of these varieties. Note in particular that \mathcal{B} is the α_2 -axis; \mathcal{H} has cubic contact with \mathcal{B} at the origin; \mathcal{F} has quadratic contact with \mathcal{B} ; and \mathcal{E} , \mathcal{F} , and \mathcal{H} share a point of quadratic contact. The simple form (4.4) facilitates the discovery of these facts.

Let $(G_1, G_2)(v, \lambda, \beta_1, \beta_2, \beta_3)$ be an unfolding of a germ $(g_1, g_2)(v, \lambda)$ that satisfies the recognition criteria (4.2–3). Define $G: \mathbb{R}^5 \rightarrow \mathbb{R}^5$ by $G = (G_1, G_2, G_{2v}, G_{2\lambda}, G_{2vv})$; compare (4.2). Then (G_1, G_2) is universal if and only if $DG(0)$ is invertible. If (G_1, G_2) is universal, there is a diffeomorphism of β -space to α -space that takes the transition varieties for (G_1, G_2) to those for (H_1, H_2) .

To conclude this section we shall show that the normal forms (4.1) and their universal unfoldings occur in a concrete system of conservation laws.

Consider the system of partial differential equations

$$\begin{aligned} u_t + (-u^2 + 2\mu_1 uv + v^2)_x &= 0, \\ v_t + (\mu_1 u^2 + 2uv)_x &= 0, \end{aligned} \quad (4.5)$$

μ_1 a parameter, which is a much studied simple example of a system of hyperbolic conservation laws having, for each μ_1 , an umbilic point at the origin

[5,6,17,18,24,25]. In fact, consider the perturbation

$$\begin{aligned} u_t + (-u^2 + 2\mu_1 uv + v^2)_x &= 0, \\ v_t + (\mu_1 u^2 + 2uv + \mu_2 u^3 + \mu_3 u^4)_x &= 0. \end{aligned} \quad (4.6)$$

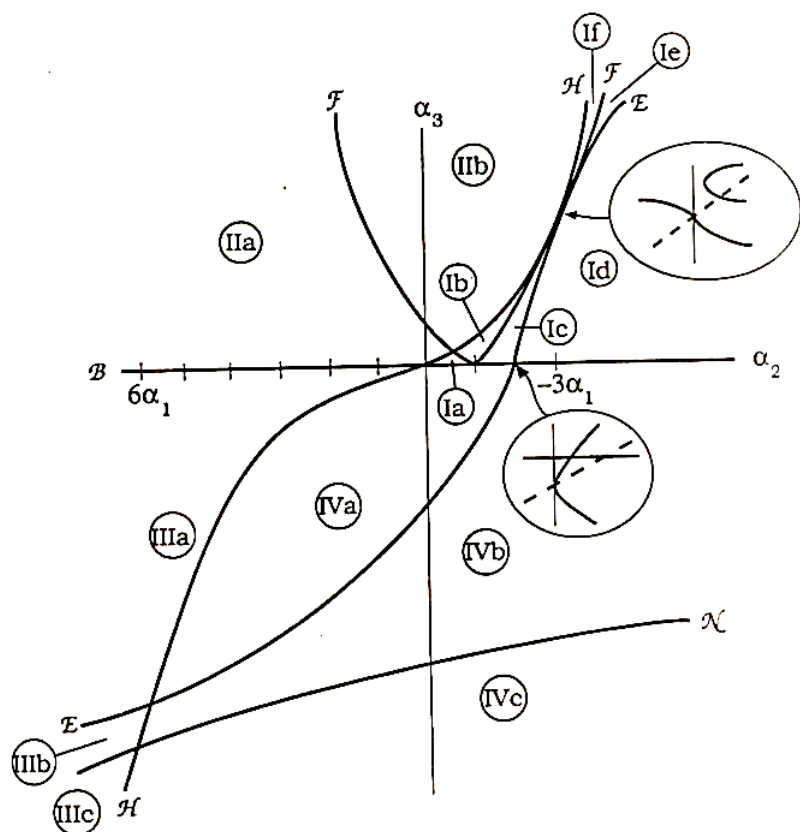


Figure 8. $H(v, \lambda, \alpha_1, \alpha_2, \alpha_3) = (-v + \lambda + \alpha_1, v^3 - v\lambda + \alpha_2 v^2 + \alpha_3)$. Transition varieties in a plane $\alpha_1 = \text{constant} < 0$ and two unstable bifurcation diagrams.

