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STABLE COMPLETE MINIMAL HYPERSURFACES IN R^4 *

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Abstract

Let $x : M \to R^4$ be a stable complete minimal hypersurface, we give a sufficient condition for $x(M)$ to be a hyperplane, improving Berard's result.

1. Introduction

Let $x : M \to R^{n+1}$ be an n-dimensional minimal immersion. Let $D \subset M$ be a domain with compact closure and piecewise smooth boundary. We say that D is *stable* if it is a minimum for the volume function of the induced metric for all variations that keep the boundary D fixed. The immersion x is *stable* if any such D is stable. We will denote by A the second fundamental form of $x(M)$ in R^{n+1}, by K the scalar curvature of $x(M)$ in the induced metric.

In [2] (also see [8] or [3]), P. Berard proved the following result

Theorem 1 (see Proposition 2.2 of [2]). *Let $x : M \to R^4$ be a 3-dimensional stable complete minimal immersion such that*

$$\int_M |A|^3 dM < +\infty, \tag{1.1}$$

then $x(M) \subset R^4$ is a hyperplane.

Set

$$B_R = \{p \in M;\ \rho(p, p_0) \le R\},$$

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where ρ is the geodesic distance from a fixed point $p_0 \in M$.

In this note, we improve the above Berard's result as following:

Theorem 2. *Let $x : M \to R^4$ be a 3-dimensional stable complete minimal immersion such that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |A|^3 dM}{R^{1+2q}} = 0, \quad q < \sqrt{\frac{2}{3}}, \tag{1.2}$$

then $x(M) \subset R^4$ is a hyperplane.

Remark 1. It is obvious that Berard's Theorem 1 is the direct consequence of our Theorem 2.

Remark 2. The strongly stable hypersurfaces with constant mean curvature have been studied by Alencar-do Carmo [1], do Carmo-Zhou [5] (also see survey paper of Berard-Santos [3]).

2. The proof of Theorem 2

We will follow the notation of [4] closely. Let $x : M \to R^4$ be a 3-dimensional stable complete minimal immersion.

By (2.11) of [4], we have

$$\int_M |A|^{4+2q} f^2 dM \le \beta_1 \int_M |A|^{2+2q} |\nabla f|^2 dM, \tag{2.1}$$

where q is any positive constant with $q < \sqrt{2/3}$, β_1 is some positive constant, and f is any function on M with compact support.

We now try to transform (2.1) so that the integrand of the right hand side only involve $|A|$ in the third power. For that, we use Young's inequality ([6], p_{111}):

$$ab \le \frac{\alpha^s a^s}{s} + \frac{\alpha^{-t} b^t}{t}, \quad \frac{1}{s} + \frac{1}{t} = 1, \tag{2.2}$$

where $\alpha > 0$ is arbitrary and $1 < s < \infty$, $1 < t < \infty$. Let p, $0 < p < 2 + 2q$, be a number yet to be determined. By use of (2.2), we obtain

$$\begin{aligned} |A|^{2+2q}|\nabla f|^2 &= f^2(|A|^{2+2q}\frac{|\nabla f|^2}{f^2}) \\ &= f^2(|A|^{2+2q-p}|A|^p\frac{|\nabla f|^2}{f^2}) \\ &\le f^2(\frac{\alpha^s}{s}|A|^{s(2+2q-p)} + \frac{\alpha^{-t}}{t}(|A|^p\frac{|\nabla f|^2}{f^2})^t). \end{aligned} \tag{2.3}$$

We now choose p to satisfy the following equations:

$$s(2+2q-p) = 4+2q, \quad pt = 3, \quad \frac{1}{s} + \frac{1}{t} = 1.$$

This is indeed possible, and the solution is

$$p = \frac{6}{1+2q}, \quad s = 1 + \frac{2}{2q-1}, \quad t = \frac{1+2q}{2}, \quad \frac{1}{2} < q < \sqrt{2/3}.$$

By use of these values and the fact that α may be made small, we obtain from (2.1) and (2.3)

$$\int_M |A|^{4+2q} f^2 dM \le \beta_2 \int_M |A|^3 \frac{|\nabla f|^{1+2q}}{f^{2q-1}} dM, \tag{2.4}$$

where β_2 is a constant depending on q. Now we use the arbitrariness of f to replace f by $f^{\frac{1+2q}{2}}$ in (2.4) and obtain

$$\int_M |A|^{4+2q} f^{2q+1} dM \le \beta_3 \int_M |A|^3 |\nabla f|^{2q+1} dM, \tag{2.5}$$

where β_3 is again a constant depending on q, $1/2 < q < \sqrt{2/3}$.

