



GLOBAL SOLUTIONS TO THE EQUATIONS FOR THE MOTION OF STRATIFIED INCOMPRESSIBLE FLUIDS

José Luiz Boldrini  Marko Antonio Rojas-Medar 

Abstract

We prove a result on global existence in time for strong solutions to the three dimensional stratified Navier-Stokes equations. These equations describe the motion of nonhomogeneous incompressible fluids. For the result, as in the usual Navier-Stokes equations, it is required small initial velocities and external force fields with a weak form of decay.

1. Introduction

In this work we will be concerned with global existence in time of strong solutions to the three dimensional stratified Navier-Stokes equations, that is, the equations for the motion of a nonhomogeneous incompressible fluid (obtained as a mixture of miscible incompressible fluids, for instance). Being $\Omega \subset \mathbb{R}^3$ a C^3 -regular bounded open set, $T > 0$ these equations are:

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u - \Delta u - \text{grad } p = \rho f, \\ \text{div } u = 0, \\ \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega \times (0, T); \\ \rho|_{t=0}(x) = \rho_0(x) \quad \text{in } \Omega; \\ u|_{t=0}(x) = u_0(x) \quad \text{in } \Omega, \end{array} \right. \quad (1.1)$$

where $[0, T)$ is the interval of time being considered; Ω is the container where the fluid is in; $u(x, t) \in \mathbb{R}^3$ denotes the velocity of the fluid at a point $x \in \Omega$ and at time $t \in [0, T)$; $\rho(x, t) \in \mathbb{R}$ and $p(x, t) \in \mathbb{R}$ denote, respectively, the density and the hydrostatic pressure of the fluid; $u_0(x)$ and $\rho_0(x)$ are the

initial velocity and density, respectively; $f(x, t)$ is the density by unit of mass of the external force acting on the fluid; here, without loosing generality, we have normalized the viscosity to be one; the fluid adheres to the wall $\partial\Omega$ of the container which is at rest. The expressions grad, Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote the gradient operator by ∇ and $\frac{\partial u}{\partial t}$ by u_t); the i th component of $u \cdot \nabla u$ is given by $(u \cdot \nabla u)_i = \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j}$; $u \cdot \nabla \rho = \sum_{j=1}^3 u_j \frac{\partial \rho}{\partial x_j}$. The first equation in (1.1) corresponds to the balance of linear momentum; the third to the balance of mass, and the second states that fluid is incompressible. The unknowns in the problem are u , ρ and p .

The classical Navier-Stokes equations corresponds to the special case where $\rho(x, t) = \rho_0$ is a positive constant; in this case the third equation in (1.1) drops out. This case has been much studied (see Ladyshenskaya [7] and Temam [12] and the references there in). Equations (1.1) have been much less studied, maybe due to their mixed parabolic-hyperbolic character. Antonzev and Kazhikov [1], Kazhikov [5], Simon [11] and Kim [6] have studied local and global existence for weak solutions to (1.1). Stronger local or global solutions were obtained by Ladyshenskaya and Solonnikov [8] by linearization and fixed point arguments, and by Okamoto [9] by using evolution operators techniques and also fixed point arguments. The more constructive spectral semi-Galerkin method was used by Salvi [10] to obtain local strong solutions and to study conditions for regularity at $t = 0$ and by Boldrini and Rojas-Medar [2] also to obtain local strong solutions and to study their regularity for $t > 0$.

In this work we present results of global existence for strong solutions (see section 2 for the exact definitions) under certain regularity assumptions on the initial data and external force field (see section 3 for the details). In particular, as in the case of the usual Navier-stokes equations, we will require smallness of the $L^2(\Omega)$ norm of the initial velocity and of the $L^2(\Omega \times (0, T))$ -norm of the force field.

Our result can be compared with the ones by Ladyshenskaya, Solonnikov

and Okamoto as follows. In their results the initial velocity can be a little less regular than in ours, but, by working with fractional powers of the Stokes operator applied to the multipliers that we used, one could weaken our regularity requirements. On the other hand, Ladyzhenskaya and Solonnikov require exponential decay in time of the (small) $L^q(\Omega)$ -norm ($q > 3$) the external force field f ; Okamoto works with f identically zero, and, in order to obtain the result with nonzero force field, at least an exponential decay in time of the $L^2(\Omega)$ -norm of f would be required. We require a weaker form of decay in the sense that we allow a (small) f belonging to $L^2([0, \infty); L^2(\Omega))$. Concerning the conditions on the initial velocity, Okamoto requires small initial velocities in the $H^{\frac{3}{2}+\varepsilon}(\Omega)$ -norm, $\varepsilon > 0$, and initial densities ρ_0 with small enough $L^\infty(\Omega)$ norm of $\nabla \rho_0$; Ladyzhenskaya and Solonnikov demand small initial velocities in the $W^{2-\frac{2}{q},q}(\Omega)$ -norm, $q > 3$. In our result it is enough to require small initial velocity in the $H^1(\Omega)$ -norm.

