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# ON THE CAUCHY PROBLEM ASSOCIATED TO THE BRINKMAN FLOW: THE ONE DIMENSIONAL THEORY

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## 1 Introduction.

In this article we are interested in the properties of the real valued solutions of the Cauchy Problem associated to the Brinkman Flow ([2]; see also [3] and[12]), namely

$$\left\{ \begin{array}{c} \phi\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = F\left(t,\rho\right), \\ (-\mu_{eff}\Delta + \frac{\mu}{k})v = -\nabla P(\rho), \\ (\rho(0), v(0)) = (\rho_0, v_0), \end{array} \right. \tag{1}$$

which models fluid flow in certain types of porous media. Here $\mu$ , $k$, and $\mu_{eff}$ denote the fluid viscosity, the porous media permeability and the pure fluid viscosity, respectively, while $\rho$ is the fluid's density, $v$ its velocity, $P$ is the pressure, $F$ is an external mass flow rate, and $\phi$ is the porosity of the medium.

In what follows, we will consider the case $n = 1$. Moreover, to simplify the notation, we will choose all the coefficients in (1) to be equal to 1. At the moment we want to consider only the mathematical structure of the system. At a later stage, the constants should be put back in, and various limiting cases should be studied. Thus our problem becomes:

$$\left\{ \begin{array}{c} \dfrac{\partial\rho}{\partial t} + \partial_x(\rho v) = F\left(t,\rho\right), \\ (-\partial_x^2 + 1)v = -\partial_x P(\rho), \\ (\rho(0), v(0)) = (\rho_0, v_0), \end{array} \right. \tag{2}$$

where $x \in \mathbb{R}$ and $t > 0$.

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To handle (2), we compute $v(t,x)$ using the second equation, (usually referred to as Brinkman's condition) to get

$$v = -\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho), \tag{3}$$

and substitute into the first one (which describes the variation of mass) to obtain the Cauchy Problem

$$\begin{cases} \partial_t \rho = \partial_x \left(\rho\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho)\right) + F(t,\rho) \\ \rho(0) = \rho_0. \end{cases} \tag{4}$$

Then we solve (4), and compute $v$ using (3). Of course, the following compatibility condition must be satisfied:

$$v_0 = -\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho_0). \tag{5}$$

In what follows we will use the following notation. If $X$ and $Y$ are Banach spaces, $\mathcal{B}(Y,X)$ denotes the set of all bounded linear operators from $Y$ into $X$. In case $X=Y$, we write simply $\mathcal{B}(Y)$. If $s\in\mathbb{R}$, we denote ($L^2$ type) Sobolev spaces by $H^s(\mathbb{R})$. Any of its usual norms will be denoted by $\|\cdot\|_s$. The corresponding inner products will be written $(\cdot|\cdot)_s$. Finally, whenever it is convenient we will use the notation $J=(1-\partial_x^2)^{1/2}$.

This paper is organized as follows. In Section 2 we use Kato's theory of quasilinear equations to prove that (4) is locally well-posed[1] in $H^s(\mathbb{R})$, $s>3/2$. This means that there exists a $T>0$ and a unique $\rho\in C([0,T],H^s(\mathbb{R}))\cap C^1([0,T],H^{s-1}(\mathbb{R}))$ such that $\rho(0)=\rho_0$, and $\rho(t)$ satisfies the differential equation in the sense that

$$\lim_{h\to 0}\left\|\frac{\rho(t+h)-\rho(t)}{h}-\partial_x\left(\rho(t)\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho(t))\right)+F(t,\rho(t))\right\|_{H^{s-1}}=0. \tag{6}$$

Moreover, $\rho$ depends continuously on the initial data in the following sense. Assume that $\rho_0^{(j)}\in H^s(\mathbb{R})$, $j=1,2,3,...,\infty$, let $\rho^{(j)}$ be the corresponding solutions. Suppose that

$$\lim_{j\to\infty}\left\|\rho_0^{(j)}-\rho_0^{(\infty)}\right\|_s=0. \tag{7}$$

[1]The notions of local end global well-posedness may be introduced in abstract settings. See for example ([7]) and ([5]).

Then, for all $T' \in (0,T)$ we have

$$\lim_{j\to\infty}\sup_{t\in[0,T']}\left\|\rho^{(j)}(t)-\rho^{(\infty)}(t)\right\|_s=0. \tag{8}$$

In Section 3 we discuss the interaction of Kato's theory and the method of parabolic regularization, and prove that the problem in question is globally well-posed in $H^s(\mathbb{R})$, $s>3/2$, in case $P(\rho)=\rho^{2k}, k=1,2,3....$ This means that the three conditions defining local well-posedness hold for all intervals $[0,T]$, $T>0$. Finally, in Section 4 we prove that the flow associated to (2), with $F\equiv 0$,is positivity preserving, that is, if $0\le\rho_0$ then, $0\le\rho(t)$ for all $t$. In the appendix we establish some estimates needed in the main body of the article.

