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GEOMETRY AND THE COMMUTABILITY OF SYMMETRIC OPERATORS

Sérgio Mendonça

Abstract

By studying the Ricatti equation, we show that the Ricci operator commutes with its derivative along geodesics. We also prove that, for any v tangent to a closed manifold of nonpositive curvature, the Ricci operator $K_{\tilde{v}}$ commutes with the second fundamental form of the horosphere at $\tilde{v}$ associated with the ray $\gamma_{\tilde{v}}$, where $\tilde{v}$ is the lifting of v to the universal covering with the induced metric.

Let M be a Riemannian manifold. Several works in Riemannian geometry have used the fact that the Ricatti equation

$$U' + U^2 + K = 0 \tag{0.1}$$

is satisfied along small geodesics γ on M or along complete geodesics or rays in some special cases. See for example [Gr], [Gu], [A], [E], [K], [FM], [OS], [MZ1], [MZ2]. The Ricci operator $K = K_{\gamma'(t)} := R(\gamma'(t), .)\gamma'(t)$ and $U = U_{\gamma'(t)}$ are symmetric operators on $\gamma'(t)^{\perp} \subset T_{\gamma(t)}M$, where R is the curvature tensor. The derivative $U' = U'_{\gamma'(t)} : \gamma'(t)^{\perp} \to \gamma'(t)^{\perp}$ is defined by $\langle U'(v), w\rangle = \langle U(v), w\rangle'$, whenever $v, w \perp \gamma'$ are parallel vector fields along γ.

Let R be a tensor in a Riemannian manifold M satisfying the usual algebraic conditions of the curvature tensor. It would be interesting to ask what conditions R should satisfy so that it is the curvature tensor of some Riemannian metric in an open subset of M. Our first result below shows that there

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is some condition on the first derivatives of R, which is a surprising fact, since the curvature is a tensor. Another surprise is that we used Ricatti equations on some closed manifold to prove this local result.

Theorem 1. *Let M be any Riemannian manifold. Then the Ricci operator K_v, defined by $K_v(u) = R(v,u)v,\ u \in \{v\}^{\perp}$, commutes with K'_v.*

To prove Theorem 1 we need to show the following general result about Ricatti equations on closed manifolds.

Theorem 2. *Let M be a complete manifold with finite volume and SM the unit tangent bundle of M with standard Liouville measure which is invariant under the geodesic flow. We consider symmetric operators U_v and K_v satisfying the Ricatti equation along geodesics as above, except that now K_v is not necessarily related to the Ricci operator. Assume that U and K are uniformly bounded. Then, for almost all $v \in SM$, U_v, K_v and U'_v commute.*

As another consequence of Theorem 2 we obtain the following result:

Corollary. *Let M be a closed manifold with nonpositive sectional curvature. Take $v \in T_pM$ and $\tilde{v} \in T_{\tilde{p}}\tilde{M}$ with $\pi(\tilde{p}) = p,\ d\pi_{\tilde{p}}\tilde{v} = v$, where $\pi\colon \tilde{M} \to M$ is the universal covering of M with the induced metric. Then the second fundamental form $U_{\tilde{v}}$ of the horosphere associated with the ray $\gamma_{\tilde{v}}$ starting at $\tilde{v}$ commutes with $K_{\tilde{v}}$, where $K_{\tilde{v}} = R(\tilde{v}, .)\tilde{v}$.*

1. Proof of Theorem 2 and its consequences

Proof of Theorem 2. Let U and K be as in Theorem 2. Let $U_c = U + cI$, where c is a positive constant and I is the identity map restricted to the orthogonal complement $\{\gamma'\}^{\perp}$. Since there exists $A > 0$ such that $|U| < A$, for sufficiently large c we have that $\langle (U_c)_v\, x, x\rangle \geq B|x|^2$, for a positive constant B, any v in the unit tangent bundle SM and any $x \perp v$. We can write

$$U_c' + U_c^2 + K_c = 0,$$

along each geodesic in M, where $K_c := K - 2cU - c^2 I$. For sufficiently large c we have that K_c is negative definite, hence $-K_c$ has a positive square root, that is, an operator $\sqrt{-K_c}$ with the same eigen spaces as $-K_c$ and whose corresponding eigenvalues are the positive square roots of the eigenvalues of $-K_c$. Applying the proof in [OS], page 459, to U_c and K_c, we get

$$\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu \le \int_{SM} \operatorname{tr} U_c\, d\mu = \int_{SM} \operatorname{tr} U\, d\mu + \int_{SM} \operatorname{tr} cI\, d\mu. \tag{1.1}$$

We will use the following trivial result.

