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FUNDAMENTAL SOLUTIONS FOR THE TRICOMI OPERATOR: POLE IN THE ELLIPTIC REGION

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Abstract

In this article we describe fundamental solutions for the classical Tricomi operator with pole in the elliptic region. The results presented here are slightly more general that the ones to be described in a forthcoming joint paper with Israel M. Gelfand.

1 Introduction

A fundamental solution for the Tricomi operator

$$\mathcal{T} = y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.1}$$

with pole at a point $(\xi, \eta) \in \mathbb{R}^2$ is a distribution $K(x, y; \xi, \eta)$ so that

$$\mathcal{T}_{x,y}K_{(\xi,\eta)}(x, y) = \delta(x - \xi, y - \eta), \tag{1.2}$$

where $\delta(x-\xi, y-\eta)$ is the Dirac distribution at (ξ, η). In this paper, we consider only the case where the pole (ξ, η) is located in the elliptic region of $\mathcal{T}$, that is, the half plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, y > 0\}$. In view of the invariance of the Tricomi operator relative to translations parallel to the x-axis, there is no loss of generality in restricting ourselves to the case $\xi = 0$ and $\eta = b > 0$.

The change of variables

$$x = x \quad \text{and} \quad s = \frac{2}{3}y^{3/2} \tag{1.3}$$

transforms $\mathcal{T}$ into

$$\mathcal{T} = \big(\frac{3s}{2}\big)^{2/3}\mathcal{T}_e, \tag{1.4}$$

where

$$\mathcal{T}_e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} + \frac{1}{3s}\frac{\partial}{\partial s}. \tag{1.5}$$

Note that if $y \geq 0$, both x and s are real and we call $\mathcal{T}_e$ the *reduced elliptic* Tricomi operator. However, if $y < 0$, then s is complex and we set $s = -2i(-y)^{3/2}/3$, with $i = \sqrt{-1}$

The change of variables

$$\ell = x + is, \quad \text{and} \quad m = x - is \tag{1.6}$$

or, in terms of x and y,

$$\ell = x + i\,\frac{2}{3}y^{3/2}, \quad \text{and} \quad m = x - i\,\frac{2}{3}y^{3/2} \tag{1.7}$$

transforms $\mathcal{T}$ into

$$\mathcal{T} = -2^{2/3}3^{2/3}(\ell - m)^{2/3}\mathcal{T}_h, \tag{1.8}$$

where

$$\mathcal{T}_h = \frac{\partial^2}{\partial \ell \partial m} - \frac{1/6}{\ell - m}\left(\frac{\partial}{\partial \ell} - \frac{\partial}{\partial m}\right). \tag{1.9}$$

Note that if $y > 0$ both ℓ and m are complex; if $y < 0$, then

$$\ell = x + \frac{2}{3}(-y)^{3/2}, \qquad m = x - \frac{2}{3}(-y)^{3/2}, \tag{1.10}$$

are the (real) *characteristic coordinates* of the Tricomi operator and we say that $\mathcal{T}_h$ is the *reduced hyperbolic* Tricomi operator.

Paying special attention to the operator $\mathcal{T}_h$ we look for homogeneous solutions to the equation $\mathcal{T}_h w = 0$. Any homogeneous function of degree λ, a complex number, in the variables ℓ and m can be written as

$$w_\lambda(\ell, m) = \ell^\lambda \phi(\zeta), \tag{1.11}$$

with ϕ is a function of a single variable $\zeta = m/\ell$. Direct substitution in (1.9) shows that $\phi(\zeta)$ is a solution of the hypergeometric equation

$$\zeta(1-\zeta)\phi''(\zeta) + \{c - (a+b+1)\zeta\}\phi'(\zeta) - ab\,\phi(\zeta) = 0, \tag{1.12}$$

with $a = -\lambda$, $b = 1/6$, and $c = 5/6 - \lambda$.

If a, b, and c are given complex numbers, with $c \neq 0, -1, -2, \cdots$, it is known [14] that the *hypergeometric series*

$$F(a,b,c;\zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \zeta^n, \tag{1.13}$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{1.14}$$

is a solution to (1.11). The series converges absolutely for $|\zeta| < 1$. In addition if $\mathrm{Re}(c-a-b) > 0$, then it also converges absolutely for $|\zeta| = 1$ and we have

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.15}$$

In the present paper we must consider not only the hypergeometric series but also its analytic continuation to the whole complex plane $\mathbb{C}$ minus the cut $[1, \infty)$. The reader should consult [7, 14] for a detailed discussion on the analytic continuation of the hypergeometric series. Here, we briefly discuss the important points needed for the comprehension of this paper.