# 2 Local Well Posedness in ($L^2$-type) Sobolev Spaces.

In this section we will use Kato's quasilinear theory to prove that (4) is locally well-posed in $H^s(\mathbb{R})$ for all $s>3/2$. Before proceeding, it is convenient to make a few remarks about Kato's theory. Its aim is to establish sufficient conditions to ensure the local well-posedness for problems of the form

$$\begin{cases}\partial_t u + A(t,u)\,u = f(t,u)\in X,\\ u(0)=\phi\in Y,\end{cases} \tag{9}$$

where $X$ and $Y$ are Banach spaces, with $Y$ continuously and densely embedded in $X$ and $A(t,u)$ is bounded from $Y$ into $X$ and is the (negative) generator of a $C^0$ semigroup for each $(t,u)\in[0.T]\times W$, $W$ open in $Y$. In its most general formulation, $X$ and $Y$ may be non-reflexive[2] Since we will deal exclusively with reflexive spaces, we will employ a simpler version, which can be found in [8]. (See also [9] and [4].) The essential assumption of the theory is the existence of an isomorphism $S$ from $Y$ onto $X$ such that

$$SA(t,u)\,S^{-1}=A(t,u)+B(t,u), \tag{10}$$

[2]This is rather important, since it allows one to show that continuous dependence can be reduced to a question of existence and uniqueness in non-reflexive Banach spaces. See [7] and the references therein.

where $B(t,u) \in \mathcal{B}(X)$. This is, in fact, a condition on the commutator $[S, A(t,u)]$ because (10) can be rewritten as

$$[S, A(t,u)]\, S^{-1} = B(t,u). \tag{11}$$

There are also lesser requirements, involving Lipschitz conditions on the operators in question. For example, $A(t,u)$ must satisfy

$$\|A(t,w) - A(t,\widetilde{w})\|_{\mathcal{B}(Y,X)} \leq \mu \|w - \widetilde{w}\|_X ., \tag{12}$$

for all pairs $(t,w), (t,\widetilde{w})$ in $[0.T] \times W$. Both $B(t,u)$ and $f(t,u)$ must satisfy similar conditions. We are now in position to state the main result of this section.

**Theorem 1** *Let*

$$A(\rho) f = -\partial_x \left( f \left(1 - \partial_x^2\right)^{-1} \partial_x P(\rho) \right) = -\partial_x \left( f J^{-2} \partial_x P(\rho) \right), \tag{13}$$

*so that the PDE in (4) can be written as*

$$\partial_t \rho + A(\rho)\rho = F(t,\rho). \tag{14}$$

*Let* $\rho_0 \in H^s(\mathbb{R})$, $s > 3/2$ *and assume that* $P$ *and* $F$ *satisfy the following assumptions.*
*(a)* $P$ *maps* $H^s(\mathbb{R})$ *into itself,* $P(0) = 0$ *and is Lipchitz in the following sense:*

$$\|P(\rho) - P(\widetilde{\rho})\|_s \leq L_s\left(\|\rho\|_s, \|\widetilde{\rho}\|_s\right) \|\rho - \widetilde{\rho}\|_s, \tag{15}$$

*where* $L_s : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is continuous and monotone non-decreasing with respect to each of its arguments.*
*(b)* $F : [0,T_0] \times H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R})$, $F(t,0) = 0$ *and satisfies the following Lipschtz condition:*

$$\|F(t,\rho) - F(t,\widetilde{\rho})\|_s \leq M_s\left(\|\rho\|_s, \|\widetilde{\rho}\|_s\right) \|\rho - \widetilde{\rho}\|_s. \tag{16}$$

*Then (4) is locally well-posed in the sense described in Section 1.*

**Proof.** Since this proof already appeared in [4], we present only a brief sketch. We take $X = L^2(\mathbb{R})$, $Y = H^s(\mathbb{R})$ and $S = (1-\partial_x^2)^{\frac{s}{2}} = J^s$. The first step is to prove begin that $A(\rho)$, $\rho \in H^s(\mathbb{R})$ is quasi-$m$-accretive in $L^2(\mathbb{R})$ (see [10]). Since we are dealing with Hilbert spaces, it is enough to show that there exists a $\beta > 0$

(a) $(A(\rho) f \,|f)_0 \geq -\beta \|f\|_0^2$ for all $f \in \mathcal{D}(A(\rho)) = H^1(\mathbb{R})$;

(b) $(A(\rho) + \lambda)$ is onto $L^2(\mathbb{R})$ for some (equivalently all) $\lambda > \beta$. (For details see [4].)

This implies that $A(\rho)$ generates a $C^0$-semigroup, $U(s) = \exp(-sA(\rho))$, $s \in [0,\infty)$, such that

$$\|U(s)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \exp(\beta s). \tag{17}$$

Next we must show that

$$\begin{gathered} SA(\rho) S^{-1} = A(\rho) + B(\rho), \\ B(\rho) \in \mathcal{B}(L^2(\mathbb{R})), \\ \|B(\rho)\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq q(\rho), \end{gathered} \tag{18}$$

where $q(\cdot)$ is a nondecreasing function. It suffices to show that

$$\begin{gathered} [S, A(\rho)] = SA(\rho) - A(\rho) S \in \mathcal{B}(H^s(\mathbb{R}), L^2(\mathbb{R})), \\ \|[S, A(\rho)]\|_{\mathcal{B}(H^s(\mathbb{R}), L^2(\mathbb{R}))} \leq q(\rho), \end{gathered} \tag{19}$$

for some nondecreasing function $q(\cdot)$. Let $f \in H^s(\mathbb{R})$ .Let $f \in H^s(\mathbb{R})$ and

$$\Theta(\rho) = J^{-2} \nabla P(\rho). \tag{20}$$

Then,

$$\begin{aligned} [S, A(\rho)] &= J^s \partial_x (f \Theta(\rho)) - \partial_x ((J^s f) \Theta(\rho)) \qquad (21) \\ &= [J^s, \partial_x \Theta(\rho)] f + [J^s, \Theta(\rho)] \partial_x f. \end{aligned}$$

