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EXAMPLES OF BIMINIMAL SURFACES OF THURSTON'S THREE-DIMENSIONAL GEOMETRIES *

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1 Introduction

Since their first work on harmonic maps [5], J. Eells and J.H. Sampson suggested the idea of studying k-harmonic maps. For $k = 2$, the idea is the following. First define harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds as critical points of the energy $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$. The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the *tension field* $\tau(\phi) = \operatorname{trace}\nabla d\phi$. Then define the *bienergy* of a map ϕ by $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$, and say that ϕ is *biharmonic* if it is a critical point of the bienergy.

In [6] G.Y. Jiang derived the first variation formula of the bienergy showing that the Euler-Lagrange equation for E_2 is

$$\tau_2(\phi) = -\left(\Delta^\phi \tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi\right) = 0. \qquad (1)$$

The equation $\tau_2(\phi) = 0$ is now called the *biharmonic equation*.

In [4] B.Y.Chen and S. Ishikawa defined biharmonic submanifolds of the Euclidean space as those with harmonic mean curvature vector, and proved that *any biharmonic surface of the Euclidean* 3*-space is harmonic, i.e. minimal.* The latter result has suggested the following

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Conjecture: *Any biharmonic submanifold of the Euclidean space is harmonic, i.e. minimal.*

If we consider the biharmonic equation $\tau_2(\phi) = 0$ for isometric immersions into the Euclidean space we recover Chen's notion of biharmonic submanifolds, so the two definitions agree.

The result of Chen-Ishikawa was generalised in [3], by R. Caddeo, S. Montaldo and C. Oniciuc, to prove the non-existence of (non-minimal) biharmonic immersed surfaces in $\mathbb{H}^3$, whether compact or not. The situation is hardly richer in $\mathbb{S}^3$, where $\mathbb{S}^2(\frac{1}{\sqrt{2}})$ is the sole possibility [2].

To have a larger and more interesting class of surfaces the authors proposed in [7] to study isometric immersions which are critical points of the bienergy for *normal variations* giving the following

Definition 1.1. An immersion $\phi : (M^m, g) \to (N^n, h)$ $(m \leq n)$ between Riemannian manifolds, or its image, is called *biminimal* if it is a critical point of the bienergy functional E_2 for variations normal to the image $\phi(M) \subset N$, that is:

$$\frac{dE_2(\phi_t)}{dt}\Big|_{t=0} = 0,$$

for any smooth variation of the map $\phi_t :]-\epsilon, +\epsilon[\times M \to N$, $\phi_0 = \phi$, such that $V = \frac{d\phi_t}{dt}|_{t=0}$ is normal to $\phi(M)$.

Note that this variational principle is close to the Willmore problem, the disparity being that we do not vary through isometric immersions.

In the instance of an isometric immersion $\phi : M \to N$, requiring that the normal part of $\tau_2(\phi)$ is zero characterises biminimal isometric immersions, that is the mean curvature vector field $\mathbf{H}$ of ϕ satisfies:

$$[\Delta^{\phi}\mathbf{H} - \operatorname{trace} R^N(d\phi, \mathbf{H})d\phi]^{\perp} = 0, \tag{2}$$

where $[.]^{\perp}$ denotes the normal component of $[.]$.

A generalisation of biminimal immersions are λ-*biminimal* immersions, they are defined as critical points, with respect to normal variations of fixed energy, of the constrained bienergy functional

$$E_{2,\lambda}(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g + \lambda \int_M |d\phi|^2 \, v_g.$$

The Euler-Lagrange equation for λ-biminimal immersions is

$$[\tau_{2,\lambda}]^\perp = [\tau_2]^\perp - 2\lambda[\tau]^\perp = 0,$$

In this paper we construct some new examples of biminimal surfaces of the Thurston's three-dimensional geometries following some basic constructions introduced, by the authors, in [7].

Notation. We shall place ourselves in the C^∞ category, i.e. manifolds, metrics, connections, maps will be assumed to be smooth. By (M^m, g) we shall mean a connected manifold, of dimension m, without boundary, endowed with a Riemannian metric g. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. For the Laplacian we shall use $\Delta(f) = \operatorname{div}\operatorname{grad} f$ for functions $f \in C^\infty(M)$ and $\Delta^\phi W = -\operatorname{trace}\nabla\nabla^\phi W$ for sections along a map $\phi : M \to N$.