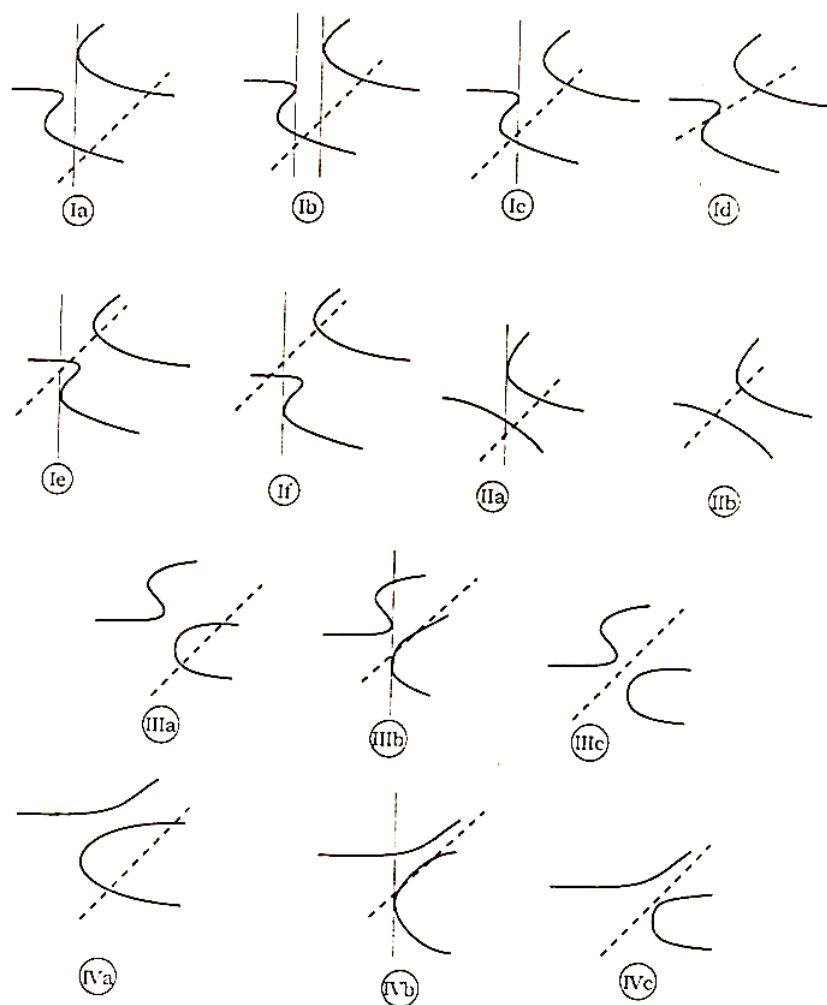


Figure 9. $H(v, \lambda, \alpha_1, \alpha_2, \alpha_3) = (-v + \lambda + \alpha_1, v^3 - v\lambda + \alpha_2 v^2 + \alpha_3)$.
Stable bifurcation diagrams that occur for $\alpha_1 < 0$.

Then a shock solution of (4.6) with left state (u_-, v_-) , speed s , and viscous profile corresponds to a heteroclinic orbit of

$$\begin{aligned}\dot{u} &= -(u^2 - u_-^2) + 2\mu_1(uv - u_-v_-) + (v^2 - v_-^2) - s(u - u_-), \\ \dot{v} &= \mu_1(u^2 - u_-^2) + 2(uv - u_-v_-) + \mu_2(u^3 - u_-^3) + \mu_3(u^4 - u_-^4) - s(v - v_-),\end{aligned}\quad (4.7)$$

from (u_-, v_-) to a second equilibrium.

