

ON THE INITIAL VALUE PROBLEM FOR THE ZAKHAROV EQUATIONS

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Abstract

In this paper the following three problems for the Zakharov equations are considered: (i) Solvability of the Zakharov equations. (ii) Smoothing effect of solutions. (iii) The nonlinear Schrödinger limit of the Zakharov equations. The authors present the results concerning the above three problems, which have recently been obtained in the papers [17] and [18], and the proofs of those results are illustrated.

1. Introduction

In the present paper we consider the initial value problem for the Zakharov equations:

$$i \frac{\partial E}{\partial t} + \Delta E = nE, \quad t > 0, \quad x \in \mathbf{R}^N, \quad (1.1)$$

$$\frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2, \quad t > 0, \quad x \in \mathbf{R}^N, \quad (1.2)$$

$$E(0, x) = E_0(x), \quad n(0, x) = n_0(x), \quad \frac{\partial}{\partial t} n(0, x) = n_1(0, x), \quad (1.3)$$

where $E(t, x)$ is a function from $\mathbf{R}_t^+ \times \mathbf{R}_x^N$ to \mathbf{C}^N , $n(t, x)$ is a function from $\mathbf{R}_t^+ \times \mathbf{R}_x^N$ to \mathbf{R} and $1 \leq N \leq 3$. (1.1)-(1.3) describe the long wave Langmuir turbulence in a plasma (see [29]). $E(t, x)$ denotes the slowly varying envelope of the highly oscillatory electric field and $n(t, x)$ denotes the deviation of the ion density from its equilibrium.

Although there are many papers concerning the physical observations and the numerical experiments for (1.1)-(1.3), there do not seem to be so many

mathematical papers treating (1.1)-(1.3). When (1.2) depends on the ion sound speed λ , that is, (1.2) is replaced by

$$\frac{1}{\lambda^2} \frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2, \quad (1.4)$$

it is thought that (1.1) and (1.4) converge to

$$i \frac{\partial E}{\partial t} + \Delta E = nE, \quad n = -|E|^2 \quad (1.5)$$

as $\lambda \rightarrow \infty$ (see [2], [6], [20] and [29]). (1.5) is just the nonlinear Schrödinger equation and this observation is one of the derivations of the nonlinear Schrödinger equation (see [29]). It is naturally conjectured that the solutions of (1.1)-(1.2) and the solution of (1.5) have some common properties. Therefore, it seems important not only to study the solvability of (1.1)-(1.3), but also to investigate the smoothing effect of solutions for (1.1)-(1.3), which is a remarkable feature of the dispersive wave equations such as the nonlinear Schrödinger equations (see, e.g., [5], [11]-[13], [22] and [23]). It is also interesting and important to study the nonlinear Schrödinger limit of the Zakharov equations, that is, the limit process of (1.1) and (1.4) as $\lambda \rightarrow \infty$.

In [25] C. Sulem and P.L. Sulem proved by using the Galerkin method that if $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2} \cap \dot{H}^{-1}$, $m \geq 3$ and $1 \leq N \leq 3$, then (1.1)-(1.3) have the unique local solutions $(E(t), n(t)) \in L^\infty(0, T; H^m) \oplus L^\infty(0, T; H^{m-1})$ for some $T > 0$. Here H^m denotes the standard Sobolev space $H^m(\mathbf{R}^N)$. \dot{H}^m denotes the homogeneous Sobolev space consisting of all tempered distributions u with $|\xi|^m \hat{u} \in L^2 \equiv L^2(\mathbf{R}^N)$, where \hat{u} is the Fourier transform of u . In [20] Schochet and Weinstein showed the similar result for (1.1), (1.4) and (1.3) by the different method, but the existence time T of the local solutions does not depend on the parameter λ in [20] (see also [2]). In both [20] and [25], the assumption $n_1 \in \dot{H}^{-1}$ is needed for the construction of the local solutions. This assumption is rather strong, because $S \not\subset \dot{H}^{-1}$ for $N = 1, 2$. For example, $e^{-|\cdot|^2}$ is not in \dot{H}^{-1} for $N = 1, 2$. Furthermore, the uniqueness of the solutions $(E(t), n(t))$ for (1.1)-(1.3) is proved only in the class $H^m \oplus H^{m-1}$, $m \geq 3$ in the previous results. In section 2 of this paper we give the unique local existence