Combining the fact that $J^{-2}\,\partial_x \in \mathcal{B}(H^s(\mathbb{R}), H^{s+1}(\mathbb{R}))$ with the assumptions on $p$ , we conclude that $\Theta(\rho) \in H^{s+1}(\mathbb{R})$ and

$$\|\Theta(\rho)\|_{s+1} \leq \left\| J^{-2}\,\partial_x \right\| L_s(\|\rho\|, 0) \, \|\rho\|_s. \tag{22}$$

The desired estimate then follows from Lemma (A4) of [11]. Finally, the assumptions on $P$ and $F$ imply the Lipchtz conditions required by Kato's theory.

This finishes the proof.

□

**Remark 2** *It should be noted that the previous result holds in* $H^s(\mathbb{R}^n)$*, with* $s > \frac{n}{2} + 1$ *and the obvious changes needed to accommodate the general case. It is also true in* $H_0^s(\Omega)$, $\Omega \subset \mathbb{R}^n$ *is a domain with a smooth boundary, and in* $H^s(S^1)$*. This is due to the fact that all the crucial estimates used here hold in theses cases. (See Lemma A4 of [11] and Appendix B of [6].*

## 3 Parabolic regularization and Global Existence.

Theorem 1 holds if we regularize the equation. More precisely, the same result holds for the Cauchy Problem

$$\left\{ \begin{array}{c} \partial_t \rho^{(\mu)} = \partial_x \left( \rho^{(\mu)} \left(1-\partial_x^2\right)^{-1} \partial_x P\left(\rho^{(\mu)}\right)\right) + F\left(t, \rho^{(\mu)}\right) + \mu \partial_x^2 \rho^{(\mu)} \\ \rho^{(\mu)}(0) = \rho_0. \end{array} \right. \tag{23}$$

for each $\mu \geq 0$. However, due to the smoothing properties os the $C^0$ semigroup $U_\mu(t) = \exp\left(\mu t \partial_x^2\right)$, $\mu > 0$, one is able to solve the problem applying Banach's Fixed Point Theorem and Gronwall's inequality to the integral equation

$$\rho^{(\mu)}(t) = U_\mu(t)\,\rho_0 + \int_0^t U_\mu(t-t')\left[A\left(\rho^{(\mu)}(t')\right)\rho^{(\mu)}(t') + F\left(t', \rho^{(\mu)}(t)\right)\right] dt'. \tag{24}$$

In fact we have a better result in this case.

**Theorem 3** *Assume that* $\mu > 0$ *and that* $P$ *and* $F$ *satisfy (15) and (16) for some fixed* $s > 1/2$*. Then (23) is locally well-posed in* $H^s(\mathbb{R})$*. Moreover, if* $(0, T_\mu]$ *is an interval of existence, then* $\rho^{(\mu)} \in C\left((0, T_\mu]; H^\infty(\mathbb{R})\right)$*, where* $H^\infty(\mathbb{R}) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R})$ *provided with its natural Frechet space topology.*

**Proof.** Local well-posedness is a routine application of Banach´s Fixed Point Theorem and Gronwall´s inequality, combined with the estimate (with $\lambda = 1$),

$$\left\| U_\mu(t)\,\phi \right\|_{r+\lambda} \leq K_\lambda \left[ 1 + \left( \frac{1}{2\mu t} \right)^\lambda \right]^{1/2} \left\| \phi \right\|_r, \tag{25}$$

where $K_\lambda > 0$ depends only on $\lambda$ and holds for all $\phi \in H^r(\mathbb{R})$, $r \in \mathbb{R}$, $\lambda \geq 0$, and $\mu$, $t > 0$. (See [5] and [6] for example.) An easy bootstrapping argument combining (24) and (25) (with $\lambda$ fixed in $(1,2)$), implies the last statement.

□

In order to take the limit as $\mu$ tends to zero, one must show that it is possible to choose intervals of existence independent of $\mu$. We have:

**Lemma 4** *Assume that $\mu$, $P$ and $F$ are as in Theorem 3. Then there exists $T = T(\phi)$, independent of $\mu > 0$ such that all solutions $\rho^{(\mu)}$ can be extended, if necessary, to $[0,T]$.*

**Proof.** Since $\rho^{(\mu)} \in C\left((0,T_\mu];H^\infty(\mathbb{R})\right)$, the following computations are entirely rigorous:

$$\begin{aligned}&\partial_t \left\|\rho^{(\mu)}\right\|_s^2 = 2\left(\rho^{(\mu)}\left|\partial_t\rho^{\mu}\right.\right)_s = \\ &= 2\mu\left(\rho^{(\mu)}\left|\partial_x^2\rho^{(\mu)}\right.\right)_s + 2\left(\rho^{(\mu)}\left|A\left(\rho^{(\mu)}\right)\rho^{(\mu)}\right.\right)_s + 2\left(\rho^{(\mu)}\left|F\left(t,\rho^{(\mu)}\right)\right.\right)_s.\end{aligned} \tag{26}$$