In (4.7) we let u_- be a fixed negative number, let $\mu = (\mu_1, \mu_2, \mu_3)$, and regard (v_-, μ) as a vector of unfolding parameters. If we then set $(v_-, \mu) = (0, 0)$, we obtain a one-parameter family of vector fields in the uv -plane,

$$\begin{aligned}\dot{u} &= -(u^2 - u_-^2) + v^2 - s(u - u_-), \\ \dot{v} &= 2uv - sv,\end{aligned}\quad (4.8)$$

with the following properties (see Figure 10):

1. There is symmetry about the u -axis. As a result, the u -axis is invariant for each s .
2. There are equilibria on the u -axis at $(u_-, 0)$ and $(-u_- - s, 0)$.
3. For $2u_- < s < -\frac{2}{3}u_-$, both equilibria are hyperbolic saddles, $u_- < -u_- - s$, and there is a heteroclinic solution from $(u_-, 0)$ to $(-u_- - s, 0)$ along the u -axis.

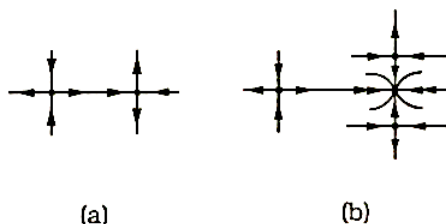


Figure 10. Flow of (4.7) near the u -axis, (a) for $2u_- < s \leq -\frac{2}{3}u_-$, and (b) for s a little larger than $-\frac{2}{3}u_-$. When $s = -\frac{2}{3}u_-$, the right hand equilibrium is semihyperbolic, with cubic behavior on its vertical center manifold.

4. At $s = -\frac{2}{3}u_-$, one eigenvalue at the right-hand equilibrium $(-\frac{1}{3}u_-, 0)$, with eigenvector $(0, 1)$ transverse to the u -axis, becomes zero; in fact, a pitchfork

bifurcation occurs. There is still a heteroclinic orbit from the hyperbolic saddle $(u_-, 0)$ to $(-\frac{1}{3}u_-, 0)$ along the u -axis, which is the stable manifold of the latter.

This one-parameter family is of infinite codimension because the connection between the equilibria does not break as s varies. To make this statement a little more precise, we note that for the corresponding bifurcation mapping (g_1, g_2) , the curve $g_1 = 0$ (a line) is contained in the curve $g_2 = 0$ (a pitchfork). No perturbation with a finite number of parameters can capture all possible perturbations of this picture.

Theorem *There is a smooth curve C through the origin in (v_-, μ) space,*

$$(v_-(t), \mu(t)) = wt + O(t^2),$$

with each component of w positive, such that if (v_-, μ) is a fixed point on C then the one-parameter family (4.7) has, at a unique s near $-\frac{2}{3}u_-$, a pitchfork bifurcation at an equilibrium near $(-\frac{1}{3}u_-, 0)$, together with a heteroclinic orbit along its stable manifold from the hyperbolic saddle (u_-, v_-) . At any point on C other than the origin, the normal form for this bifurcation is (4.1); δ is the sign of t . Moreover, a universal unfolding of (4.1) occurs on any three-dimensional slice of parameter space transverse to C through a nonzero point of C .

The proof of this theorem goes as follows. We perform center manifold reduction on (4.7) at the point

$$(u, v, s, v_-, \mu) = (-\frac{1}{3}u_-, 0, -\frac{2}{3}u_-, 0, 0),$$

then let

$$s = -\frac{2}{3}u_- + \lambda,$$

obtaining

$$\dot{v} = G_2(v, \lambda, v_-, \mu). \quad (4.9)$$

It turns out that

$$\frac{\partial G_2}{\partial v_-}(0, 0, 0, 0) \neq 0,$$

so that equilibria of (4.9) are parameterized by (v, λ, μ) . Thus there is a separation function $S(v, \lambda, \mu)$, and we define

$$G_1(v, \lambda, v_-, \mu) = S(v, \lambda, \mu).$$

G_2 is easily computed to any order using a symbolic manipulation program, and first derivatives of G_1 with respect to v and the μ_i can be computed at any point on the λ -axis as Melnikov integrals.