2 Codimension-one biminimal submanifolds

Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension-one. We denote by B the second fundamental form of ϕ, by $\mathbf{N}$ a unit normal vector field to $\phi(M) \subset N$ and by $\mathbf{H} = H\mathbf{N}$ the mean curvature vector field of ϕ (H the mean curvature function). Then we have

Proposition 2.1. *[7] Let $\phi : M^n \to N^{n+1}$ be an isometric immersion of codimension-one and $\mathbf{H} = H\mathbf{N}$ its mean curvature vector. Then ϕ is biminimal if and only if:*

$$\Delta^M H = (\|B\|^2 - \operatorname{Ricci}(\mathbf{N}))H. \tag{3}$$

If the ambient space has constant sectional curvature, Condition (3) takes a simpler form as shown in the following

Corollary 2.2. *[7] An isometric immersion $\phi : M^n \to N^{n+1}(c)$ into a space form of constant curvature c is biminimal if and only if:*

$$\Delta^M H - H(n^2H^2 - s + n(n-2)c) = 0,$$

where s is the scalar curvature of M^n. Moreover, an isometric immersion $\phi : M^2 \to N^3(c)$ from a surface of Gaussian curvature G to a three dimensional space form is λ-biminimal if and only if:

$$\Delta^M H - 2H(2H^2 - G + \lambda) = 0, \tag{4}$$

Remark 2.3. Condition (4), for $\lambda = 0$, is very similar to the equation of the Willmore problem ($\Delta H + 2H(H^2 - G) = 0$) but the minus sign in (4) rules out the existence of compact solutions when $c \le 0$.

In [7] the authors, using Corollary 2.2, have described some constructions to produce examples of biminimal immersions in a space form. To explain the constructions, let's first recall that a submersion $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a *horizontally homothetic submersion* if there exists a function $\lambda : M \to \mathbb{R}$, called the *dilation function*, such that:

- at each point $p \in M$ the differential $d\phi_p : H_p \to T_{\phi(p)}N$ is a conformal map with factor $\lambda(p)$, i.e.

  $$\lambda^2(p) g(X, Y)(p) = h(d\phi_p(X), d\phi_p(Y))(\phi(p))$$

  for all $X, Y \in H_p = Ker_p(d\phi)^{\perp}$;
- $X(\lambda^2) = 0$ for all horizontal vector fields, X being horizontal if $X_p \in H_p$ for all $p \in M$.

The idea is then to start with a horizontally homothetic submersion $\phi : (M, g) \to (N^2, h)$ to a surface, consider on N a differentiable curve $\gamma : I \subset \mathbb{R} \to N$ and take the inverse image $S = \phi^{-1}(\gamma(I)) \subset M$ of γ in M via the map ϕ. The set S is a hypersurface of M and its mean curvature can be related to the signed curvature of γ as shown in the following

Lemma 2.4. *[7] Let $\phi : (M^n, g) \to (N^2, h)$ be a horizontally homothetic submersion with dilation λ and minimal fibres and let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrised by arc-length of signed curvature k_γ. Then the codimension-one submanifold $S = \phi^{-1}(\gamma(I)) \subset M$ has mean curvature $H_S = \lambda k_\gamma/(n-1)$.*

Combining (4) and Lemma 2.4 the main theorems in [7] can be stated as follows.

Theorem 2.5. *[7] Let $\phi : M^3(c) \to (N^2, h)$ be a horizontally homothetic submersion with dilation λ, minimal fibres and integrable horizontal distribution from a space form of constant sectional curvature c to a surface. Let $\gamma : I \subset \mathbb{R} \to N^2$ be curve parametrised by arc-length such that the surface $S = \phi^{-1}(\gamma) \subset M^3$ has constant Gaussian curvature c. Then $S = \phi^{-1}(\gamma) \subset M^3$ is a c-biminimal surface if and only if γ is a biminimal curve.*

If the horizontal space is not integrable, Theorem 2.5 can be reformulated for Riemannian submersions.