result in $H^2 \oplus H^1$ for (1.1)-(1.3). $H^2 \oplus H^1$ seems more natural than the class in the previous results, because the solutions in $H^2 \oplus H^1$ are the so-called strong solutions. The difficulty of solving (1.1)-(1.3) is that when we use the standard iteration scheme, we meet with the loss of derivative, which comes from the second derivatives of $|E(t)|^2$ in (1.2). In the case of the single nonlinear Schrödinger equation the $L^p - L^q$ estimate and the Strichartz estimate play an important role (see [8], [9], [15] and [28]). However, in the previous papers [20] and [25] they are not used, because the loss of derivative prevents us from using them directly. In order to overcome this difficulty, we first transform (1.1)-(1.2) into the system which does not have the derivative loss. For that purpose, we apply the technique developed by Shibata and Y. Tsutsumi [21], which was used to solve the fully nonlinear wave equation. After that we apply the $L^p - L^q$ estimate and the Strichartz estimate to the resulting system, following Kato [15]. In section 2 we also state several remarks concerning the global existence and the asymptotic behavior of solutions for (1.1)-(1.3).

We next consider the smoothing effect of the solutions for (1.1)-(1.3). It is well known that the linear or nonlinear Schrödinger equation has the drastic smoothing effect (see, e.g., [5], [11]-[13], [14], [22] and [23]). In [11] and [12] it is proved that if $E(0) = E_0 \in H^1$, $|x|^k E_0 \in L^2$, $k \geq 1$ and $1 \leq N \leq 3$, then the solution $E(t)$ of (1.5) is in $H_{loc}^k \equiv H_{loc}^k(\mathbf{R}^N)$ for $t > 0$ as long as $E(t)$ exists. In [5] and [23] the smoothing effect of the different type for (1.5) is proved, that is, if $E(0) = E_0 \in H^k$, $k \geq 1$ and $1 \leq N \leq 3$, then the solution $E(t)$ of (1.5) satisfies

$$\int_0^T \|\varphi(1 - \Delta)^{k/2+1/4} E(t)\|_{L^2}^2 dt \leq C$$

for $\varphi \in C_0^\infty(\mathbf{R}^N)$ and $0 < T < T_{max}$, where T_{max} is the maximal existence time of $E(t)$. On the other hand, there seems to be no result concerning the smoothing property of (1.1)-(1.3). The Zakharov system consists of the Schrödinger equation (1.1) and the wave equation (1.2), and we can not expect the smoothing effect of the wave equation part. Accordingly, we can not expect the drastic smoothing effect for (1.1)-(1.3) like the single nonlinear Schrödinger equation. However, we can prove that the solution $E(t)$ of the Schrödinger

part for (1.1)-(1.3) has some smoothing properties similar to those of the single nonlinear Schrödinger equation. In section 3 we state the results concerning the smoothing properties for (1.1)-(1.3).

Finally we consider the nonlinear Schrödinger limit and the initial layer of (1.1), (1.4) and (1.3) as $\lambda \rightarrow \infty$. Since this limit process is a singular perturbation problem, it can happen that the initial condition (1.3) is not compatible with the limit equation (1.5). In that case, the singularity occurs at $t = 0$ as $\lambda \rightarrow \infty$. This singularity is called an initial layer. It is important to investigate the relation between the limit process and the formation of initial layer. In [20] Schochet and Weinstein studied the nonlinear Schrödinger limit of (1.1), (1.4) and (1.3) as $\lambda \rightarrow \infty$. But they treated only the case of no initial layer. In [2] H. Added and S. Added have shown that if $(E_0, n_0, n_1) \in \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S} \cap \dot{H}^{-1}$ and $1 \leq N \leq 3$, then for any T with $0 < T < T_{max}$ and any positive integer m there exist two positive constants C_0 and λ_0 such that the solutions $(E_\lambda(t), n_\lambda(t))$ of (1.1), (1.4) and (1.3) exist on $[0, T]$ for $\lambda \geq \lambda_0$ and satisfy