For fixed (v^*, μ^*) , the germ of the C^p mapping (G_1, G_2) at $(v^*, \lambda^*, v_-^*, \mu^*)$ is C^* \mathcal{U} -equivalent to one of the normal forms (4.1), s sufficiently smaller than p , provided at $(v^*, \lambda^*, v_-^*, \mu^*)$,

$$G_1 = G_2 = G_{2v} = G_{2\lambda} = G_{2vv} = 0,$$

$$G_{1v} < 0, \quad G_{2v\lambda} < 0, \quad G_{2vvv} > 0, \quad G_{1v}G_{2\lambda\lambda} - 2G_{1\lambda}G_{2v\lambda} \neq 0;$$

these are the recognition criteria (4.2-3).

For $(v_-^*, \mu^*) = (0, 0)$, all these criteria are satisfied at $(v^*, \lambda^*) = (0, 0)$ except the last. The failure of the last criterion is due to the fact that when $(v_-^*, \mu^*) = (0, 0)$, the heteroclinic orbit does not break as λ varies.

Now $\mathbf{G} = (G_1, G_2, G_{2v}, G_{2\lambda}, G_{2vv})$ is a function from \mathbf{R}^6 ($v\lambda v_- \mu$ -space) to \mathbf{R}^5 . It turns out that $D\mathbf{G}(0)$ is surjective, so by the implicit function theorem there is a unique smooth curve \hat{C} through the origin in \mathbf{R}^6 , tangent at the origin to \hat{w} , a null vector for $D\mathbf{G}(0)$, on which $\mathbf{G} = 0$.

By continuity, points on \hat{C} near the origin have $G_{1v} < 0$, $G_{2v\lambda} < 0$, and $G_{2vvv} > 0$. The projection of \hat{C} to $v_- \mu$ -space is the curve C whose existence is asserted in the theorem. The statement about the normal form at nonzero points of C is proved by showing that $D(G_{1v}G_{2\lambda\lambda} - 2G_{1\lambda}G_{2v\lambda})(0, 0, 0, 0)\hat{w} > 0$; thus the last recognition criterion is satisfied at nonzero points of \hat{C} . The statement about universal unfoldings follows from the recognition criterion for universal unfoldings and the fact that \mathbf{G} is transverse to \hat{C} .

5. Discussion

Let us consider a system of conservation laws (2.1) with associated ordinary differential equation (2.7) and corresponding heteroclinic bifurcation diagram.

The stable bifurcations that occur in these diagrams should in turn correspond to local existence-uniqueness theorems for solutions of Riemann problems, provided additional assumptions are made. A Riemann problem is of course (2.1) with piecewise constant initial conditions,

$$u(x, 0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0. \end{cases} \quad (5.1)$$

Such an existence-uniqueness theorem is worked out for the nondegenerate saddle-to-saddle heteroclinic bifurcation in [23]. In this bifurcation, (2.7), with $U_- = U_o$ and $s = s_o$, has a heteroclinic orbit from a hyperbolic saddle at U_o to another hyperbolic saddle U_o^* . The orbit breaks in a nondegenerate manner as s varies. A normal form for the associated bifurcation mapping is

$$\begin{aligned} h_1(v, \lambda) &= -v, \\ h_2(v, \lambda) &= v + \delta\lambda, \end{aligned}$$

$\delta = \pm 1$, which has codimension 0.

To describe the result of [23], let us assume in addition that (2.1) is strictly hyperbolic and genuinely nonlinear near U_o and U_o^* . For U_L near U_o^* let $W_1(U_L)$ be the usual slow wave curve through U_L . For each $U_- \in W_1(U_L)$ there is a unique pair (s, U_+) near (s_o, U_o^*) such that the differential equation (2.7) has a heteroclinic orbit from U_- to U_+ . The collection of such U_+ forms an *undercompressive shock curve* $\Sigma(U_L)$. Let us make one final assumption, that $\Sigma(U_o)$ and the fast wave curve through U_o^* are transverse. Then for each U_L near U_o^* and U_R near U_o , there is a unique solution of the Riemann problem (2.1), (5.1) that consists of a slow wave from U_L to some U_- in $W_1(U_L)$, a shock from U_- to U_+ near U_o^* (of course $U_+ \in \Sigma(U_L)$), and a fast wave from U_+ to U_R . See Figure 11.