Integrating by parts and using the assumptions on $P$ and $F$ we obtain

$$2\mu\left(\rho^{(\mu)}\left|\partial_x^2\rho^{(\mu)}\right.\right)_s = -2\mu\left\|\partial_x\rho^{(\mu)}\right\|_s^2 \leq 0, \tag{27}$$

$$\left|\left(\rho^{(\mu)}\left|F\left(t,\rho^{(\mu)}\right)\right.\right)_s\right| \leq M_s\left(\left\|\rho^{(\mu)}\right\|_s, 0\right)\left\|\rho^{(\mu)}\right\|_s^2, \tag{28}$$

and

$$\begin{aligned}&\left|\left(\rho^{(\mu)}\left|A\left(\rho^{(\mu)}\right)\rho^{(\mu)}\right.\right)_s\right| = \\ &= \left|\left(\left(\rho^{(\mu)}\right)^2\left|\partial_x^2 J^{-2}P\left(\rho^{(\mu)}\right)\right.\right)_s\right| \leq L_s\left(\left\|\rho^{(\mu)}\right\|_s\right)\left\|\rho^{(\mu)}\right\|_s^3.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}&\partial_t\left\|\rho^{(\mu)}\right\|_s^2 \leq \\ &\leq M_s\left(\left\|\rho^{(\mu)}\right\|_s, 0\right)\left\|\rho^{(\mu)}\right\|_s^2 + L_s\left(\left\|\rho^{(\mu)}\right\|_s\right)\left\|\rho^{(\mu)}\right\|_s^3 = G\left(\left\|\rho^{(\mu)}\right\|_s^2\right).\end{aligned} \tag{30}$$

Let $h(t)$ be the maximal solution of the problem

$$\left\{\begin{array}{c}\partial_t h(t) = G(h(t)) \\ h(0) = \left\|\rho_0\right\|_s^2.\end{array}\right. \tag{31}$$

Then

$$\left\|\rho^{(\mu)}(t)\right\|_s^2 \leq h(t), \tag{32}$$

whenever both sides are defined. This finishes the proof since $h$ does not depend on $\mu$.

□

Arguing as in the proof of Theorem 2.1 of [6], we are able to show existence and uniqueness of solutions in $AC([0,T]; H^{s-1}(\mathbb{R})) \cap L^\infty([0,T]; H^s)$. Due to technical reasons lack of invariance under certain changes of variables), so far we were unable to prove that the solution we obtain in this way actually belongs to $C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$, $s > 1/2$. We do believe this to be true. However, combining this remark with the results of the previous section we see that the solution constructed using parabolic regularization must coincide with the obtained using Kato´s theorem in case $s > 3/2$.

Kato's theory is extremely convenient to establish local well-posedness because continuous dependence follows automatically from the assumptions[3]. As far as we know, no such result exists in the parabolic regularization context. Although we can obtain existence and uniqueness, in order to prove continuous dependence (in the case $\mu = 0$), one must resort to the Bona-Smith approximations ([1]; see also [5]). However, parabolic regularization may be extremely useful in establishing the *a priori* estimates that imply global well-posedness. Let's illustrate the point by deriving a rather amusing inequality.

From now on, we assume, for simplicity's sake, that $F(t,\rho) = 0$. All the following arguments can be easily modified, under convenient assumptions, for the case of nonzero $F$. In view of 3, the following computation is entirely rigorous if $\mu > 0$[4]. To simplify the notation write $\rho^{(\mu)} = \rho$

$$\begin{aligned} \partial_t \|\rho(t)\|_0^2 &= 2\left(\rho(t)\,|\partial_t \rho(t)\right)_0 = \\ &= 2\mu\left(\rho(t)\left|\partial_x^2 \rho(t)\right.\right)_0 + 2\left(\rho(t)\left|\partial_x(\rho(t))\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho)\right.\right)_0. \end{aligned} \tag{33}$$

[3]Continuous dependence can be a formidable problem in the case of nonlinear evolution equations.

[4]Recall that we are interested in real valued solutions.

Integration by parts shows that $\mu\left(\rho(t)\,|\partial_x^2\rho(t)\right) \le -\mu\|\partial_x\rho(t)\|_s^2 \le 0$, so that we can discard the first term in the third member of (33). But then,

$$\begin{aligned}
&\partial_t\|\rho(t)\|_0^2 = 2\left(\rho(t)\,|\partial_t\rho(t)\right)_0 \\
&\le 2\left(\rho(t)\left|\partial_x(\rho(t))\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho(t))\right.\right)_0 = -\left(\partial_x\rho(t)^2\left|\left(1-\partial_x^2\right)^{-1}\partial_x P(\rho(t))\right.\right)_0 \\
&= \left(\rho(t)^2\left|\left(1-\partial_x^2\right)^{-1}\partial_x^2 P(\rho(t))\right.\right)_0 \\
&= -\left(\rho(t)^2\,|P(\rho(t))\right)_0 + \left(\rho(t)^2\left|\left(1-\partial_x^2\right)^{-1}P(\rho(t))\right.\right)_0 \qquad (34)\\
&\le -\left(\rho(t)^2\,|P(\rho(t))\right)_0 + \left\|\rho(t)^2\right\|_0\left\|\left(1-\partial_x^2\right)^{-1}P(\rho(t))\right\|_0 \\
&\le -\left(\rho(t)^2\,|P(\rho(t))\right)_0 + \frac{\left\|\rho(t)^2\right\|_0^2 + \left\|\left(1-\partial_x^2\right)^{-1}P(\rho(t))\right\|_0^2}{2}.
\end{aligned}$$