Lemma 1.1. *Consider the function* $f(x) = \sqrt{-ax^2 + 2bx + 1} - 1$*, where* $|a|$ *and* $|b|$ *are small compared with* 1*. Then we have* $f(0) = 0,\ f'(0) = b,\ f''(0) = -(b^2 + a),\ f'''(0) = 3(b^2 + a)b.$

Now consider, at each point $v \in SM$, an orthonormal basis $\{e_i\} \subset \{v\}^\perp$ which diagonalizes the positive symmetric matrix $-K_c$. We choose $\{e_i\}$ in such a way that $g(v) = \sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle^2$ is a minimum between all orthonormal bases which diagonalize K_c. This will imply that g is measurable (and integrable, by the uniform boundedness of $\langle U(e_i), e_i\rangle$). To see this, consider the fiber bundle $F \to SM$ where the fiber over $v \in SM$ is the set of orthonormal bases of $\{v\}^\perp \subset T_vM$. The set $S = \{(v, \{e_i\}) \in F |\, \sum_i(\langle K_c^2 e_i, e_i\rangle - \langle K_c e_i, e_i\rangle^2) = 0\}$ is a measurable subset of F and consists of the set of orthonormal bases that diagonalize K_c. The function g can be defined as $g(v) = \min\{\sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle^2 \,|(v, \{e_i\}) \in S\}$. So g is clearly measurable. We have

$$\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu - \int_{SM} \operatorname{tr} cI\, d\mu \tag{1.2}$$
$$= \int_{SM} \sum_{i=0}^{n-1} \left[\sqrt{-\langle K(e_i), e_i\rangle + 2c\langle U(e_i), e_i\rangle + c^2} - c\right] d\mu$$

$$= \int_{SM} \sum_{i=0}^{n-1} c \left[\sqrt{\frac{-\langle K(e_i), e_i\rangle}{c^2} + \frac{2\langle U(e_i), e_i\rangle}{c} + 1} - 1 \right] d\mu.$$

From (1.2), Lemma 1.1 and the fact that $\langle U(e_i), e_i\rangle$ and $\langle K(e_i), e_i\rangle$ are uniformly bounded, we obtain

$$\int_{SM} \operatorname{tr} \sqrt{-K_c}\, d\mu - \int_{SM} \operatorname{tr} cI\, d\mu \tag{1.3}$$
$$= \int_{SM} \sum_{i=0}^{n-1} \left[\langle U(e_i), e_i\rangle + \frac{-\langle K(e_i), e_i\rangle - \langle U(e_i), e_i\rangle^2}{2c} + O\left(\frac{1}{c^2}\right) \right].$$

Since $\int_{SM} \sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle\, d\mu = \int_{SM} \operatorname{tr} U\, d\mu$, we obtain

$$\int_{SM} \sum_{i=0}^{n-1} \left[-\langle K(e_i), e_i\rangle - \langle U(e_i), e_i\rangle^2 \right] d\mu \leq 0. \tag{1.4}$$

In fact, if the integral in (1.4) would be positive, this would contradict (1.1) and (1.3) for sufficiently large c.