Note that this result requires two assumptions in addition to those encoded in the normal form of the bifurcation mapping. This illustrates the principle that information is lost in passing from the PDE (2.1) to the bifurcation diagram, or, more precisely, to the normal form of the bifurcation mapping. At least some of the loss of information can be traced to the largeness of the class of

transformations allowed in putting the bifurcation mapping (g_1, g_2) into normal form. In particular, in addition to information about the Hugoniot locus, as was mentioned in Section 3, all information about the undercompressive shock curves is lost in passage to the normal form. Unfortunately, it is not clear how to get by with a significantly smaller class of transformations.

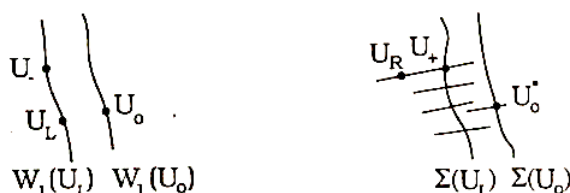


Figure 11. Solution of a Riemann in the presence of undercompressive shocks. Fast wave curves are shown as light curves. Given U_L and U_R , (1) locate U_+ on $\Sigma(U_L)$ such that the fast wave curve through U_+ meets U_R ; (2) locate U_- on $W_1(U_L)$ such that there is an undercompressive shock from U_- to U_+ . The solution is then a slow wave from U_L to U_- , a shock from U_- to U_+ , and a fast wave from U_+ to U_R .

A result similar to that in [23] can presumably be proved for saddle-node bifurcation at U_+ when the unstable manifold of U_- does not meet the stable manifold of U_+ . The Riemann problem solution would use *shock-rarefactions* and *rarefaction-shocks*, constructions introduced by Liu for problems without genuine nonlinearity [8].

In the same spirit, it would be interesting to try to relate low codimension bifurcation diagrams to bifurcations in the pattern of solutions of Riemann problems, under additional hypotheses.

Next we remark that in a concrete problem, even if there is no normal form, it may be possible to produce descriptions of transition varieties like those in Section 4. An example occurs in [21], where cubic perturbations of the flux function in (4.5) were studied. As noted in Section 4, the bifurcation mapping of (4.8) has infinite codimension, so as far as I know there is no normal form. In

[22] we put G_2 into normal form, but G_1 could not be reduced to a polynomial. Nevertheless it was possible to compute the transition varieties rather explicitly.

In this problem we also computed to low order the coordinate changes that converted the problem to partial normal form. Thus we were able to recover information about the Hugoniot loci and undercompressive shock curves. This indicates some of the additional work that may be necessary in using heteroclinic bifurcation theory to study Riemann problems.

Finally we note that since the normal forms for bifurcation mappings correspond to transversality conditions, heteroclinic bifurcation theory should have a natural role to play in the new geometric theory of Riemann problems being developed by Isaacson, Marchesin, Palmeira, and Plohr [7].

References

- [1] Battelli, F., *Bifurcation from heteroclinic orbits to semihyperbolic equilibria*, preprint, University of Urbino (Italy), 1991.
- [2] Golubitsky, M. and Schaeffer, D.G., *Singularities and Groups in Bifurcation Theory* Vol. I, Springer-Verlag, Berlin, 1985.
- [3] Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, Berlin, 1983.
- [4] Holmes, P., *A nonlinear oscillator with a strange attractor*, Phil. Trans. Roy. Soc. A., 292 (1979), 419-448.
- [5] Isaacson, E., Marchesin, D., Plohr, B., and Temple, J.B., *The Riemann problem near a hyperbolic singularity: The classification of quadratic Riemann problems I*, SIAM J. Appl. Math. 48 (1988), 1-24.
- [6] Isaacson, E., Marchesin, D. and Plohr, B., *Transitional waves for conservation laws*, SIAM J. Math. Anal. 21 (1990), 837-866.
- [7] Isaacson, E., Marchesin, D., Palmeira, C. F., and Plohr, B., *A global formalism for nonlinear waves in conservation laws*, preprint, 1991.