Choose a family of subsets B_R on M given by

$$B_R = \{p \in M : \, d(p, p_0) = \rho(p) \le R\}, \tag{2.6}$$

where $\rho(p)$ is the geodesic distance on M from p to a fixed point $p_0 \in M$. By completeness, B_R is compact and $\bigcup_{R \in (0,\infty)} B_R = M$. It is well known that $|\nabla \rho| \le 1$ almost everywhere on M (see [7]). Now fix numbers R and θ, $0 < \theta < 1$, and choose f in (2.5) by

$$\begin{aligned} &f(p) = 1, \quad \text{for} \quad \rho(p) \le \theta R, \\ &f(p) = \frac{R - \rho(p)}{(1-\theta)R}, \quad \text{for} \quad \theta R \le \rho(p) \le R, \\ &f(p) = 0, \quad \text{for} \quad \rho(p) \ge R. \end{aligned}$$

Then from (2.5) we have

$$\int_{B_{\theta R}} |A|^{4+2q} dM \le \beta_3 \frac{\int_{B_R} |A|^3 dM}{((1-\theta)R)^{2q+1}}, \quad \frac{1}{2} < q < \sqrt{\frac{2}{3}}.$$

By letting $R \to \infty$, we have, by assumption, that the right hand side of the above inequality approaches zero. Thus

$$\int_M |A|^{4+2q} dM = 0,$$

which implies that $|A| = 0$, i.e., $x(M) \subset R^4$ is a hyperplane. We complete the proof of Theorem 2.

3. A Related Problem

In [4], M.do Carmo and C.K.Peng proved the following well-known result

Theorem 3 (Theorem 1.3 of [4]). *Let $x : M \to R^{n+1}$ be an n-dimensional ($n \ge 3$) stable complete minimal immersion and assume that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |A|^2 dM}{R^{2+2q}} = 0, \quad q < \sqrt{\frac{2}{n}}, \tag{3.1}$$

then $x(M) \subset R^{n+1}$ is a hyperplane.

By Gauss equation, we know $|A|^2 = -K$, the total curvature $\int_M |K| dM = \int_M |A|^2$, thus we have from Theorem 3

Corollary 1 (Theorem 1.2 of [4]). *Let $x : M \to R^{n+1}$ be an n-dimensional ($n \ge 3$) stable complete minimal immersion such that the total curvature is finite, that is*

$$\int_M |A|^2 dM < +\infty, \tag{3.2}$$

then $x(M) \subset R^{n+1}$ is a hyperplane.

In [8], Y. B. Shen and X. H. Zhu proved the following result, which generalized Theorem 1 for general dimension n.

Theorem 4 (Main Theorem of [8]). *Let $x : M \to R^{n+1}$ be an n-dimensional ($n \geq 3$) stable complete minimal immersion such that*

$$\int_M |A|^n dM < +\infty, \tag{3.3}$$

then $x(M) \subset R^{n+1}$ is a hyperplane.

Observing that Theorem 4 is the counterpart of Corollary 1 if we use $\int_M |A|^n dM$ instead of $\int_M |A|^2 dM$, the following problem seems to be interesting

Problem. *Can we establish the counterpart of Theorem 3 for n-dimensional ($n \geq 3$) stable complete minimal immersion $x : M \to R^{n+1}$?*

Here we would like to formulate the above problem as following:

Conjecture: *Let $x : M \to R^n$ be an n-dimensional stable complete minimal immersion and assume that*

$$\lim_{R\to\infty} \frac{\int_{B_R} |A|^n dM}{R^{1+2q}} = 0, \quad q < \sqrt{\frac{2}{n}}, \tag{3.4}$$

then $x(M) \subset R^n$ is a hyperplane.

Remark 3. Our Theorem 2 only has checked the conjecture for the case $n = 3$. In this moment, we can not get any result about the conjecture for the general case $n \geq 4$.

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