But $\left\|\left(1-\partial_x^2\right)^{-1}P(\rho(t))\right\|_0 \le \|P(\rho(t))\|_0$ so that (34) implies

$$\partial_t\|\rho(t)\|_0^2 \le \frac{1}{2}\left\|\rho(t)^2 - P(\rho(t))\right\|_0^2. \qquad (35)$$

Thus if $P(\rho) = \rho^2$, it follows that $\partial_t\|\rho(t)\|_0^2 \le 0$ which, in turn implies that $\|\rho(t)\|_0^2 \le \|\rho_0\|_0^2$. Moreover this argument shows that $P(\rho) = \rho^2$ is a natural choice for the function $P(\rho)$. In fact one can show that all Sobolev norms remain finite as $t \to \infty$. Combining this with the local well posedness result of the previous section, we see that if $P(\rho) = \rho^2$ then (25) is globally well-posed in $H^s(\mathbb{R})$ for all $s > 3/2$. However, one can do better than that.

**Theorem 5** *Let $P(\rho) = \rho^{2k}$, $k = 1, 2, 3....$ Then (25) is globally well-posed for all $s > 3/2$ and $\mu \ge 0$*[5].

**Proof.** Consider $\mu > 0$, first. Note that the case $k = 1$ is the subject of 8, assume that $k > 2$. Since $\rho \in C\left((0,T], H^\infty(\mathbb{R})\right)$, we have

$$\begin{aligned}
&\partial_t\|\rho(t)\|_s^2 = \partial_t\|J^s\rho(t)\|_0^2 \qquad (36)\\
&= 2\mu\left(J^s\rho(t)\,|\partial_x^2 J^s\rho(t)\right)_0 + 2\left(J^s\rho(t)\,|J^s\partial_x\rho\left(J^{-2}\partial_x P(\rho)\right)\right)_0.
\end{aligned}$$

[5]Recall that we are now assuming that $F(t,\rho) = 0$.

Now, $\left(J^s\rho(t)\,|\partial_x^2 J^s\rho(t)\right) = -\|\partial_x J^s\rho(t)\|_0^2 \le 0$ so that

$$\begin{aligned}\partial_t \|\rho(t)\|_s^2 &\le 2\left(J^s\rho(t)\,\middle|\,J^s\partial_x\rho\left(J^{-2}\partial_x P(\rho)\right)\right)_0 \\ &= -2\left(J^s\partial_x\rho(t)\,\middle|\,J^s\rho\left(J^{-2}\partial_x P(\rho)\right)\right)_0 \\ &= -2\left(J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s+1}\rho\left(J^{-2}\partial_x P(\rho)\right)\right)_0 = A + B,\end{aligned} \tag{37}$$

where

$$A = -2\left(J^{s-1}\partial_x\rho(t)\,\middle|\,\left[J^{s+1},\left(J^{-2}\partial_x P(\rho)\right)\right]\rho\right)_0, \tag{38}$$

and

$$B = -2\left(\left\{J^{-2}\partial_x P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}J^2\rho(t)\right)_0. \tag{39}$$

Consider $B$ first:

$$B = -2\left(\left\{J^{-2}\partial_x P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}\left(\rho(t)-\partial_x^2\rho(t)\right)\right)_0 = B_1 + B_2, \tag{40}$$

where

$$B_1 = -2\left(\left\{J^{-2}\partial_x P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}\rho(t)\right)_0, \tag{41}$$

and

$$\begin{aligned}B_2 &= 2\left(\left\{J^{-2}\partial_x P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}\partial_x^2\rho(t)\right)_0 \\ &= -2\left(\left\{J^{-2}\partial_x^2 P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}\partial_x\rho(t)\right)_0 \\ &= -2\left(\left\{-P(\rho(t)) + \left(1-\partial_x^2\right)^{-1}P(\rho(t))\right\}J^{s-1}\partial_x\rho(t)\,\middle|\,J^{s-1}\partial_x\rho(t)\right)_0.\end{aligned} \tag{42}$$

Since $J^{-2}\partial_x \in \mathcal{B}\left(L^1(\mathbb{R}), L^\infty(\mathbb{R})\right)$ we have

$$|B_1| \le \|P(\rho(t))\|_{L^1}\int_{\mathbb{R}}\left|J^{s-1}\partial_x\rho(t)\right|\cdot\left|J^{s-1}\rho(t)\right|dx \le \|P(\rho_0)\|_{L^1}\|\rho(t)\|_s^2. \tag{43}$$

Next, in view of Lemmas 7 and 8,

$$|B_2| \le 2\left(\left(\|\rho_0\|_1^2\exp\left(C\|P(\rho_0)\|_{L^1}t\right)\right)^{2k} + \|P(\rho_0)\|_{L^1}\right)\|\rho(t)\|_s^2. \tag{44}$$

It follows that

$$|B| \le K(\rho_0, t)\|\rho(t)\|_s^2, \tag{45}$$

where $K(\rho_0, t)$ is defined and finite for all $t \in [0, \infty)$.