$$\sup_{0 \leq t \leq T} [\|E_\lambda(t) - E(t)\|_{H^{m+1}} + \|n_\lambda(t) + |E_\lambda(t)|^2 - \cos(\lambda t(-\Delta)^{1/2})(n_0 + |E_0|^2)\|_{H^m}] \leq \begin{cases} C_0 \lambda^{-1/2} & \text{if } n_0 + |E_0|^2 \neq 0 \text{ and } N = 1, 2, \\ C_0 \lambda^{-1} \log \lambda & \text{if } n_0 + |E_0|^2 \neq 0 \text{ and } N = 3, \\ C_0 \lambda^{-1} & \text{if } n_0 + |E_0|^2 = 0 \text{ and } 1 \leq N \leq 3, \end{cases} \quad (1.6)$$

where $E(t)$ is the solution of (1.5) with $E(0) = E_0$ and T_{max} is the maximal existence time of $E(t)$. There is a discrepancy between the non-compatible case $n_0 + |E_0|^2 \neq 0$ and the compatible case $n_0 + |E_0|^2 = 0$ concerning the rate of convergence as $\lambda \rightarrow \infty$. This corresponds to the initial layer phenomenon and the term $\cos(\lambda t(-\Delta)^{1/2})(n_0 + |E_0|^2)$ represents the initial layer. But the rate of convergence in (1.6) does not seem natural for the compatible case $n_0 + |E_0|^2 = 0$, because (1.4) depends on the square of λ (see [6] and [16]). In section 4 we state the results concerning the precise rate of convergence as $\lambda \rightarrow \infty$, which give a clear relation between the limit process and the formation of initial layer.

We conclude this section by giving several notations. Let $W^{m,p}$ denote the

Sobolev space

$$W^{m,p} = \{f \in \mathcal{S}'; \|f\|_{W^{m,p}} \equiv \|(1 - \Delta)^{m/2} f\|_{L^p} < \infty\}$$

for $m \in \mathbf{R}$ and $1 < p < \infty$. We put $H^m \equiv W^{m,2}$. Let $H^{m,s}$ denote the weighted Sobolev space

$$H^{m,s} = \{f \in \mathcal{S}'; \|f\|_{H^{m,s}} \equiv \|(1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} f\|_{L^2} < \infty\}$$

for $m, s \in \mathbf{R}$. For a Banach space X and $T > 0$, we define $W^{m,p}(0, T; X)$ by

$$W^{m,p}(0, T; X) \equiv \{f(t) \in L^p(0, T; X); (\sum_{j=0}^m \int_0^T \|\frac{d^j}{dt^j} f(t)\|_X^p dt)^{1/p} < \infty\},$$

if $1 \leq p < \infty$.

2. Solvability of the Zakharov equations

In this section we describe the results concerning the local solvability of (1.1)-(1.3). We also state several remarks concerning the global existence and the asymptotic behavior of solutions for (1.1)-(1.3).

We first describe the main theorem in this section.

Theorem 2.1 Assume that $1 \leq N \leq 3$.

(1) Let $(E_0, n_0, n_1) \in H^2 \oplus H^1 \oplus L^2$. Then for some $T > 0$ there exist the unique strong solutions $(E(t), n(t))$ of (1.1)-(1.3) such that

$$E(t) \in \bigcap_{j=0}^1 C^j([0, T]; H^{2-2j}), \quad (2.1)$$

$$E(t) \in \bigcap_{j=0}^1 W^{j,8/N}(0, T; W^{2-2j,4}), \quad (2.2)$$

$$n(t) \in \bigcap_{j=0}^2 C^j([0, T]; H^{1-j}), \quad (2.3)$$

where T depends only on $\|E_0\|_{H^2}, \|n_0\|_{H^1}, \|n_1\|_{L^2}$ and N .

(2) Let m be an even integer with $m \geq 4$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then the solutions $(E(t), n(t))$ of (1.1)-(1.3) given by part (1) satisfy

$$E(t) \in \bigcap_{j=0}^{m/2} C^j([0, T]; H^{m-2j}), \quad (2.4)$$

$$E(t) \in \bigcap_{j=0}^{m/2} W^{j, 8/N}(0, T; W^{m-2j, 4}), \quad (2.5)$$

$$n(t) \in \bigcap_{j=0}^3 C^j([0, T]; H^{m-1-j}), \quad (2.6)$$

and if $m \geq 6$,

$$n(t) \in \bigcap_{j=4}^{m/2+1} C^j([0, T]; H^{m+2-2j}). \quad (2.7)$$

(3) Let m be an odd integer with $m \geq 3$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then the solutions $(E(t), n(t))$ of (1.1)-(1.3) given by part (1) satisfy

$$E(t) \in \bigcap_{j=0}^{(m-1)/2} C^j([0, T]; H^{m-2j}), \quad (2.8)$$

$$E(t) \in \bigcap_{j=0}^{(m-1)/2} W^{j, 8/N}(0, T; W^{m-2j, 4}), \quad (2.9)$$

$$n(t) \in \bigcap_{j=0}^3 C^j([0, T]; H^{m-1-j}), \quad (2.10)$$

and if $m \geq 7$,

$$n(t) \in \bigcap_{j=4}^{(m+1)/2} C^j([0, T]; H^{m+2-2j}). \quad (2.11)$$

Remark 2.1.