As it is shown in [14], Barnes' contour integral defines a single-valued analytic function of ζ in the region $|\arg(-\zeta)| < \pi$, that is, the ζ-plane minus the positive real axis, and gives the analytic continuation of the series. Such a function which standard convention still denotes by $F(a,b,c;\zeta)$ is the *principal branch* of the analytic continuation generated by the hypergeometric series. If $a-b$ is not an integer or zero, then by conveniently choosing the contour of integration, one obtains from Barnes' integral the following representation for the analytic continuation:

$$\begin{aligned} F(a,b,c;\zeta) &= A(-\zeta)^{-a} F(a, 1-c+a, 1-b+a; \zeta^{-1}) \\ &+ B(-\zeta)^{-b} F(b, 1-c+b, 1-a+b; \zeta^{-1}), \end{aligned} \tag{1.16}$$

where $F(a, 1-c+a, 1-b+a; \zeta^{-1})$ and $F(b, 1-c+b, 1-a+b; \zeta^{-1})$ are hypergeometric series in ζ^{-1}, absolutely convergent for $|\zeta| > 1$, and $|\arg(-\zeta)| < \pi$ [14]. If $a-b$ is an integer or zero, then (1.16) must be modified because of the presence of double poles in the integrand of Barnes' integral. In the case $a = b$ the following result to be found in [7] holds:

$$F(a, a, c; \zeta) = (-\zeta)^{-a}[\log(-\zeta)U(\zeta) + V(\zeta)], \tag{1.17}$$

where $|\arg(-\zeta)| < \pi$, and $U(\zeta)$ and $V(\zeta)$ are power series in ζ^{-1} absolutely convergent for $|\zeta| > 1$. The explicit expressions of the series $U(\zeta)$ and $V(\zeta)$ can be found in [4, 7].

Going back to (1.11) the functions

$$w_\lambda(\ell, m) = \ell^\lambda F(-\lambda, \frac{1}{6}, \frac{5}{6} - \lambda; \frac{m}{\ell}) \tag{1.18}$$

are then homogeneous solutions of degree λ of $\mathcal{T}_h w = 0$. They were already considered in the papers [3, 4], where it was remarked that if one is looking for fundamental solutions to the Tricomi operator, then the appropriate degree of homogeneity must be $\lambda = -1/6$.

From now on, $\lambda = -1/6$ and we define

$$\mathcal{E}(\ell, m) = \ell^{-1/6} F(\frac{1}{6}, \frac{1}{6}, 1, \frac{m}{\ell}). \tag{1.19}$$

This is a homogeneous solution of degree $-1/6$ to $\mathcal{T}_h w = 0$. Let now (ℓ_0, m_0) be an arbitrary point in $\mathbb{C}^2$ and consider the change of variables

$$\ell = \frac{\ell' - m_0}{\ell' - \ell_0}, \qquad m = \frac{m' - m_0}{m' - \ell_0}.$$

Direct verification or by following Darboux in [5] one can show, after a relabeling of variables, that

$$\mathcal{E}(\ell, m; \ell_0, m_0) = (\ell - m_0)^{-1/6}(\ell_0 - m)^{-1/6} F(\frac{1}{6}, \frac{1}{6}, 1; \frac{(\ell - \ell_0)(m - m_0)}{(\ell - m_0)(m - \ell_0)}) \tag{1.20}$$

is also a solution to $\mathcal{T}_h w = 0$. This function was used in [4] to obtain fundamental solutions relative to an arbitrary point located in the hyperbolic region of $\mathcal{T}$.

We point out that the homogeneous functions (1.19) had already been considered by Germain and Bader in [10]. Also J. Leray in [11], after restricting himself to the hyperbolic region of $\mathcal{T}$ and localizing the problem, produced fundamental solutions in terms of the hypergeometric function $F(1/6, 1/6, 1; \zeta)$.

The plan of this paper is as follows. We show in Section 2 how to obtain from (1.20) fundamental solutions to the Tricomi operator with pole at $(0, b), b > 0$. In Section 3, by passing to the limit, we obtain the fundamental solutions of the joint paper [3].