2. Preliminaries

In what follows we will assume Ω of class C^3 . We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D); \|\partial_f^\alpha\|_{L^q(D)} < +\infty, (|\alpha| \leq m)\},$$

$m = 0, 1, 2, \dots, 1 \leq q \leq \infty, D = \Omega$ or $\Omega \times (0, T), 0 < T \leq +\infty$, with the usual norm. When $q = 2$, we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(\Omega)$ in $H^m(D)$. If B is a Banach-space, we denote by $L^q([0, T], B)$ the Banach space of the B -valued functions defined in the interval $[0, T]$ that are L^q -integrable in the sense of Bochner.

Let $C_{0,\sigma}^\infty(\Omega) = \{v = (v_1, v_2, v_3) \in C_0^\infty(\Omega)^3; \operatorname{div} v = 0 \text{ in } \Omega\}$; $V =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $H_0^1(\Omega)^3$, and $H =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$

Let P be the orthogonal projection from $L^2(\Omega)^3$ onto H obtained by the usual Helmholtz decomposition. Then the operator $A : H \rightarrow H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega)^3 \cap V$ is called the Stokes operator. It is well known that A is a positive definite self-adjoint operator and is characterized

by the relation

$$(Aw, v) = (\nabla w, \nabla v) \text{ for all } w \in D(A), v \in V.$$

From now on, we denote the inner product in H (i.e., the L^2 -inner product) by (\cdot, \cdot) . The L^p -norm will be denoted by $\|\cdot\|_{L^p(\Omega)}$.

The following assumptions on the initial data will hold throughout this paper.

(A.1) The initial value for the density ρ_0 belongs to $W^{1,\infty}(\Omega)$ and

$$\text{satisfies } 0 < \alpha \leq \rho_0(x) \leq \beta < +\infty \text{ a.e. in } \Omega.$$

(A.2) The initial value u_0 belongs to $V \cap H^2(\Omega)$

Now, using the properties of P , we can reformulate problem (1.1) as follows: find $\rho \in W^{1,\infty}(\Omega \times (0, T))$ and $u \in C^1([0, T], H) \cap C([0, T], D(A))$ such that

$$\begin{cases} \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 & \text{for a.e. } (x, t) \in \Omega \times (0, T) \\ (\rho u_t, v) + (\rho u \cdot \nabla u, v) + (Au, v) = (\rho f, v), & 0 < t < T, \quad \forall v \in H \\ u(0) = u_0, \quad \rho(0, x) = \rho_0(x) \end{cases} \quad (2.1)$$

By using spectral semi-Galerkin approximations, Boldrini, Rojas-Medar [2] proved the following local existence theorem:

Theorem 2.1. *Suppose (A.1) and (A.2) are true and that $f \in L^2(0, T, H^1(\Omega))$, $f_t \in L^2(0, T, L^2(\Omega))$. Then, there is $0 < T_1 \leq T$ such that the problem (2.1) has a unique solution in the interval $[0, T_1]$.*

Remark.

(i) Actually the solution has a little better regularity. For instance, it is proved that $u \in L^2(0, T_1, H^3(\Omega))$.

(ii) From the proof (2), one sees that

$$T_1 = T_1(\|u_0\|_{H^1(\Omega)}, \|\rho_0\|_{W^{1,\infty}(\Omega)}) \text{ increases as } \|u_0\|_{H^1(\Omega)} \text{ decreases.}$$

In the next section we will prove that under more stringent conditions, the above solution is global in time.

3. Global Existence

We have the following result:

Theorem 3.1: *Suppose that (A.1) and (A.2) are true and that*

$$f \in L^2_{loc}([0, \infty); H^1(\Omega)) \cap L^\infty([0, \infty); L^2(\Omega)) \cap L^2([0, \infty), L^2(\Omega)),$$

$f_t \in L^2_{loc}(0, \infty, L^2(\Omega))$. Then, if $\|u_0\|_{H^1(\Omega)}$ and $\|f\|_{L^2(\Omega \times (0, \infty))}$ are small enough, the solution described in Theorem 2.1 exists globally in time, that is, it exists for all $t \in \mathbb{R}$ and for any $0 < T < +\infty$ it is in the required spaces.