Nest we turn to $A$. We have

$$|A| \leq 2\left\|\rho\left(t\right)\right\|_{s}\left\|\left[J^{s+1},v\right]\rho\left(\left(t\right)\right)\right\|_{0}, \tag{46}$$

where $v=-J^{-2}\partial_{x}P\left(\rho\right)$. To estimate the commutator, we note that

$$\left\|\left[J^{s+1},v\right]\rho\left(\left(t\right)\right)\right\|_{0} \leq C\left\{\left\|v_{x}\right\|_{L^{\infty}}\left\|J^{s}\rho\right\|_{0}+\left\|J^{s+1}v\right\|_{0}\left\|\rho\right\|_{L^{\infty}}\right\}. \tag{47}$$

In view of the results in the Appendix, we have,

$$\begin{aligned} &\left\|v_{x}\right\|_{L^{\infty}}=\left\|P\left(\rho\right)-\left(1-\partial_{x}^{2}\right)^{-1}P\left(\rho\right)\right\|_{L^{\infty}} \leq \\ &\leq\left\|P\left(\rho\right)\right\|_{L^{\infty}}+\left\|P\left(\rho\right)\right\|_{L^{1}} \leq \left(\left\|\rho_{0}\right\|_{1}^{2}\exp\left(C\left\|P\left(\rho_{0}\right)\right\|_{L^{1}}t\right)\right)^{2k}+\left\|P\left(\rho_{0}\right)\right\|_{L^{1}}. \end{aligned} \tag{48}$$

Next,

$$\left\|J^{s+1}v\left(t\right)\right\|_{0}=\left\|J^{s+1}J^{-2}\partial_{x}P\left(\rho\left(t\right)\right)\right\|_{0} \leq \left\|P\left(\rho\left(t\right)\right)\right\|_{s}. \tag{49}$$

Since $\rho\in C\left(\left(0,T\right],H^{\infty}\left(\mathbb{R}\right)\right)$, and $P\left(\rho\right)=\rho^{2k}$, it follows that

$$\begin{aligned} &\left\|P\left(\rho\right)\right\|_{s}^{2}=\left(J^{s}\rho^{2k}\left(t\right)\middle|J^{s}\rho^{2k}\left(t\right)\right)_{0}=\left(\rho^{2k}\left(t\right)\middle|J^{2s}\rho^{2k}t\left(t\right)\right)_{0} \leq \\ &\leq\left\|\rho^{2k-2}\right\|_{L^{\infty}}\left(\rho^{2}\left(t\right)\middle|J^{2s}\rho^{2k}\left(t\right)\right)_{0}=\left\|\rho^{2k-2}\left(t\right)\right\|_{L^{\infty}}\left(J^{2s}\rho^{2}\left(t\right)\middle|\rho^{2k}\left(t\right)\right)_{0} \\ &\leq\left\|\rho^{2k-2}\left(t\right)\right\|_{L^{\infty}}^{2}\left(J^{s}\rho^{2}\left(t\right)\middle|J^{s}\rho^{2}\left(t\right)\right)_{0} \leq \left\|\rho\left(t\right)\right\|_{1}^{4k-4}\left\|\rho^{2}\left(t\right)\right\|_{s} \\ &\leq\left\|\rho\left(t\right)\right\|_{1}^{4k-4}\left\|\rho\left(t\right)\right\|_{s}^{2} \leq \left(\left\|\rho_{0}\right\|_{1}^{2}\exp\left(C\left\|P\left(\rho_{0}\right)\right\|_{L^{1}}t\right)\right)^{4k-4}\left\|\rho\left(t\right)\right\|_{s}^{2}. \end{aligned} \tag{50}$$

Therefore, if $\mu>0$ we have

$$\partial_{t}\left\|\rho\left(t\right)\right\|_{s}^{2} \leq \widetilde{K}\left(\rho_{0},t\right)\left\|\rho\left(t\right)\right\|_{s}^{2}, \forall t\in\left[0,T\right]. \tag{51}$$

Gronwall's inequality implies that the problem is globally well-posed if $\mu>0$. Now, the limiting argument employed in Theorem 5 can be used in any interval $\left[0,T\right], T>0$, so that we get global well-posedness for $\mu=0$. This finishes the proof.