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2 Fundamental solutions: pole at $(0, b), b > 0$

For b an arbitrary *positive* number, let $a = 2b^{3/2}/3$. Going back to formula (1.20) replace ℓ_0 by $-ia$, and m_0 by $-\ell_0$. From (1.7) it follows that

$$(\ell - m_0)(m - \ell_0) = \frac{1}{9}[9(x^2 + a^2) + 4y^3 - 12ay^{3/2}],$$

for $y > 0$, and from (1.10) it follows that

$$(\ell - m_0)(m - \ell_0) = \frac{1}{9}[9(x^2 + a^2) + 4y^3 + i12a(-y)^{3/2}],$$

for $y < 0$. Similarly, we have

$$(\ell - \ell_0)(m - m_0) = \frac{1}{9}[9(x^2 + a^2) + 4y^3 + 12ay^{3/2}]$$

if $y > 0$ and

$$(\ell - \ell_0)(m - m_0) = \frac{1}{9}[9(x^2 + a^2) + 4y^3 - i12a(-y)^{3/2}]$$

if $y < 0$. Thus, we derive from (1.20) the following solution to $\mathcal{T}w = 0$:

$$\mathcal{E}(x, y; 0, b) = (-v)^{-1/6} F(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{v}), \tag{2.1}$$

where

$$u(x, y) = \begin{cases} 9(x^2 + a^2) + 4y^3 + 12ay^{3/2} & \text{if } y \geq 0 \\ 9(x^2 + a^2) + 4y^3 - i12a(-y)^{3/2} & \text{if } y < 0 \end{cases} \tag{2.2}$$

and

$$v(x, y) = \begin{cases} 9(x^2 + a^2) + 4y^3 - 12ay^{3/2} & \text{if } y \geq 0 \\ 9(x^2 + a^2) + 4y^3 + i12a(-y)^{3/2} & \text{if } y < 0 \end{cases} \tag{2.3}$$

In order not to overload our notations we are keeping the same letter $\mathcal{E}$ to denote two different expressions, namely, (1.20) and (2.1). However the presence of the variables (ℓ, m) in one case and (x, y) in the other should dispel any confusion.

In formulas (2.2) and (2.3), $u(x, y)$ and $v(x, y)$ are complex conjugate of each other whenever $y < 0$. We wish to analyze the variation of their arguments in terms of x, y, and b (or a recalling that $a = 2b^{3/2}/3$). It suffices to consider $v(x, y)$. For a fixed b, v maps $\mathbb{R}^2$ into the half-plane $(\mathrm{Re}(v), \mathrm{Im}(v)), \mathrm{Im}(v) > 0$. Define in the hyperbolic region of $\mathcal{T}$ the following two sets:

$$D_{+,a} = \{(x, y) \in \mathbb{R}^2 : 9(x^2 + a^2) + 4y^3 > 0, y < 0\} \tag{2.4}$$

and

$$D_{-,a} = \{(x, y) \in \mathbb{R}^2 : 9(x^2 + a^2) + 4y^3 < 0, y < 0\}, \tag{2.5}$$

as shown in Figure 1. In the region $D_{+,a}$, both $\mathrm{Re}(v) = 9(x^2 + a^2) + 4y^3$ and

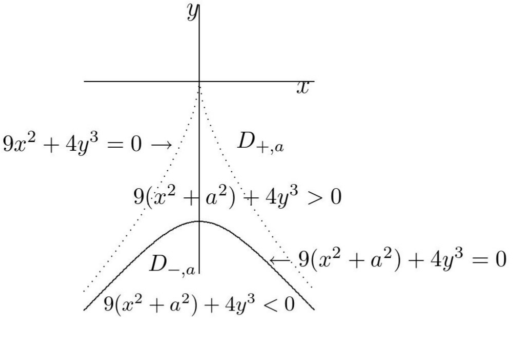

Figure 1

$\mathrm{Im}(v) = 12a(-y)^{3/2}$ are positive. If θ is such that

$$\tan(\theta) = \frac{12a(-y)^{3/2}}{9(x^2 + a^2) + 4y^3},$$

we may choose θ to vary from 0 to $\pi/2$. In the region $D_{-,a}$, $\mathrm{Re}(v) < 0$ and $\mathrm{Im}(v) > 0$, and we choose θ to vary from $\pi/2$ to π. We say that θ is the *principal argument* of v. In general the argument of v is

$$\arg(v) = \theta + 2k\pi, \; k \in \mathbb{Z}. \tag{2.6}$$

Note that if $(x, y) \in D_{+,a}$ and $y \to 0$, then $\theta \to 0$, hence $\arg(v) \to 2k\pi$, and we set $\arg(v) = 2k\pi$ whenever $y \leq 0$.