Theorem 2.6. *[7] Let $\phi : M^3(c) \to N^2(\bar{c})$ be a Riemannian submersion with minimal fibres from a space form of constant sectional curvature c to a surface of constant Gaussian curvature $\bar{c}$. Let $\gamma : I \subset \mathbb{R} \to N^2$ be a curve parametrised by arc-length. Then $S = \phi^{-1}(\gamma) \subset M^3$ is a biminimal surface if and only if γ is a $\bar{c}/2$-biminimal curve.*

Theorem 2.5 and Theorem 2.6 have been used in [7] to produce examples of biminimal surfaces in a 3-dimensional space form. For instance we have:

- a vertical cylinder in $\mathbb{R}^3$ with generatrix a biminimal curve of $\mathbb{R}^2$ is a biminimal surface in $\mathbb{R}^3$;
- the cone in $\mathbb{R}^3$ on a biminimal curve on $\mathbb{S}^2$ is a biminimal surface of $\mathbb{R}^3$;
- a Hopf cylinder of $\mathbb{S}^3$ is a biminimal surface if and only if the base curve γ on $\mathbb{S}^2(1/2)$ is a 2-biminimal curve of $\mathbb{S}^2(1/2)$;

- a vertical cylinder in the hyperbolic 3-space (half space model) with generatrix a biminimal curve of $\mathbb{R}^2$ (plane to infinity) is a (-1)-biminimal surface in the hyperbolic space;

3 Examples of biminimal surfaces of Thurston's eight geometries

Of Thurston's eight geometries (cf. [1]), three have constant sectional curvature, $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, and contain biminimal surfaces as described in the previous section, two are Riemannian products, $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, and will be our first class of examples, two are line bundles, over $\mathbb{R}^2$ for $\mathbb{H}_3$ and over $\mathbb{R}^2_+$ for $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, and one, Sol, does not allow Riemannian submersion or horizontally homothetic maps with minimal fibres to a surface, even locally, and therefore does not fit our framework.

Before describing the examples we recall, cf. [7], that a curve γ on a surface N^2 of Gaussian curvature G is λ-biminimal if and only if its signed curvature k satisfies the ordinary differential equation:

$$k'' - k^3 + kG - 2\lambda k = 0 \tag{5}$$

3.1 Biminimal surfaces of $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$

In both cases, consider the Riemannian submersion given by the projection onto the first factor, call it $\pi : N^2\times\mathbb{R}\to N^2$, which has totally geodesic fibres. Given a curve $\gamma : I\subset\mathbb{R}\to N^2$ parametrised by arc length, take its Frenet frame $\{T,N\}$, and consider $\{e_1,e_2\}\in T(N^2\times\mathbb{R})$ its horizontal lift. The unit vertical vector e_3 completes $\{e_1,e_2\}$ into an orthonormal frame of $T(N^2\times\mathbb{R})$, such that $\{e_1,e_3\}$ is a basis of TS, for $S=\pi^{-1}(\gamma(I))$, with e_2 the normal to the surface. From Lemma 2.4 then the mean curvature of S is $H=k/2$, where k is the signed curvature of γ, and, from Proposition 2.1, S is biminimal in $N^2\times\mathbb{R}$ if:

$$\Delta H = (\|B\|^2 - \mathrm{Ricci}(e_2))H.$$

With respect to the frame $\{e_1, e_3\}$ the second fundamental form B is

$$\begin{pmatrix} <\nabla_{e_1} e_1, e_2> & <\nabla_{e_1} e_3, e_2> \\ <\nabla_{e_3} e_1, e_2> & <\nabla_{e_3} e_3, e_2> \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$

Besides ,

$$\text{Ricci}^{N^2\times\mathbb{R}}(e_2) = \text{Ricci}^{N^2}(e_2) = \begin{cases} +1 & \text{if } N^2 = \mathbb{S}^2 \\ -1 & \text{if } N^2 = \mathbb{H}^2. \end{cases}$$

In both cases, (using Equation (11) in [7]), $\Delta H = \Delta(k/2) = \frac{1}{2}k''$, so S is biminimal in $N^2 \times \mathbb{R}$ if:

$$k'' = k^3 - k \quad \text{if } N^2 = \mathbb{S}^2$$

and

$$k'' = k^3 + k \quad \text{if } N^2 = \mathbb{H}^2.$$

Now comparing with (5), we have the following

Proposition 3.1. *The cylinder $S = \pi^{-1}(\gamma)$ is a biminimal surface in $N^2 \times \mathbb{R}$ if and only if γ is a biminimal curve on N^2 ($\mathbb{S}^2$ or $\mathbb{H}^2$) .*

