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LOCAL ENERGY DECAY OF SOLUTIONS TO THE WAVE EQUATION FOR NONTRAPPING METRICS

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We will begin by recalling Vainberg's results [5], [6] on the local energy decay of solutions to the wave equation. Let $\mathcal{O} \subset \mathbf{R}^n$ be a bounded domain with a C^∞-smooth boundary and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let g be a Riemannian metric in Ω of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}),$$

such that $g_{ij}(x) = \delta_{ij}$ outside some big compact, where δ_{ij} denotes the Kronecker symbol. Denote by Δ_g the positive Laplace-Beltrami operator on (Ω, g) and let ∇_g be the corresponding gradient. Consider the wave equation

$$\begin{cases} (\partial_t^2 + \Delta_g) u(t,x) = 0 \quad \text{in} \quad \mathbf{R} \times \Omega, \\ Bu(t,x) = 0 \quad \text{on} \quad \mathbf{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where B denotes either Dirichlet or Neumann boundary conditions. Define the local energy of the solution of (1) as follows

$$E_{loc}(t) = \int_\Omega \left(|\partial_t u(t,x)|^2 + |\nabla_g u(t,x)|^2 \right) \chi dx,$$

where $\chi \in C^\infty(\overline{\Omega})$, $\chi = 0$ outside some compact. Denote by G the selfadjoint realization of Δ_g on the Hilbert space $H = L^2(\Omega, d\mathrm{Vol}_g)$ with boundary conditions $Bf = 0$. Then the solution u of the equation (1) can be expressed by the formula

$$u = \cos\left(t\sqrt{G}\right) f_1 + \frac{\sin\left(t\sqrt{G}\right)}{\sqrt{G}} f_2. \tag{2}$$

We will say that the *Generalized Huyghens Principle* (GHP) for the equation (1) holds if for every function $\chi \in C^\infty(\overline{\Omega})$ of compact support, there exists $T > 0$ such that the distribution kernels of the operators $\chi \cos\left(T\sqrt{G}\right)\chi$ and $\chi\frac{\sin\left(T\sqrt{G}\right)}{\sqrt{G}}\chi$ are of class $C^\infty(\overline{\Omega}\times\overline{\Omega})$. It follows from the results of Melrose-Sjöstrand [3], [4] on propagation of C^∞ singularities that GHP holds for non-trapping metrics. Recall that the metric g is said *nontrapping* if every generalized geodesics (see [3], [4] for the definition) leaves any compact in a finite time. Vainberg [5], [6] proved the following

Theorem 1. *Assume GHP fulfilled. Then, for $t \gg 1$, the following estimates hold*

$$E_{loc}(t) \leq \begin{cases} Ce^{-\gamma t}E(0), & n \geq 3 \text{ odd}, \quad C, \gamma > 0, \\ Ct^{-2n}E(0), & n \geq 2 \text{ even}, \quad C > 0, \end{cases} \tag{3}$$

provided f_1 and f_2 are of compact support, where

$$E(0) = \int_\Omega \left(|f_2|^2 + |\nabla_g f_1|^2\right) dx.$$

Vainberg's proof is based on the fact that GHP implies that the cutoff resolvent

$$R_\chi(\lambda) = \chi(G - \lambda^2)^{-1}\chi : H \to H, \quad \operatorname{Im}\lambda < 0,$$

extends analytically (if $n \geq 3$ is odd) to the strip $\operatorname{Im}\lambda \leq \gamma_0$, $\gamma_0 > 0$, has a logarithmic singularity at $\lambda = 0$ if n is even, and satisfies in this region the estimate

$$\|R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq \frac{C}{|\lambda|}.$$

Given $s > 0$, define $E_s(t)$ by replacing χ by $\langle x\rangle^{-s}$ in the definition of $E_{loc}(t)$. There are several open problems concerning Vainberg's result stated above. The first one is to see if one still has estimates like (3) for more general Riemannian metrics as for example longe-range perturbations of the Euclidien metric, or for perturbations by long-range potentials. Another intersting question is to ask if GHP implies estimates analogues to (3) for $E_s(t)$ even in the setting described above. The purpose of this work is to study this kind of problems. We will state our main results in the case of the Scrödinger operator $\Delta_g + V(x)$, where the metric g is as above and V is a real-valued, non-negative, long-range smooth potential. Note that the same results still hold for more general Riemannian metrics on a large class of non-compact complete Riemannian manifolds. The proofs as well as more details are presented in the article [7].