Since u is the complex conjugate of v we may write

$$\arg(u) = -\theta + 2k\pi, \ k \in \mathbb{Z}. \tag{2.7}$$

If $(x, y) \in D_{+,a}$ and $y \to 0$, then $\arg(u) \to 2k\pi$ and we set $\arg(u) = 2k\pi$ whenever $y \geq 0$. Next if

$$\arg(-v) = \arg(v) + \pi \qquad \text{and} \qquad \arg(\frac{u}{v}) = \arg(u) - \arg(v),$$

one obtains from (2.6) and (2.7) that

$$-\pi < \arg(-\frac{u}{v}) < \pi.$$

Therefore, no matter how one chooses the arguments of u and v, the hypergeometric function $F(1/6, 1/6, 1; u/v)$ in (2.1) represents the principal branch of the analytic continuation of its series.

Now, for $y > 0$, u and v defined by formulas (2.2) and (2.3) are real and ≥ 0. Moreover, $u/v \geq 1$ (to see this, write u and v in terms of the variables x and s, as in formula (2.13) below.) By virtue of (1.17), we may then write (2.1) as

$$\begin{aligned} \mathcal{E}(x, y; 0, b) &= (-v)^{-1/6} F(\frac{1}{6}, \frac{1}{6}, 1, \frac{u}{v}) \\ &= u^{-1/6}(-\frac{u}{v})^{1/6} F(\frac{1}{6}, \frac{1}{6}, 1, \frac{u}{v}) \\ &= u^{-1/6}\{\log(-\frac{u}{v})U(\frac{u}{v}) + V(\frac{u}{v})\}. \end{aligned} \tag{2.8}$$

By recalling that $\arg(u) = 2k\pi$ we rewrite the last formula as

$$\mathcal{E}(x, y; 0, b) = e^{-ik\pi/3}|u|^{-1/6}\{\log(-\frac{u}{v})U(\frac{u}{v}) + V(\frac{u}{v})\}. \tag{2.9}$$

We now state the following

Theorem 2.1. *For $0 \leq k \leq 5$ the distribution*

$$\mathcal{F}_k(x, y; 0, b) = -\frac{e^{ik\pi/3}}{2^{1/3}}\mathcal{E}(x, y; 0, b) \tag{2.10}$$

is a fundamental solution of $\mathcal{T}$ with pole at $(0, b)$.

Proof. 1. Consider first the case $k = 0$. We must show that

$$\langle \mathcal{T}\mathcal{F}, \phi\rangle = \langle \mathcal{F}, \mathcal{T}\phi\rangle = \phi(0, b), \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2). \tag{2.11}$$

It follows from our brief review on hypergeometric functions in Section 1 and particularly from formula (1.17) that $\mathcal{F}$ is locally integrable and everywhere smooth except at $(0, b)$ where it has a logarithmic singularity. Thus, we may write the second bracket in (2.11) as an integral:

$$\langle \mathcal{F}, \mathcal{T}\phi\rangle = -\frac{1}{2^{1/3}} \iint_{\mathbb{R}^2} \mathcal{E}(x, y; 0, b)\mathcal{T}\phi(x, y)\, dx\, dy. \tag{2.12}$$

We may assume that the support of ϕ is contained in the half plane $y > 0$ because, away from $(0, b)$, $\mathcal{F}$ is a solution of $\mathcal{T}w = 0$.