Proof: We will combine arguments used by Kim [6] with a variant of arguments used by Heywood and Rannacher [4].

We take $\bar{u}_0 \in V \cap H^2(\Omega)$, and will prove that for small enough $\lambda \in (0, 1]$, the solution $(u_\lambda, \rho_\lambda)$ to (2.1), with initial data $(\lambda \bar{u}_0, \rho_0)$ and external force λf , exists globally in time. The crucial estimate will be the one for $\|\nabla u_\lambda(t)\|_{L^2(\Omega)}$; and to obtain it, we proceed as follows: from the proof of the local existence theorem (Theorem 2.1), for any s in the interval of existence, we have the estimate

$$\frac{\alpha}{2} \|u_\lambda(s)\|_{L^2(\Omega)}^2 + \int_0^s \|\nabla u_\lambda(\tau)\|_{L^2(\Omega)}^2 d\tau \leq \lambda^2 \left[\frac{\beta}{2} \|\bar{u}_0\|_{L^2(\Omega)}^2 + \beta \int_0^s \|f(\tau)\|_{L^2(\Omega)}^2 d\tau \right] \quad (3.1)$$

(this was obtained by taking u_λ as a multiplier in (2.1) (ii) with λf in place of f and $\lambda \bar{u}_0$ in place of u_0).

Also, working as in Kim [6], (Proposition 2.4, p. 93), for t in the interval of existence of the solution, one obtains

$$\frac{d}{dt} \|\nabla u_t(t)\|_{L^2(\Omega)}^2 \leq C \|\nabla u_t(t)\|_{L^2(\Omega)}^{10} + C \lambda^2 \|f(t)\|_{L^2(\Omega)}^2. \quad (3.2)$$

We will show by contradiction that for λ small enough, the above two inequalities imply that $\|\nabla u_\lambda(t)\|$ is bounded for finite times.

In fact, suppose the opposite, that is, that for any $\lambda \in (0, 1]$ the function $\psi_\lambda(t) = \|\nabla u_\lambda(t)\|_{L^2(\Omega)}^2$ blows-up in a finite time $t^*(\lambda)$ (which, according to the

Remark (ii) above, is necessarily larger than the $T_1 > 0$ given in Theorem 2.1, corresponding to the initial data (\bar{u}_0, ρ_0) .

Now, we observe that $c\psi_\lambda^5 + c_1\lambda^2 \leq 2c\psi_\lambda^5$ for $\psi_\lambda \geq (c_1/c)^{1/5}\lambda^{2/5} = l(\lambda)$, where $c_1 = c \sup_{t \in [0, T_1]} \|f(t)\|_{L^2(\Omega)}$. Therefore, we have $0 \leq \psi_\lambda(t) < l(\lambda)$ or $\frac{d\psi_\lambda}{dt} \leq 2c\psi_\lambda^5$.

Now we consider the equation $\frac{d}{dt}\Phi_\lambda = 2c\Phi_\lambda^5$ and its solution $\Phi_\lambda(t)$ that blows-up exactly at $t^*(\lambda)$. We will prove that the graph of $\Phi_\lambda(t)$ stays below the graph of $\psi_\lambda(t)$ for $t \in \Delta_\lambda = \{t \in [0, t^*(\lambda)), \psi_\lambda(t) \geq l(\lambda)\}$. In fact, take a sequence $\bar{t}_n > t^*(\lambda)$, $n \in \mathbb{N}$, converging to $t^*(\lambda)$, and consider the solutions $\varphi_{\bar{t}_n}(t)$ to the problem $\varphi' = 2c\varphi^5$ that blows-up exactly at \bar{t}_n . Since $\psi_\lambda(t) \rightarrow +\infty$ as $t \rightarrow t^*(\lambda)^-$ and $\varphi_{\bar{t}_n}(t)$ are finite for such t , we have $\varphi_{\bar{t}_n}(t) < \psi_\lambda(t)$ for $t < t^*(\lambda)$, close enough to $t^*(\lambda)$. On the other hand, by using well known results on differential inequalities, as well as the definitions of $\psi_\lambda(t)$ and $\varphi_{\bar{t}_n}(t)$, we see that there cannot exist $\tau \in \Delta_\lambda$ such that $\psi_\lambda(\tau) = \varphi_{\bar{t}_n}(\tau)$, and, therefore, $\varphi_{\bar{t}_n}(t) < \psi_\lambda(t)$ for $t \in \Delta_\lambda$. Since, for $0 \leq t < t^*(\lambda)$, we have $\varphi_{\bar{t}_n}(t) \rightarrow \Phi_\lambda(t)$ as n goes to $+\infty$, we finally conclude that $\Phi_\lambda(t) \leq \psi_\lambda(t)$ for $t \in \Delta_\lambda$.