(1) The solutions $(E(t), n(t))$ of (1.1)-(1.3) in Theorem 2.1(1) satisfy (1.1) in the L^2 sense, while they satisfy (1.2) in the distribution sense. Therefore, the solutions in the class of Theorem 2.1(1) are called the strong solutions (for the weak solutions, see [20, Theorem 4] and [25, Theorem 1]).

(2) Theorem 2.1 (1) shows that the solutions of (1.1)-(1.3) are unique in the class of the strong solutions.

(3) In Theorem 2.1 we do not need the condition $n_1 \in \dot{H}^{-1}$, which was always assumed in the previous papers [20] and [25].

(4) In Theorem 2.1(2) and (3) the existence time T of the more regular solutions than the strong solutions is the same as that of the strong solutions. In the previous papers [20] and [25], T depends on the higher order Sobolev norms of the initial data, when the solutions are regular. Theorem 2.1(2) and (3) imply that if $(E_0, n_0, n_1) \in \cap_{k=1}^{\infty} H^k$, then the solutions $(E(t), n(t)) \in C^\infty([0, T] \times \mathbf{R}^N)$.

(5) (2.2), (2.5) and (2.9) show that $E(t)$ has a smoothing property in a certain sense like the solution of the single nonlinear Schrödinger equation (see [9], [15], [24] and [28]).

(6) (2.7) and (2.11) imply that $\frac{\partial^j}{\partial t^j} n, j \geq 3$ lose the regularity of Sobolev order 2 with respect to the spatial variables, each time we differentiate them in t . This may seem strange, since $n(t)$ is a solution of the wave equation. But (1.2) contains the solution $E(t)$ of the Schrödinger equation as the external force, which is why (2.7) and (2.11) occur.

When we use the standard iteration scheme to solve (1.1)-(1.3), the loss of derivative occurs, as stated in section 1. In fact, if $1 \leq N \leq 3$ and $E(t) \in L^\infty(0, T; H^m)$ for some $m \geq 2$ and $T > 0$, we solve (1.2) to have $n(t) \in L^\infty(0, T; H^{m-1})$. However, we have only $E(t) \in L^\infty(0, T; H^{m-1})$ by (1.1), when $n(t) \in L^\infty(0, T; H^{m-1})$.

Thus, we first consider the following system:

$$i \frac{\partial F}{\partial t} + \Delta F - nF - \frac{\partial n}{\partial t} (E_0 + \int_0^t F ds) = 0, \quad (2.12)$$

$$\frac{\partial^2 n}{\partial t^2} - \Delta n - \Delta |E|^2 = 0, \quad (2.13)$$

$$(-\Delta + 1)E = iF - (n - 1)(E_0 + \int_0^t F ds), \quad (2.14)$$

$$F(0) = i(\Delta E_0 - n_0 E_0), \quad n(0) = n_0, \quad \frac{\partial n}{\partial t}(0) = n_1. \quad (2.15)$$

If we formally differentiate (1.1) in t and put $F = \frac{\partial}{\partial t} E$, we obtain (2.12). (1.1) is also rewritten as (2.14) in terms of F . The loss of derivative does not

occur for (2.12)-(2.14). This technique was used to solve the fully nonlinear wave equation in [21]. In order to obtain Theorem 2.1, we can directly apply the $L^p - L^q$ estimate and the Strichartz estimate to the system (2.12)-(2.15), following Kato [15]. For the details of the proof of Theorem 2.1, see Ozawa and Y. Tsutsumi [17].

We also have the following theorem concerning the global existence of solutions for (1.1)-(1.3).