2. It is now more convenient to introduce the change of variables (1.3) where, for $y \geq 0$, both x and s real. In these variables we have

$$u = 9[x^2 + (s + a)^2] \qquad \text{and} \qquad v = 9[x^2 + (s - a)^2]$$

and $\mathcal{E}(x, y; 0, b)$, given by (2.1), can be written as

$$\mathcal{E}(x, s; 0, a) = 3^{-1/3}[-(x^2 + (s - a)^2)]^{-1/6} F(\frac{1}{6}, \frac{1}{6}, 1; \frac{x^2 + (s + a)^2}{x^2 + (s - a)^2}). \tag{2.13}$$

Again the abuse of notation $\mathcal{E}(x, y; 0, b) = \mathcal{E}(x, s; 0, a)$ should not cause any confusion in what follows. After taking into account (1.4) and noting that the Jacobian determinant of the transformation is

$$\frac{\partial(x, y)}{\partial(x, s)} = -(3/2)^{-1/3} s^{-1/3},$$

replacement into (2.12) yields

$$\langle \mathcal{F}, \mathcal{T}\phi\rangle = -\frac{1}{2^{2/3}} \int\int_{\mathbb{R}^2} s^{1/3}\mathcal{E}(x, s; 0, a)\mathcal{T}_e\psi(x, s)\, dx\, ds, \tag{2.14}$$

where $\psi(x, s) = \phi(x, (3s/2)^{2/3})$. To complete the proof it suffices to show that the last integral is equal to $-2^{2/3}\psi(0, a)$.

3. To evaluate the double integral in (2.14) we proceed as follows. For f and g smooth functions in $\mathbb{R}^2$ we have the identity

$$f\mathcal{T}_e g - g\mathcal{T}_e^* f = (fg_x - gf_x)_x + (fg_s - gf_s + \frac{1}{3s} fg)_s,$$

with $\mathcal{T}_e^*$ the formal adjoint of $\mathcal{T}_e$. If $f, g \in \mathcal{C}^2(\bar{D})$ where D is an open set in $\mathbb{R}^2$ with smooth boundary, then we have *Green's identity* for the reduced elliptic operator $\mathcal{T}_e$:

$$\iint_D (f\mathcal{T}_e g - g\mathcal{T}_e^* f)\, dx\, ds = \int_\Gamma (fg_x - gf_x)\, ds - (fg_s - gf_s + \frac{1}{3s} fg)\, dx. \quad (2.15)$$

We apply this identity with $f = s^{1/3}\mathcal{E}(x, s; 0, a)$, $g = \psi(x, s)$, D an annulus centered at $(0, a)$ of radii ϵ and R, with R large enough so that the support of ψ is contained in the ball of center $(0, a)$ and radius R. Since, away from $(0, a)$, $s^{1/3}\mathcal{E}(x, s; 0, a)$ is a solution to $\mathcal{T}_e^* w = 0$, it follows from (2.11) and (2.15), after setting $\mathcal{E}^* = s^{1/3}\mathcal{E}(x, s; 0, a)$, that

$$\iint_D \mathcal{E}^*\mathcal{T}_e\psi\, dx\, ds = \int_{\gamma(\epsilon)} (\mathcal{E}^*\psi_x - \psi\mathcal{E}^*_x)\, ds - (\mathcal{E}^*\psi_s - \psi\mathcal{E}^*_s + \frac{1}{3s}\psi\mathcal{E}^*)\, dx, \quad (2.16)$$

where $\gamma(\epsilon)$ is the circumference centered at $(0, a)$ with radius ϵ. If n denotes the exterior normal (pointing towards $(0, a)$) and $d\sigma$ the line element along γ_ϵ, then we may write the line integral in (2.16) as

$$I(\epsilon) = \int_{\gamma(\epsilon)} (\mathcal{E}^*\frac{d\psi}{dn} - \psi\frac{d\mathcal{E}^*}{dn} + \frac{1}{3s}\psi\mathcal{E}^*\frac{ds}{dn})\, d\sigma, \quad (2.17)$$

where $d\sigma$ is the infinitesimal arc length element. We must find the limit of $I(\epsilon)$ as $\epsilon \longrightarrow 0$.

4. It is convenient to rewrite the expression of $\mathcal{E}^*$ as follows. Consider the points $P_0(0, a)$ and $P_1(0, -a)$ and let

$$r = d(P_0, P) = \sqrt{x^2 + (s - a)^2} \quad \text{and} \quad r_1 = d(P_1, P) = \sqrt{x^2 + (s + a)^2}$$

denote, respectively, the distances from P_0 and P_1 to $P(x, y)$. Taking into account formulas (2.8) and (2.9) and recalling that $k = 0$, we obtain

$$\mathcal{E}^* = (\frac{s}{r_1})^{1/3}\{\log(\frac{r_1^2}{r^2})U(\frac{r_1^2}{r^2}) + V(\frac{r_1^2}{r^2}) \pm i\pi U(\frac{r_1^2}{r^2})\}. \quad (2.18)$$

Remark that $\mathcal{E}^*$ has a logarithmic singularity at $r = 0$, that is, at $(x, s) = (0, a)$.