Now, $\Phi_\lambda(t) = [8c(t^*(\lambda) - t)]^{-1/4}$ and we observe that $\Phi_\lambda(t) \geq l(\lambda)$ for $t \geq t_1(\lambda) = t^*(\lambda) - (8cl^4(\lambda))^{-1}$, and for such t we necessarily have $\psi_\lambda(t) \geq \Phi_\lambda(t)$. Moreover, since $l(\lambda) \rightarrow 0+$ as $\lambda \rightarrow 0+$, for small enough λ we have $[8cl^4(\lambda)]^{-1} > T_1$, and we also know that $t^*(\lambda)$ grows as $\lambda \rightarrow 0+$. We conclude that $\psi_\lambda(t) \geq \Phi_\lambda(t)$ for $t \geq t^*(\lambda) - T_1$; therefore,

$$\frac{4T_1^{3/4}}{3(8c)^{1/4}} = \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \Phi_\lambda(\tau) d\tau \leq \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \psi_\lambda(\tau) d\tau.$$

Now, if we take λ satisfying

$$\lambda^2 \left[\frac{\beta}{2} \|\bar{u}_0\|^2 + \beta \int_0^\infty \|f(\tau)\|_{L^2(\Omega)}^2 d\tau \right] < \frac{4T_1^{3/4}}{3(8c)^{1/4}},$$

estimate (3.1) implies that

$$\begin{aligned} \int_0^t \psi_\lambda(\tau) d\tau &\leq \lambda^2 \left[\frac{\beta}{2} \|\bar{u}_0\|_{L^2(\Omega)}^2 + \beta \int_0^\infty \|f(\tau)\|_{L^2(\Omega)}^2 d\tau \right] \\ &< \frac{4T_1^{3/4}}{3(8c)^{1/4}} \leq \int_{t^*(\lambda)-T_1}^{t^*(\lambda)} \psi_\lambda(\tau) d\tau, \end{aligned}$$

for $0 \leq s < t^*(\lambda)$, which by taking $s \rightarrow t^*(\lambda)^-$ is a contradiction. Therefore, for small enough λ , $\|\nabla u_\lambda(t)\|_{L^2(\Omega)}$ does not blow up in finite time and $\nabla u_\lambda \in L_{loc}^\infty([0, \infty), L^2(\Omega))$. Now, with this estimate, the proof of the local existence Theorem will furnish that $\rho_\lambda \in L^\infty([0, \infty), L^\infty(\Omega))$, $u_{\lambda,t}, \Delta u_\lambda \in L_{loc}^\infty([0, \infty), L^2(\Omega))$ and $\nabla u_{\lambda,t} \in L_{loc}^2([0, \infty), L^2(\Omega))$. These results imply that $\rho_\lambda(u_{\lambda,t} + u_\lambda \cdot \nabla u_\lambda - \lambda f) \in L_{loc}^\infty([0, \infty), L^{3+\epsilon}(\Omega))$ for some $\epsilon > 0$. Thus, the equivalent form of (2.1) (ii):

$$Au_\lambda = P(\rho_\lambda(u_{\lambda,t} + u_\lambda \cdot \nabla u_\lambda - \lambda f))$$

and Cattabriga's estimate for the Stokes operator, [3], will give that $u_\lambda \in L_{loc}^2([0, \infty); W^{2,3+\epsilon}(\Omega))$. Therefore, by Sobolev Imbedding we have $\nabla u_\lambda \in L_{loc}^1([0, \infty); L^\infty(\Omega))$ which, by Ladyzhenskaya and Solonnikov estimates [8], imply that $\nabla \rho_\lambda$ and $\rho_{\lambda,t} \in L_{loc}^\infty([0, \infty); L^\infty(\Omega))$. Finally, from the above argument, we observe that how small λ must be depends only on $\|\nabla \bar{u}_0\|_{L^2(\Omega)}$ (for fixed ρ_0 and f), that is, if we have another initial condition \bar{u}_1 such that $\|\nabla \bar{u}_0\| = \|\nabla \bar{u}_1\|$, the above condition on λ will be the same. In other words, the result depends only on the H^1 -norm of the initial velocity.

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UNICAMP - IMECC,

C.P. 6065

13081, Campinas, SP, Brazil