It is a well known consequence of (0.1) and the fact that μ is invariant under the geodesic flow that $\int_{SM} \operatorname{tr}(-K) = \int_{SM} \operatorname{tr} U^2$ (see [Gr]). So we have

$$\int_{SM} \sum_{i=0}^{n-1} \left[-\langle K(e_i), e_i\rangle - \langle U(e_i), e_i\rangle^2 \right] d\mu = \int_{SM} \left[\operatorname{tr}(-K) - \sum_{i=0}^{n-1} \langle U(e_i), e_i\rangle^2 \right] d\mu$$
$$= \int_{SM} \left[\sum_{i=0}^{n-1} \langle U^2(e_i), e_i\rangle - \langle U(e_i), e_i\rangle^2 \right] d\mu. \tag{1.5}$$

From the symmetry of U and the Theorem of Cauchy-Schwarz we have

$$\sum_{i=0}^{n-1} \left[\langle U^2(e_i), e_i\rangle - \langle U(e_i), e_i\rangle^2 \right] = \sum_{i=0}^{n-1} \left[\langle U(e_i), U(e_i)\rangle - \langle U(e_i), e_i\rangle^2 \right] \geq 0, \tag{1.6}$$

By (1.4) and (1.6) we conclude that equality holds in both inequalities. By the rigidity part in the Theorem of Cauchy-Schwarz we obtain $U(e_i) = \lambda_i e_i$. So the basis $\{e_i\}$ diagonalizes U and K_c almost everywhere on SM. It diagonalizes K from the definition of K_c, and it also diagonalizes U', because of (0.1). Theorem 2 is proved.

□

Proof of Theorem 1. Fix $v \in T_pM$. We want to prove that K_v and K'_v commute, where $K_v = R(v, .)v$. We consider a closed Riemannian manifold (N, g) containing a neighborhood V of p in M such that the Riemannian metric g is an extension of the Riemannian metric of $V \subset M$. We consider on N the trivial Ricatti equation

$$K' + K^2 + (-K' - K^2) = 0.$$

From Theorem 2 we conclude that K and K' commutes almost everywhere on SM, hence they commute everywhere because of the continuity of the map $u \mapsto K_u$. In particular K_v and K'_v commute. The proof is complete.

□

Proof of the Corollary. In the case that M is closed with nonpositive curvature it is well known that there exists a global solution U for the Ricatti equation (with $K_v(u) = R(v, u)v$). Let $\tilde{U}$ and $\tilde{K}$ be the liftings of U and K to $\tilde{M}$. It is also known that the horosphere $S_{\tilde{v}}$, associated to the ray $\gamma_{\tilde{v}}$ in $\tilde{M}$, is C^2 and that U can be taken so that $\tilde{U}$ becomes the second fundamental form of $S_{\tilde{v}}$ at $\tilde{v}$. In particular $\tilde{U}$ and U are continuous. Thus Theorem 2 implies that U, U' and K commute everywhere on SM, hence $\tilde{U}$ and $\tilde{K}$ also commute. This concludes the proof of the corollary.

□

References

[**A**] Ambrose, W., *A Theorem of Myers*, Duke Math J 24 (1057), 345–348.

[**E**] Eberlein, P., *When is a geodesic flow of Anosov type? I*, J. Differential Geometry 8 (1973), 437–463.

[**FM**] Freire, A., Mañé, R., *On the entropy of the geodesic flow in manifolds without conjugate points*, Invent. Math. 69 (1982), 375–392.

[**Gr**] Green, L. A., *A theorem of E. Hopf*, Michigan Math. J. 5 (1958), 31–34.

[**Gu**] Guimarães, F. F., *The integral of the scalar curvature of complete manifolds without conjugate points*, J. Differential Geom. 36 (1992) 651–662.

[**K**] Klingenberg, W., *Riemannian manifolds with geodesic flows of Anosov type*, Ann. Math. 99 (1974), 1–13.

[**MZ1**] Mendonça, S., Zhou, D., *Curvature conditions for immersions of submanifolds and applications*, Compositio Math. 137 (2003) 211–226.

[**MZ2**] Mendonça, S., Zhou, D., *Maximal ergodic theorems and applications to Riemannian geometry*, Israel J. of Math. 139 (2004), 319–335.

Instituto de Matemática
Departamento de Análise
Universidade Federal Fluminense (UFF)
Rua Mário Santos Braga S/N, Valonguinho
24.020-140, Niterói, RJ, Brazil
E-mail addresses: sxmendonca@hotmail.com , mendonca@mat.uff.br