Denote by G the self-adjoint realization of the operator $\Delta_g + V(x)$, where $V \in C^\infty(\overline{\Omega})$, $V(x) \geq 0$, $\forall x \in \Omega$. We will be interested in studing the following mixed problem

$$\begin{cases} (\partial_t^2 + \Delta_g + V(x))u(t,x) = 0 \quad \text{in} \quad \mathbf{R} \times \Omega, \\ Bu(t,x) = 0 \quad \text{on} \quad \mathbf{R} \times \partial\Omega, \\ u(0,x) = \varphi(G)f_1(x), \partial_t u(0,x) = \varphi(G)f_2(x), \quad x \in \Omega, \end{cases} \tag{4}$$

where $\varphi \in C^\infty(\mathbf{R})$, $\varphi(\sigma) = 0$ for $\sigma \leq a$, $\varphi(\sigma) = 1$ for $\sigma \geq a+1$, $a > 0$ being a large constant to be fixed later on. The potential V satisfies

$$\left| \frac{\partial^k V}{\partial r^k}(x) \right| \leq C\langle x\rangle^{-k-\delta}, \quad k = 0, 1, \tag{5}$$

with some constants $C > 0$, $0 < \delta \leq 1$, where r denotes the radial variable. Given a real $s > 0$, we define $E_s(t)$ for the solution to (4) as above, and $\widetilde{E}_{-s}(0)$ by replacing in the definition of $E(0)$ above dx by $\langle x\rangle^{2s}dx$ with f_1 and f_2 being as in (4). Our main result is the following

Theorem 2. *Assume (5) and GHP fulfilled. Then, if the parameter a above is taken large enough, we have, for* $t \gg 1$,

$$E_{(1+\delta)/2}(t) \leq O_\epsilon\left(t^{-2\delta+\epsilon}\right)\widetilde{E}_{-(1+\delta/2)}(0), \quad \forall 0 < \epsilon \ll 1. \tag{6}$$

In particular, if f_1 and f_2 are of compact support, we have

$$E_{loc}(t) \leq O_\epsilon\left(t^{-2\delta+\epsilon}\right)\widetilde{E}(0), \quad \forall 0 < \epsilon \ll 1. \tag{7}$$

Note that the estimates (6) and (7) hold for more general asymptotically Euclidean manifolds (see [7]).

The proof of Theorem 2 is based on the following properties of the resolvent of G.

Proposition 3. *Assume (5) and GHP fulfilled. Then, for every $s > 1/2$, there exist constants $C_0, C > 0$ such that for $z \geq C_0$, $0 < \varepsilon \leq 1$, we have*

$$\|\langle x\rangle^{-s}(G - z \pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Cz^{-1/2}. \tag{8}$$

Moreover, we have with $s = 1 + \delta/2$ and $\forall\, 0 < \mu < \delta$, $z \geq C_0$,

$$\|\langle x\rangle^{-s}(G - z \pm i\varepsilon z^{1/2})^{-2}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Cz^{-1}\varepsilon^{-1+\mu}. \tag{9}$$

Note that the parameter a above can be taken $a = \sqrt{C_0}$. It is worth also noticing that a better rate of the energy decay can be achieved if one has information for higher negative powers of the resolvent. More precisely, we have the following

Proposition 4. *Suppose that there exist $s > 1/2$, an integer $m \geq 0$, $0 < \mu \leq 1$, $C_0 > 0$, $C > 0$ so that for $z \geq C_0$ we have*

$$\|\langle x\rangle^{-s}(G - z \pm i\varepsilon)^{-k}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Cz^{-k/2}, \quad k = 1, ..., m+1, \tag{10}$$

$$\|\langle x\rangle^{-s}\left(G - z \pm i\varepsilon z^{1/2}\right)^{-m-2}\langle x\rangle^{-s}\|_{\mathcal{L}(H)} \leq Cz^{-(m+2)/2}\varepsilon^{-1+\mu}, \tag{11}$$

Then, we have (with $a = \sqrt{C_0}$), for $t \gg 1$,

$$E_{s-1/2}(t) \leq O\left(t^{-2m-2\mu}\right)\widetilde{E}_{-s}(0). \tag{12}$$

In particular, if f_1 and f_2 are of compact support, we have

$$E_{loc}(t) \leq O\left(t^{-2m-2\mu}\right)\widetilde{E}(0). \tag{13}$$

Remark. If (10) holds for every integer $k \geq 1$ with $s = s_k > 1/2$ and $C = C_k > 0$, then (13) holds with $O_N\left(t^{-N}\right)$, $\forall N \gg 1$, in the RHS.

The proofs of the above results can be found in [7].

References

[1] Cardoso, F.; Vodev, G., *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II*, Ann. H. Poincaré 3 (2002), 673-691.

[2] Cardoso, F.; Vodev, G., *High frequency resolvent estimates and energy decay of solutions to the wave equation*, Canadian Math. Bull., to appear.

[3] Melrose, R. B.,; Sjöstrand, J., *Singularities of boundary value problems. I*, Commun. Pure Appl. Math. 31 (1978), 593-617.

[4] Melrose, R. B.; Sjöstrand, J., *Singularities of boundary value problems. II*, Commun. Pure Appl. Math. 35 (1982), 129-168.

[5] Vainberg, B., *On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of nonstationary problems*, Russian Math. Surveys 30 (1975), 1-53.

[6] Vainberg, B., *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach, New York, 1988.

[7] Vodev, G., *Local energy decay of solutions to the wave equation for non-trapping metrics*, Ark. Math., to appear.

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