Theorem 2.2

(1) Assume $N = 1$. Let m be an integer with $m \geq 2$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$ and $n_1 \in \dot{H}^{-1}$, then the existence time T of the solutions in Theorem 2.1 can be chosen as $T = +\infty$. Furthermore, if $E_0, n_0, n_1 \in \cap_{m=1}^\infty H^m$ and $n_1 \in \dot{H}^{-1}$, then the solutions $E(t, x)$ and $n(t, x)$ are in $C^\infty([0, \infty) \times \mathbb{R})$.

(2) Assume $N = 2$. Let m be an integer with $m \geq 2$. There exists $\delta > 0$ such that if $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, $n_1 \in \dot{H}^{-1}$ and $\|E_0\|_{L^2} < \delta$, then the existence time T of the solutions in Theorem 2.1 can be chosen as $T = +\infty$. In addition, if E_0, n_0 , and n_1 are in $\cap_{m=1}^\infty H^m$, then the solutions $E(t, x)$ and $n(t, x)$ are in $C^\infty([0, \infty) \times \mathbb{R}^2)$.

The a priori estimates needed for the proof of existence of global solutions are already established by C. Sulem and P.L. Sulem [25, Proof of Theorem 2] and by H. Added and S. Added [1, Proof of Theorem] (see also [20]). The proof of the a priori estimates requires the assumption $n_1 \in \dot{H}^{-1}$, because the energy identity of (1.1)-(1.3) contains the \dot{H}^{-1} norm of n_1 . Those a priori estimates and Theorem 2.1 show Theorem 2.2.

We conclude this section by stating the following remark concerning the asymptotic behavior of solutions for (1.1)-(1.3).

Remark 2.2. There seems to be no result concerning the asymptotic behavior of global solutions for (1.1)-(1.3). It is difficult to treat the scattering problem for (1.1)-(1.2), since the nonlinearity is quadratic. However, in the case of $N = 3$, we can construct the wave operators for certain scattered data, which are

not necessarily small. The proof is based on the improved decay estimates of the interaction term which take account of the difference between the propagation of the Schrödinger wave and that of the acoustic wave. The proof is similar to that of the result in [19] for the coupled Klein-Gordon-Schrödinger equations, but it is more complicated. The details will appear elsewhere.

3. Smoothing effect of solutions for the Zakharov equations

In this section we describe the result concerning the smoothing effect of solutions for (1.1)-(1.3). We have the following theorem.

Theorem 3.1 *Let m be an integer with $m \geq 2$. Assume that $1 \leq N \leq 3$ and $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$. Let $(E(t), n(t))$ and $T_{\max} > 0$ be the solutions of (1.1)-(1.3) given by Theorem 2.1 and their maximal existence time, respectively.*

(1) *Let $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$. Then, $E(t)$ satisfies*

$$\varphi E(t) \in L^2(0, T; H^{m+1/2}) \quad (3.1)$$

for any T with $0 < T < T_{\max}$.

(2) *In addition, let $m \geq 4$. Put $k = 1$ if $m \geq 4$ and $k = 1$ or 2 if $m \geq 6$. If $E_0 \in H^{m,k}$, then*

$$E(t) \in H_{loc}^{m+k}, \quad 0 < t < T_{\max}. \quad (3.2)$$

Remark 3.1. Theorem 3.1 shows the smoothing properties of the Zakharov equations (1.1)-(1.3). Part (1) is completely the same as in the case of the single nonlinear Schrödinger equation (see [5] and [22]). On the other hand, part (2) is not so good as in the case of the single nonlinear Schrödinger equation (see [11]-[13]). This is because the Zakharov system contains the wave equation and it has the form such that the derivative loss occurs.

In order to show Theorem 3.1(1), we rewrite (1.1) as the integral form and evaluate the resulting equation by using (2.2), (2.5) and (2.9). In the proof

of Theorem 3.1(2), we use the commutator method (see [14], [11]-[13]) and the difference of the derivative in t between the Schrödinger equation and the wave equation. For the details of the proof of Theorem 3.1, see Ozawa and Y. Tsutsumi [17].

4. The nonlinear Schrödinger limit of the Zakharov equations

In this section we state the result concerning the limit process of (1.1), (1.4) and (1.3) as $\lambda \rightarrow \infty$. We have the following theorem.