- [8] Liu, T.-P., *The Riemann problem for general 2×2 conservation laws*, Trans. Amer. Math. Soc., 199 (1974), 89-112.
- [9] Martinet, J., *Singularities of Smooth Functions and Maps*, London Mathematical Society Lecture Notes Series, 58, Cambridge University Press, Cambridge, 1982.
- [10] Mather, J. N., *Stability of C^∞ mappings. I: The division theorem*, Annals of Math., 87 (1968), 89-104.
- [11] Mather, J. N., *Stability of C^∞ mappings. II: Infinitesimal stability implies stability*, Annals of Math., 89 (1969), 254-291.
- [12] Mather, J. N., *Stability of C^∞ mappings. III: Finitely determined map germs*, Publ. Math. I.H.E.S., 35 (1968), 127-156.
- [13] Mather, J. N., *Stability of C^∞ mappings. IV: Classification of stable germs by R -algebras*, Publ. Math. I.H.E.S., 37 (1969), 223-248.
- [14] Mather, J. N., *Stability of C^∞ mappings. V: Transversality*, Advances in Mathematics, 4 (1970), 301-336.
- [15] Mather, J. N., *Stability of C^∞ mappings. VI: The nice dimensions*, in Proceedings of Liverpool Singularities Symposium, Vol. 1, (C.T.C. Wall, ed.), Lecture Notes in Mathematics, 192, Springer-Verlag, Berlin, 1971.
- [16] Melnikov, V. K., *On the stability of the center for time-periodic perturbations*, Trans. Moscow Math. Soc., 12 (1963), 1-57.
- [17] Schaeffer, D.G. and Shearer, M., *The classification of 2×2 systems of nonstrictly hyperbolic conservation laws, with application to oil recovery*, Comm. Pure Appl. Math., 40 (1987), 141-178.
- [18] Schaeffer, D.G., and Shearer, M., *Riemann problems for nonstrictly hyperbolic 2×2 systems of conservation laws*, Trans. Amer. Math. Soc., 301 (1987), 267-306.

- [19] Schecter, S., *Simultaneous equilibrium and heteroclinic bifurcation of planar vector fields via the Melnikov integral*, Nonlinearity, 3 (1990), 79–99.
- [20] Schecter, S., *C^p singularity theory and heteroclinic bifurcation with a distinguished parameter*, to appear in J. Differential Equations.
- [21] Schecter, S., *Pitchfork bifurcation with a heteroclinic orbit: normal form, recognition criteria, and universal unfolding*, to appear in J. Differential Equations.
- [22] Schecter, S. and Shearer, M., *Undercompressive shocks for nonstrictly hyperbolic conservation laws*, J. Dynamics and Differential Equations, 3 (1991), 199–271.
- [23] Schecter, S. and Shearer, M., *Transversality for undercompressive shocks in Riemann problems*, in Viscous Profiles and Numerical Methods for Shock Waves (M. Shearer, ed.), SIAM, Philadelphia, 1991.
- [24] Shearer, M., Schaeffer, D.G., Marchesin, D. and Paes-Leme, P., *Solution of the Riemann problem for a prototype 2×2 system of non-strictly hyperbolic conservation laws*, Arch. Rat. Mech. Anal., 97 (1987), 299–320.
- [25] Shearer, M., *The Riemann problem for 2×2 systems of hyperbolic conservation laws with case I quadratic nonlinearities*, J. Differential Equations, 80 (1989), 343–363.
- [26] Sotomayor, J., *Estabilidade estrutural de primeira ordem e variedades de Banach*, thesis, IMPA, Rio de Janeiro, 1964.
- [27] Vegter, G., *Bifurcations of gradient vectorfields*, Astérisque, 98–99 (1982), 39–73.
- [28] Whitney, H., *On singularities of mappings of Euclidean spaces I. Mappings of the plane into the plane*, Ann. of Math., 62 (1955), 374–410.

Mathematics Department
North Carolina State University
Raleigh, NC 27695, USA