3.2 Biminimal surfaces of the Heisenberg space

The 3-dimensional Heisenberg space $\mathbb{H}_3$ is the two-step nilpotent Lie group standardly represented in $Gl_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

with $x, y, z \in \mathbb{R}$. Endowed with the left-invariant metric

$$g = dx^2 + dy^2 + (dz - xdy)^2, \tag{6}$$

$(\mathbb{H}_3, g)$ has a rich geometric structure, reflected by the fact that its group of isometries is of dimension 4, which is the maximal possible dimension for a non constant curvature metric on a 3-manifold. Also, from the algebraic point of view, it is a 2-step nilpotent Lie group, i.e. "almost Abelian". An orthonormal basis of left-invariant vector fields is given, with respect to the coordinates vector fields, by

$$E_1 = \frac{\partial}{\partial x}; \quad E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}; \quad E_3 = \frac{\partial}{\partial z}. \tag{7}$$

Let now $\pi : \mathbb{H}_3 \to \mathbb{R}^2$ be the projection $(x, y, z) \mapsto (x, y)$. At a point $p = (x, y, z) \in \mathbb{H}_3$ the vertical space of the submersion π is $V_p = \mathrm{Ker}(\mathrm{d}\pi_\mathrm{p}) = \mathrm{span}(\mathrm{E}_3)$ and the horizontal space is $H_p = \mathrm{span}(\mathrm{E}_1, \mathrm{E}_2)$. An easy computation shows that π is a Riemannian submersion with minimal fibres. In fact we have that the non zero covariant derivatives of the left invariant vector fields are:

$$\begin{aligned} \nabla_{E_1} E_2 = -\nabla_{E_2} E_1 &= \frac{1}{2} E_3 \\ \nabla_{E_1} E_3 = \nabla_{E_3} E_1 &= -\frac{1}{2} E_2; \\ \nabla_{E_2} E_3 = \nabla_{E_3} E_2 &= \frac{1}{2} E_1 \end{aligned} \tag{8}$$

Now let $\gamma(t) = (x(t), y(t))$ be a curve in $\mathbb{R}^2$ parametrized by arc length with signed curvature k and consider the flat cylinder $S = \pi^{-1}(\gamma)$ in $\mathbb{H}_3$. Since the left invariant vector fields are orthonormal the vector fields

$$e_1 = x' E_1 + y' E_2; \quad e_2 = E_3$$

give an orthonormal frame tangent to S and

$$N = -y' E_1 + x' E_2$$

is a unit normal vector field of S in $\mathbb{H}_3$.
We now determine the second fundamental form B of the surface $S = \pi^{-1}(\gamma)$, given by:

$$\begin{pmatrix} < \nabla_{e_1} e_1, N > & < \nabla_{e_1} e_2, N > \\ < \nabla_{e_2} e_1, N > & < \nabla_{e_2} e_2, N > \end{pmatrix}.$$

From (8) we have

$$< \nabla_{e_1} e_1, N > = < x'' E_1 + y'' E_2, -y' E_1 + x' E_2 > = x' y'' - x'' y' = k$$

$$< \nabla_{e_2} e_2, N > = 0$$

$$< \nabla_{e_1} e_2, N > = < \nabla_{e_2} e_1, N > = \frac{1}{2} < -x' E_2 + y' E_1, -y' E_1 + x' E_2 > = -\frac{1}{2}.$$

Thus the second fundamental form B is

$$B = \begin{pmatrix} k & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

From the expression of B we see that $H = \text{trace}(B)/2 = k/2$ and that $\|B\|^2 = k^2 + 1/2$.

To write down the biminimality condition for S, we need to compute $\text{Ricci}(N)$. For this, let first recall that the non zero components of the Riemann tensor of $\mathbb{H}_3$, with respect to the left invariant vector fields, are

$$R_{1212} = R(E_1, E_2, E_1, E_2) = -\frac{3}{4}$$

and

$$R_{1313} = R(E_1, E_3, E_1, E_3) = \frac{1}{4} = R_{2323} = R(E_2, E_3, E_2, E_3).$$

Then,

$$\begin{aligned}
\text{Ricci}(N) =& R(e_1, N, e_1, N) + R(e_2, N, e_2, N) \\
=& R(x'E_1 + y'E_2, -y'E_1 + x'E_2, x'E_1 + y'E_2, -y'E_1 + x'E_2) \\
&+ R(E_3, -y'E_1 + x'E_2, E_3, -y'E_1 + x'E_2) \\
=& x'^4 R_{1212} + y'^4 R_{1212} + 2x'^2 y'^2 R_{1212} + y'^2 R_{3131} + x'^2 R_{3232} \\
=& R_{1212}(x'^2 + y'^2)^2 + R_{3131}(x'^2 + y'^2) \\
=& -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}
\end{aligned}$$