Theorem 4.1 *Assume that $1 \leq N \leq 3$ and $(E_0, n_0, n_1) \in \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S} \cap \dot{H}^{-1}$. Let $E(t)$ be the solution of (1.5) with $E(0) = E_0$ and let T_{\max} be the maximal existence time of $E(t)$. Let $(E_\lambda(t), n_\lambda(t))$ be the solutions of (1.1), (1.4) and (1.3).*

(1) *For any T with $0 < T < T_{\max}$ and any positive integer m there exist two positive constants C and λ_0 such that for any $\lambda \geq \lambda_0$ $(E_\lambda(t), n_\lambda(t))$ exist on $[0, T]$ and satisfy*

$$\sup_{0 \leq t \leq T} \|n_\lambda(t) + |E_\lambda(t)|^2 - Q_\lambda^{(1)}(t) - Q_\lambda^{(2)}(t)\|_{H^m} \leq C\lambda^{-1},$$

where

$$\begin{aligned} Q_\lambda^{(1)}(t) &= \cos(\lambda t(-\Delta)^{1/2})(n_0 + |E_0|^2), \\ Q_\lambda^{(2)}(t) &= \lambda^{-1}(-\Delta)^{-1/2} \sin(\lambda t(-\Delta)^{1/2})(n_1 + 2\Im(\bar{E}_0 \cdot \Delta E_0)). \end{aligned}$$

In particular, for any $\lambda \geq \lambda_0$

$$\sup_{0 \leq t \leq T} \|n_\lambda(t) + |E_\lambda(t)|^2 - Q_\lambda^{(1)}(t)\|_{H^m} \leq C\lambda^{-1}.$$

(2) *Assume $n_0 + |E_0|^2 \neq 0$. Then, for any T with $0 < T < T_{\max}$ and any positive integer m there exist two positive constants C and λ_0 such that for any $\lambda \geq \lambda_0$*

$$\sup_{0 \leq t \leq T} \|E_\lambda(t) - E(t)\|_{H^m} \leq C\lambda^{-1}.$$

(3) Assume $n_0 + |E_0|^2 = 0$. In addition, when $N = 1, 2$, assume that $n_1 = \nabla \cdot \phi$ for some $\phi \in \mathcal{S}$. Then, for any T with $0 < T < T_{\max}$ and any positive integer m there exist two positive constants C and λ_0 such that for any $\lambda \geq \lambda_0$

$$\sup_{0 \leq t \leq T} \|E_\lambda(t) - E(t)\|_{H^m} \leq C\lambda^{-2}.$$

Remark 4.1.

(1) The assumption $n_1 \in \mathcal{S} \cap \dot{H}^{-1}$ is redundant when $N = 3$, since $\mathcal{S} \subset \dot{H}^{-1}$ for $N = 3$. This fact follows by using the Hardy inequality in the Fourier space.

(2) $Q_\lambda^{(j)}(t), j = 1, 2$ in part (1) of Theorem 4.1 solve the wave equation

$$\frac{\partial^2 Q_\lambda^{(j)}}{\partial t^2} - \Delta Q_\lambda^{(j)} = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad j = 1, 2$$

with the initial conditions $Q_\lambda^{(1)}(0) = n_0 + |E_0|^2$, $\frac{\partial}{\partial t} Q_\lambda^{(1)}(0) = 0$ and $Q_\lambda^{(2)}(0) = 0$, $\frac{\partial}{\partial t} Q_\lambda^{(2)}(0) = n_1 + 2\Im(\bar{E}_0 \cdot \Delta E_0)$, respectively. The terms $Q_\lambda^{(1)}(t)$ and $Q_\lambda^{(2)}(t)$ represent the first initial layer and the second initial layer, respectively, in the nonlinear Schrödinger limit of (1.1), (1.4) and (1.3). While the analysis of the first initial layer is important in the proof of Theorem 4.1(2), the analysis of the second initial layer is important in the proof of Theorem 4.1(3).

(3) For $t > 0$ $Q_\lambda^{(1)}(t)$ tends to zero locally in space for $N = 1$ and globally in space for $N = 2, 3$ as $\lambda \rightarrow \infty$. Accordingly, Theorem 4.1 implies that for $0 < t < T_{\max}$, n_λ behaves like $-|E_\lambda|^2$ and so like $-|E|^2$ as $\lambda \rightarrow \infty$. In the compatible case $n_0 + |E_0|^2 = 0$, $Q_\lambda^{(1)}(t) \equiv 0$ and so n_λ behaves like $-|E|^2$ on the time interval $[0, T_{\max})$ including $t = 0$ as $\lambda \rightarrow \infty$. This difference between the time intervals for convergence is due to the initial layer phenomenon. The formation of initial layer is also reflected in the convergence rate of the solution E_λ as $\lambda \rightarrow \infty$ (see Theorem 4.1(2) and (3)).