□

# 4 Preservation of Positivity.

Let $\rho\left(t,x\right)$ be the global solution constructed in the previous section, with $P\left(\rho\right)=\rho^{2k}$. In this section we prove that the flow associated to (2) is positivity preserving. We have

**Theorem 6** *Let $\rho_0 \in H^s(\mathbb{R})$, $s > 3/2$, be such that $\rho_0(x) \geq 0$ for all $x \in \mathbb{R}$. Then the corresponding solution satisfies*

$$\rho(t,x) \geq 0, \ \forall t \geq 0, \ x \in \mathbb{R}. \tag{52}$$

**Proof.** Let $\phi(t,y)$ be the solution of the following initial value problem

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t,y) &= v(t,\phi(t,y)), \\ \phi(0,y) &= y. \end{aligned} \tag{53}$$

where

$$v(t,\phi(t,y)) = -\partial_x \left(1-\partial_x^2\right)^{-1} P(t,\rho(t,\phi(t,y))) \tag{54}$$

Define $S(t,y) = \rho(t,\phi(t,y))$. Then,

$$\frac{dS}{dt} = \rho_t + \rho_x \frac{\partial \phi}{\partial t} = -(\rho v)_x + \rho_x v = -\rho v_x = v_x S(t). \tag{55}$$

Moreover

$$S(0,y) = \rho(0,y) = \rho_0(y). \tag{56}$$

It follows that

$$S(t,y) = \rho_0(y) \exp\left(-\int_0^t v_x(\tau,\phi(\tau,y))\, d\tau\right), \tag{57}$$

so that $S(t,y) = \rho(t,\phi(t,y)) \geq 0$ if $\rho_0(y) \geq 0$. To finish the proof it remains to show that $y \in \mathbb{R} \longmapsto \phi(t,y)$ is onto. But this follows from the inequality[6]

$$|\phi(t,y) - y| \leq \int_0^t |v(\tau,\phi(\tau,y))|\, d\tau \leq \int_0^t \|P(\rho(\tau,\phi(\tau,y)))\|_{L^1} d\tau, \tag{58}$$

and Lemma 7.

□

# 5 Appendix: Some Technical Estimates.

Note that if $\rho \in H^s(\mathbb{R})$, $s > 1/2$, then $P(\rho) = \rho^{2k} = \rho^{2k-2}\rho^2 \in L^1(\mathbb{R})$, $k = 1,2,3....$ Indeed, $H^s(\mathbb{R})$, $s > 1/2$ is a Banach algebra continuously and

[6]Note that $\partial_x\left(1-\partial_x^2\right)^{-1} \in \mathcal{B}\left(L^1(\mathbb{R}), L^\infty(\mathbb{R})\right)$.

densely embedded in $L^{\infty}(R)$ and $\rho^2 \in L^1(\mathbb{R})$. In this appendix we will estimate the $L^1(\mathbb{R})$ and $L^{\infty}(\mathbb{R})$ norms of $P(\rho((t)))$, where $\rho(t)$ is a solution of (4) on some interval $[0,T]^7$ (as the one constructed in Section 2). To simplify the notation we will write sometimes $\rho(t) = \rho$ and, $v(\rho) = -J^{-2}\partial_x P(\rho)$ as in the original problem (2).

**Lemma 7** *Let $\rho(t)$ be as above. Then for all $(t,x) \in [0,\infty) \times \mathbb{R}$.*

$$\|P(\rho(t))\|_{L^1} \leq \|P(\rho_0)\|_{L^1}, \ \forall t \in [0,T]. \tag{59}$$

**Proof.** We have

$$\begin{aligned} \partial_t \|P(\rho(t))\|_{L^1} &= \partial_t \int_{\mathbb{R}} \rho^{2k}(t,x)\,dx \\ &= \partial_t \left\|\rho^k\right\|_0^2 = 2\left(\rho^k \left|\partial_t \rho^k\right.\right)_0 = 2k\left(\rho^k \left|\rho^{k-1}\partial_t \rho\right.\right)_0. \end{aligned} \tag{60}$$

It follows that

$$\partial_t \|P(\rho(t))\|_{L^1} = -2k\left(\rho^{2k-1}\left|\partial_x(\rho v)\right.\right)_0 + 2k\mu\left(\rho^{2k-1}\left|\partial_x^2 \rho\right.\right)_0. \tag{61}$$

But

$$2\mu\left(\rho^{2k-1}\left|\partial_x^2\rho\right.\right)_0 = -2\int_{\mathbb{R}} \rho^{2k-2}(\partial_x \rho)^2\,dx \leq 0, \tag{62}$$

so that we must concentrate on estimating the first term on the RHS of (61). We integrate by parts several times to obtain

$$-2k\left(\rho^{2k-1}\left|\partial_x(\rho v)\right.\right)_0 = 2k(2k-1)\left(\rho^{2k-2}\partial_x\rho\left|\rho v\right.\right) = -2k\left(\rho^{2k}\left|\partial_x v\right.\right). \tag{63}$$

Now,

$$\partial_x v = -\left(1-\partial_x^2\right)^{-1}\partial_x^2 P(\rho) = P(\rho) - \left(1-\partial_x^2\right)^{-1} P(\rho). \tag{64}$$

It follows that

$$-2k\left(\rho^{2k-1}\left|\partial_x(\rho v)\right.\right)_0 = -2k\left(\rho^{2k}\left|P(\rho) - \left(1-\partial_x^2\right)^{-1}P(\rho)\right.\right)_0. \tag{65}$$

Since $P(\rho) = \rho^{2k}$ we have

$$\left(\rho^{2k}\left|\left(1-\partial_x^2\right)^{-1}\rho^{2k}\right.\right)_0 = \left\|\left(1-\partial_x^2\right)^{-1/2}\rho^{2k}\right\|_0^2 \leq \left\|\rho^{2k}\right\|_0^2. \tag{66}$$

[7] With $F(t,x) = 0$.