5. Along γ_ϵ the terms $d\psi/dn$ and $(1/3s)\psi ds/dn$ remain bounded hence their integrals in (2.17) tend to zero, as $\epsilon \to 0$. Thus, one is left with the evaluation of the integral

$$J(\epsilon) = -\int_{\gamma(\epsilon)} \psi \frac{d\mathcal{E}*}{dn}\, d\sigma = \int_{\gamma(\epsilon)} \psi \frac{d\mathcal{E}*}{dr}|_{r=\epsilon}\, d\sigma. \tag{2.19}$$

After differentiating $\mathcal{E}^*$ with respect to r, one verifies that the only term to be integrated is the one resulting from the derivative of $\log(r_1^2/r^2)$, namely,

$$-2(\frac{s}{r_1})^{1/3}\frac{1}{r}U(\frac{r_1^2}{r^2}).$$

All other terms give integrals that tend to zero as $\epsilon \to 0$. By introducing polar coordinates $x = \epsilon\cos\theta$, $s - a = \epsilon\sin\theta$, $0 \le \theta \le 2\pi$, we then obtain

$$\begin{aligned} \lim_{\epsilon\to 0} J(\epsilon) &= -\lim_{\epsilon\to 0}\int_0^{2\pi} 2(\frac{s}{r_1})^{1/3}\frac{1}{r}U(\frac{r_1^2}{r^2})|_{r=\epsilon}\psi(\epsilon\cos\theta, a+\epsilon\sin\theta)\epsilon\, d\theta \\ &= -\frac{2}{2^{1/3}}\frac{2\pi}{\Gamma(1/6)\Gamma(5/6)}\psi(0,a) = -2^{2/3}\psi(0,a), \end{aligned} \tag{2.20}$$

because $\Gamma(1/6)\Gamma(5/6) = 2\pi$. Hence the case $k = 0$ is proved.

6. In the general case, the factor $e^{ik\pi/3}$ in formula (2.10) cancels out the factor $e^{-ik\pi/3}$ in formula (2.9) and so we end up with the same expression for $\mathcal{E}^*$ in (2.18). This completes the proof.

□

3 Fundamental solutions: pole on the real axis

We recall that the case of fundamental solutions with pole on the real axis was studied in the joint paper [3] where we showed that the distributions

$$F_+(x,y) = \begin{cases} -\dfrac{1}{2^{1/3}3^{1/2}}F(\dfrac{1}{6},\dfrac{1}{6},1;1)(9x^2+4y^3)^{-1/6} & \text{in } D_+ \\ \\ 0 & \text{elsewhere,} \end{cases} \tag{3.1}$$

where

$$D_+ = \{(x,y)\in\mathbb{R}^2 : 9x^2+4y^3 > 0\}, \tag{3.2}$$

and

$$F_-(x,y) = \begin{cases} \dfrac{1}{2^{1/3}} F(\dfrac{1}{6}, \dfrac{1}{6}, 1; 1)|9x^2+4y^3|^{-1/6} & \text{in } D_- \\ 0 & \text{elsewhere,} \end{cases} \tag{3.3}$$

where

$$D_- = \{(x,y) \in \mathbb{R}^2 : 9x^2+4y^3 < 0\}, \tag{3.4}$$

are two distinct fundamental solutions of $\mathcal{T}$ relative to the origin. Note that $9x^2+4y^3 = 0$ is the equation of the two characteristic curves of $\mathcal{T}$ emerging from the origin. The first solution F_+ is supported by the closure of the region "outside" these two characteristics while the second one F_- is supported by the closure of the region "inside" these characteristics.

A natural question to consider is whether one can obtain either F_+ or F_- or a suitable linear combination of both as a limit, in the sense of distributions, when $b \to 0$, of the fundamental solutions $\mathcal{F}_k$. For simplicity, we restrict ourselves to the case $k = 0$ and set $\mathcal{F}_0 = \mathcal{F}$.