(4) The results in Theorem 4.1 (2) and (3) are optimal concerning the rate of convergence with respect to λ . Indeed, there exist nontrivial solutions E_λ satisfying

$$\liminf_{\lambda \rightarrow \infty} \lambda \sup_{0 \leq t \leq T} \|E_\lambda(t) - E(t)\|_{H^m} > 0$$

in the non-compatible case $n_0 + |E_0|^2 \neq 0$ and

$$\liminf_{\lambda \rightarrow \infty} \lambda^2 \sup_{0 \leq t \leq T} \|E_\lambda(t) - E(t)\|_{H^m} > 0$$

in the compatible case $n_0 + |E_0|^2 = 0$ (see [18, Theorem 1.3]).

(5) The maximal existence time T_{\max} of $E(t)$ is infinity for $N = 1$, but T_{\max} can be finite for $N = 2, 3$ (see Glassey [10] and M. Tsutsumi [26]). It is conjectured that for $N = 2, 3$ there are also solutions of the Zakharov equations blowing up in finite time (see [29]).

Our proof of Theorem 4.1 depends essentially on the special propagation properties of the Schrödinger wave and the acoustic wave. An outline of the proof is roughly illustrated as follows. We put $Q_\lambda = n_\lambda + |E_\lambda|^2$. Let $\omega = (-\Delta)^{1/2}$. (1.1), (1.4) and (1.3) are rewritten as the system of integral equations:

$$E_\lambda(t) = e^{it\Delta} E_0 + i \int_0^t e^{i(t-s)\Delta} (|E_\lambda|^2 E_\lambda - Q_\lambda E_\lambda)(s) ds,$$

$$Q_\lambda(t) = Q_\lambda^{(1)}(t) + Q_\lambda^{(2)}(t) + \int_0^t (\lambda\omega)^{-1} \sin(\lambda(t-s)\omega) \frac{\partial^2}{\partial s^2} |E_\lambda|^2(s) ds,$$

where $Q_\lambda^{(1)}$ and $Q_\lambda^{(2)}$ are defined as in Theorem 4.1(1). Therefore, we have

$$\begin{aligned} E_\lambda(t) - E(t) &= i \int_0^t e^{i(t-s)\Delta} (|E_\lambda|^2 E_\lambda - |E|^2 E)(s) ds, \\ &\quad - i \int_0^t e^{i(t-s)\Delta} Q_\lambda(s) E_\lambda(s) ds. \end{aligned} \quad (4.1)$$

Our main task is to evaluate the second integral in the right hand side of (4.1). In [2] H. Added and S. Added use the time decay estimate of the solution of the acoustic wave equation to show that $Q_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$ and so $Q_\lambda E_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$. This argument is the same one as used for the problem of the incompressible limit of the compressible Euler equation (see Asano [3] and Ukai [27]). But this argument is not sufficient for the optimal result in our problem. In fact, the product of Q_λ and E_λ tends to zero faster as $\lambda \rightarrow \infty$ than Q_λ alone tends to zero as $\lambda \rightarrow \infty$.

The integrand $Q_\lambda E_\lambda$ in the right hand side of (4.1) corresponds to the interaction between the acoustic wave Q_λ and the Schrödinger wave E_λ . Q_λ

propagates according to the Huygens principle and is localized in a neighborhood of the sphere $|x| = \lambda t$. Hence Q_λ propagates very fast as $\lambda \rightarrow \infty$. On the other hand, E_λ propagates with group velocity independent of λ and is well localized in a neighborhood of the origin. Therefore, the main part of the support of $Q_\lambda(t)$, say $\{x; |Q_\lambda(t, x)| \geq \varepsilon\}$ for some $\varepsilon > 0$ and that of $E_\lambda(t)$ become almost disjoint and so the product $Q_\lambda E_\lambda$ can be proved to go to 0. This is a main idea of our proof and a different point from the proof of [2]. For the details of the proof of Theorem 4.1, see Ozawa and Y. Tsutsumi [18].