Thus, from (3), S is biminimal if and only if

$$\Delta H = (\|B\|^2 - \text{Ricci}(N))H$$

and using the computations if and only if

$$k'' = (k^2 + 1/2 + 1/2)k = k^3 + k$$

Finally, taking into account (5), we have the following

Proposition 3.2. *The flat cylinder $S = \pi^{-1}(\gamma) \subset \mathbb{H}_3$ is a biminimal surface of $\mathbb{H}_3$ if and only if γ is a $1/2$-biminimal curve of $\mathbb{R}^2$.*

3.3 Biminimal surfaces of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$

Following [1, page 301] we identify $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with

$$\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 \,:\, z > 0\}$$

endowed with the metric

$$ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}. \tag{9}$$

Then the projection $\pi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{R}^2_+$ defined by $(x,y,z) \mapsto (y,z)$ is a submersion and if we denote, as usual, by $\mathbb{H}^2$ the space $\mathbb{R}^2_+$ with the hyperbolic metric $\frac{dy^2+dz^2}{z^2}$, the submersion $\pi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{H}^2$ becomes a Riemannian submersion with minimal fibres. The vertical space at a point $p = (x,y,z) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is $V_p = \mathrm{Ker}(\mathrm{d}\pi_\mathrm{p}) = \mathrm{span}(\mathrm{E}_1)$ and the horizontal space at p is $H_p = \mathrm{span}(\mathrm{E}_2, \mathrm{E}_3)$, where

$$E_1 = \frac{\partial}{\partial x}; \quad E_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}; \quad E_3 = z\frac{\partial}{\partial z} \tag{10}$$

give an orthonormal frame on $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with respect to the metric (9). In this case the non zero covariant derivatives of the vector fields in (10) are:

$$\begin{aligned} &\nabla_{E_2} E_2 = E_3; \quad \nabla_{E_1} E_2 = \nabla_{E_2} E_1 = \frac{1}{2} E_3; \\ &\nabla_{E_1} E_3 = \nabla_{E_3} E_1 = -\frac{1}{2} E_2; \\ &\nabla_{E_2} E_3 = -\frac{1}{2} E_1 - E_2; \quad \nabla_{E_3} E_2 = \frac{1}{2} E_1. \end{aligned} \tag{11}$$

Now let $\gamma(t) = (y(t), z(t))$ be a curve in $\mathbb{H}^2$ parametrized by arc length and consider the flat cylinder $S = \pi^{-1}(\gamma)$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. Since the vector fields in (10) are orthonormal the vector fields

$$e_1 = \frac{y'}{z} E_2 + \frac{z'}{z} E_3; \quad e_2 = E_1 \tag{12}$$

give an orthonormal frame tangent to S and

$$N = -\frac{z'}{z} E_2 + \frac{y'}{z} E_3$$

is a unit normal vector field of S in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
With calculations similar to that of the previous examples we find that the second fundamental form B, with respect to the orthonormal frame (12), is

$$B = \begin{pmatrix} k & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

For Ricci(N), taking into account that

$$R_{2323} = R(E_2, E_3, E_2, E_3) = -\frac{7}{4}$$

and

$$R_{1212} = R(E_1, E_2, E_1, E_2) = \frac{1}{4} = R_{1313} = R(E_1, E_3, E_1, E_3),$$

we have

$$\begin{aligned} \mathrm{Ricci}(N) =& R(e_1, N, e_1, N) + R(e_2, N, e_2, N) \\ =& R_{3232} \left(\frac{y'^2 + z'^2}{z^2} \right)^2 + R_{1212} \left(\frac{y'^2 + z'^2}{z^2} \right) \\ =& -\frac{7}{4} + \frac{1}{4} = -\frac{3}{2}. \end{aligned}$$

Thus we have the following

Proposition 3.3. *The flat cylinder* $S = \pi^{-1}(\gamma) \subset \widetilde{\mathrm{SL}_2(\mathbb{R})}$ *is a biminimal surface of* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *if and only if* γ *is a* $1/2$*-biminimal curve of* $\mathbb{H}^2$.

Remark 3.4. These links between biminimal cylinders and biminimal curves that we have described in the Thurston geometries are very similar to the link between Willmore Hopf cylinders of $\mathbb{S}^3$ and elastic curves on $\mathbb{S}^2$ proved by U. Pinkall in [8].

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