Theorem 3.1. *We have*

$$\lim_{b\to 0} \mathcal{F}(x,y;0,b) = \begin{cases} -\dfrac{1}{2^{1/3}}(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2})F(\dfrac{1}{6}, \dfrac{1}{6}, 1; 1)(9x^2+4y^3)^{-1/6} & \textit{in } D_+ \\ -\dfrac{1}{2^{1/3}}(\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2})F(\dfrac{1}{6}, \dfrac{1}{6}, 1; 1)|9x^2+4y^3|^{-1/6} & \textit{in } D_- \end{cases} \tag{3.5}$$

Proof. Note that as b (or a) tend to 0, the sets $D_{+,a}$ and $D_{-,a}$ defined by (2.4) and (2.5) tend, respectively, to the sets D_+ and D_-.

1. Let $(x,y) \in D_+$ with $y < 0$. For b small enough $(x,y) \in D_{+,a}$ and $\mathcal{F}$ is given by the formula

$$\mathcal{F}(x,y;0,b) = -\frac{(-v)^{-1/6}}{2^{1/3}} F(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{v}). \tag{3.6}$$

As $b \to 0$, $u/v \to 1$, and $F(1/6, 1/6, 1; u/v)$ tends to $F(1/6, 1/6, 1; 1)$, whose value is given by (1.15) with $a = 1/6$, $b = 1/6$, and $c = 1$. It then suffices to consider the limit of $(-v)^{-1/6}$. Since $\arg(v)=\theta, 0\le\theta<\pi/2$, with $\arg(v)\to 0$, as $b\to 0$,it follows that $\arg(-v)\to\pi$, and we have

$$(-v)^{-1/6}=e^{-i\arg(-v)/6}|v|^{-1/6}\to(\frac{\sqrt{3}}{2}-\frac{i}{2})(9x^2+4y^3)^{-1/6}.$$

Thus, at the limit, we obtain the top expression in formula (3.5).

2. Assume that $(x,y)\in D_+$ with $y\ge 0$. In this case, both u and v are real and, as $b\to 0$, $F(1/6,1/6,1;u/v)$ still tends to $F(1/6,1/6,1;1)$. Since $(-v)^{-1/6}$ tends to the same value as in item 1. we also obtain, at the limit, the top expression in formula (3.5).

3. Let $(x,y)\in D_-$ with $y<0$. In this case, $(x,y)\in D_{-,a}$, for all a, and the expression of $\mathcal{F}$ is given by the same formula (3.6), where u and v are complex conjugate of each other. It follows, as in item 2., that the limit of $F(1/6,1/6,1;u/v)$, as $b\to 0$, is $F(1/6,1/6,1;1)$. On the other hand (see Section 2), $\arg(v)=\theta$, $\pi/2<\theta<\pi$, is such that $\theta\to\pi$, as $b\to 0$. This implies that $\arg(-v)\to 2\pi$ and so

$$(-v)^{-1/6}\to(\frac{1}{2}-\frac{i\sqrt{3}}{2})|9x^2+4y^3|^{-1/6},$$

as $b\to 0$. From these results, we get the bottom expression in formula (3.5), which completes the proof.

□

We remark that Theorem 3.1 is also true for $\mathcal{F}_k, 0\le k\le 5$. Returning to these fundamental solutions, observe that they are complex valued functions, consequently their real parts are also fundamental solutions to the Tricomi operator $\mathcal{T}$. From Theorem 3.1 applied to $\operatorname{Re}\mathcal{F}$, the real part of $\mathcal{F}$, we derive the following

Corollary 3.1. *We have*

$$\lim_{b\to 0}\operatorname{Re}\mathcal{F}(x,y;0,b)=\begin{cases}\frac{3}{2}(-\frac{1}{2^{1/3}3^{1/2}}F(\frac{1}{6},\frac{1}{6},1;1)(9x^2+4y^3)^{-1/6}) & \text{in } D_+\\ -\frac{1}{2}(\frac{1}{2^{1/3}}F(\frac{1}{6},\frac{1}{6},1;1)|9x^2+4y^3|^{-1/6}) & \text{in } D_-\end{cases} \tag{3.7}$$

Formula (3.7) can be rewritten as follows

$$\lim_{b\to 0} \operatorname{Re} \mathcal{F}(x,y;0,b) = \frac{3}{2}F_+(x,y) - \frac{1}{2}F_-(x,y),$$

that is, the limit of the real part of $\mathcal{F}(x,y;0,b)$ is a fundamental solution with pole at the origin given by a linear combination of the fundamental solutions F_+ and F_-.

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