References

- [1] H. Added and S. Added, *Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2*, C. R. Acad. Sci. Paris 299 (1984), 551–554.
- [2] H. Added and S. Added, *Equations of Langmuir turbulence and nonlinear Schrödinger equation: Smoothness and approximation*, J. Funct. Anal. 79 (1988), 183–210.
- [3] K. Asano, *On the incompressible limit of the compressible Euler equation*, Japan J. Appl. Math. 4 (1987), 455–488.
- [4] H. Brezis and T. Gallouet, *Non linear Schrödinger evolution equations*, Nonlinear Analysis, TMA 4 (1980), 677–681.
- [5] P. Constantin and J. C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. 1 (1988), 413–439.
- [6] J. Gibbons, *Behavior of slow Langmuir solitons*, Phys. Letter 67A (1978), 22–24.
- [7] J. Gibbons, S. G. Thornhill, M. J. Wardrop and D. Ter Haar, *On the theory of Langmuir solitons*, J. Plasma Phys. 17 (1977), 153–170.

- [8] J. Ginibre and G. Velo , *On a class of nonlinear Schrödinger equations I: the Cauchy problem* , J. Funct. Anal. 32 (1979), 1-32.
- [9] J. Ginibre and G. Velo , *Scattering theory in the energy space for a class of nonlinear Schrödinger equations* , J. Math. Pures Appl. 64 (1985), 363-401.
- [10] R. T. Glassey , *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations* , J. Math. Phys. 18 (1977), 1794-1797.
- [11] N. Hayashi, K. Nakamitsu and M. Tsutsumi , *On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension* , Math. Z. 192 (1986), 637-650.
- [12] N. Hayashi, K. Nakamitsu and M. Tsutsumi , *On solutions of the initial value problem for the nonlinear Schrödinger equations* , J. Funct. Anal. 71 (1987), 218-243.
- [13] N. Hayashi and T. Ozawa , *Smoothing effect for some Schrödinger equations* , J. Funct. Anal. 85 (1989), 307-348.
- [14] A. Jensen , *Commutator methods and a smoothing property of the Schrödinger evolution group* , Math. Z. 191 (1986), 53-59.
- [15] T. Kato , *On nonlinear Schrödinger equations* , Ann. Inst. Henri Poincaré, Phys. Théor. 46 (1987), 113-129.
- [16] T. Ozawa and Y. Tsutsumi , *The nonlinear Schrödinger limit and the initial layer of the Zakharov equations* , Proc. Japan Acad., Ser. A 67 (1991), 113-116.
- [17] T. Ozawa and Y. Tsutsumi , *Existence and smoothing effect of solutions for the Zakharov equations*, to appear in Publ. RIMS. Kyoto Univ.
- [18] T. Ozawa and Y. Tsutsumi , *The nonlinear Schrödinger limit and the initial layer of the Zakharov equations*, to appear in Differential and Integral Equations.

- [19] T. Ozawa and Y. Tsutsumi , *Asymptotic behavior of solutions for the coupled Klein-Gordon- Schrödinger equations* Preprint.
- [20] S. H. Schochet and M. I. Weinstein , *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence* , Comm. Math. Phys. 106 (1986), 569-580.
- [21] Y. Shibata and Y. Tsutsumi , *Local existence of solutions for the initial boundary value problem of fully nonlinear wave equation* , Nonlinear Analysis, TMA 11 (1987), 335-365.
- [22] P. Sjölin , *Regularity of solutions to the Schrödinger equation* , Duke Math. J. 55 (1987), 699-715.
- [23] P. Sjölin , *Local regularity of solutions to nonlinear Schrödinger equations*, Ark. Mat. 28 (1990), 145-157.
- [24] R. S. Strichartz , *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations* , Duke Math. J. 44 (1977), 705-714.
- [25] C. Sulem and P. L. Sulem , *Quelques résultats de régularité pour les équations de la turbulence de Langmuir* , C. R. Acad. Sci. Paris 289 (1979), 173-176.
- [26] M. Tsutsumi , *Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equation* , SIAM J. Math. Anal. 15 (1984), 357-366.
- [27] S. Ukai , *The incompressible limit and the initial layer of the compressible Euler equation* , J. Math. Kyoto Univ. 26 (1986), 323-331.
- [28] K. Yajima , *Existence of solutions for Schrödinger evolution equations* , Comm. Math. Phys. 110 (1987), 415-426.

- [29] V. E. Zakharov , *Collapse of Langmuir waves* , Sov. Phys. JETP 35 (1972), 908-914.

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