Therefore,

$$-2k\left(\rho^{2k-1}\left|\partial_x(\rho v)\right)_0 \leq 0. \tag{67}$$

so that

$$\partial_t \left\|P(\rho(t))\right\|_{L^1} \leq 0, \tag{68}$$

and we get the following a priori estimate

$$\left\|P(\rho(t))\right\|_{L^1} \leq \left\|P(\rho_0)\right\|_{L^1}, \tag{69}$$

for all $t$ where $\rho(t)$ is defined.

□

To handle the $L^\infty$norm of $P((\rho(t)))$, we first obtain an estimate for the $H^1(\mathbb{R})$ of $\rho(t)$.

**Lemma 8** *Let $\rho(t)$ be as in Lemma 7 and $s_0 > 1/2$. Then,*

$$\left\|\rho(t)\right\|_1 \leq \left\|\rho_0\right\|_1^2 \exp\left(C\left\|P(\rho_0)\right\|_{L^1} t\right), \forall t \in [0,T]. \tag{70}$$

**Proof.** Note that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \left(\rho^2 + \rho_x^2\right) dx = 2\int_{\mathbb{R}} \left(\rho\partial_t\rho + \rho_x\partial_t\rho_x\right) dx \\
&= 2\int_{\mathbb{R}} \left[\rho\left(-\partial_x(\rho v) + \mu\partial_x^2\rho\right) + \rho_x\left(-\partial_x^2(\rho v) + \mu\partial_x^2\rho_x\right)\right] dx \\
&= -2\int_{\mathbb{R}} \left[\rho\partial_x(\rho v) + \rho_x\partial_x^2(\rho v)\right] dx + 2\mu\int_{\mathbb{R}} \left[\rho\partial_x^2\rho + \partial_x^2\rho_x\right].
\end{aligned} \tag{71}$$

Writing $v(\rho) = -J^{-2}\partial_x P(\rho)$, using (64) and integrating by parts several times, we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \left(\rho^2 + \rho_x^2\right) dx + 3\int_{\mathbb{R}} P(\rho)\rho_x^2 dx + 2\int_{\mathbb{R}} \rho P'(\rho)\rho_x^2 dx = \\
&= 3\int_{\mathbb{R}} \rho_x^2\left(1 - \partial_x^2\right)^{-1} P(\rho)\, dx + 2\mu\int_{\mathbb{R}} \left[\rho\partial_x^2\rho + \partial_x^2\rho_x\right] dx.
\end{aligned} \tag{72}$$

Since $P(\rho) = \rho^{2k}$ we have $P(\rho) \geq 0$ and $\rho P'(\rho) \geq 0$. Integration by parts shows that $2\mu\int_{\mathbb{R}} \left[\rho\partial_x^2\rho + \partial_x^2\rho_x\right] dx \leq 0$. Combining these facts with (72) we obtain

$$\partial_t \int_{\mathbb{R}} \left(\rho^2 + \rho_x^2\right) dx \leq 3\int_{\mathbb{R}} \rho_x^2\left(1 - \partial_x^2\right)^{-1} P(\rho)\, dx. \tag{73}$$

Since

$$\left(\left(1-\partial_x^2\right)^{-1} f\right)(x)=\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} f(y)\,dy, \tag{74}$$

Young's inequality for the convolution (see [4] or [13]) implies

$$\left\|\left(1-\partial_x^2\right)^{-1} P(\rho)\right\|_{L^\infty} \leq \frac{1}{2}\left\|e^{-|\cdot|}\right\|_{L^\infty}\|P(\rho)\|_{L^1}. \tag{75}$$

In view of Lemma 7 it follows that

$$\partial_t \int_{\mathbb{R}}\left(\rho^2+\rho_x^2\right) dx \leq \frac{3}{2}\|P(\rho)\|_{L^1}\int_{\mathbb{R}}\left(\rho_x\right)^2 dx \leq C\left\|P\left(\rho_0\right)\right\|_{L^1}\|\rho\|_1^2. \tag{76}$$

Gronwall's inequality then implies that

$$\|\rho(t)\|_1^2 \leq \|\rho_0\|_1^2 \exp\left(C\left\|P\left(\rho_0\right)\right\|_{L^1} t\right),\ \forall t\in[0,T]. \tag{77}$$

□

**Corollary 9** *Let $P(\rho)=\rho^{2k}$ as before. Then*

$$\|P(\rho(t))\|_{L^\infty} \leq \left(\|\rho_0\|_1^2 \exp\left(C\left\|P\left(\rho_0\right)\right\|_{L^1} t\right)\right)^{2k},\ \forall t\in[0,T]. \tag{78}$$

**Proof.** Since $\rho(t)\in H^s(\mathbb{R})$, $s>3/2$ and $H^s(\mathbb{R})\hookrightarrow H^1(\mathbb{R})$ are Banach algebras,

$$\|P(\rho(t))\|_{L^\infty}=\left\|\rho(t)^{2k}\right\|_{L^\infty} \leq \left\|\rho(t)^{2k}\right\|_1 \leq \|\rho(t)\|_1^{2k}. \tag{79}$$

The corollary follows combining this inequality with the previous lemma.

□

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