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PREIMAGE, IMAGE, AND ITERATED IMAGE OF THE CLIQUE OPERATOR

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Abstract

The clique operator K maps a graph G into its *clique graph*, which is the intersection graph of $\mathcal{C}(G)$ (family of cliques of G). Let $\mathcal{G}$ be the class of all graphs and $\mathcal{H}$ that of all graphs satisfying that $\mathcal{C}(G)$ is a Helly family. In this work we will be interested in the following question: is $K(\mathcal{G})$ the same as $K^2(\mathcal{G})$? The present paper represents an effort toward its solution.

Since $K(\mathcal{H}) = \mathcal{H}$, we focus our study to graphs in $\mathcal{G} \setminus \mathcal{H}$ and we obtain a necessary condition for a graph to be in $K(\mathcal{G}) \setminus \mathcal{H}$ in terms of the presence of a certain subgraph A. Then A and all graphs obtained from A by the addition of extra edges are the smaller that could be in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$, i. e. separate $K(\mathcal{G})$ from $K^2(\mathcal{G})$. To analyze all these graphs we must develop general results which are presented in this work: we show an algorithm for recognition of clique graphs, we give a constructive characterization of $K^{-1}(G)$ for a fixed but generic G. Finally we study all graphs derived from A proving that none of them is in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ and that if there is a graph which separates both classes it must have at least eight vertices.

1 Introduction

The *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. The graph $K(G)$ is called the *clique graph* of G. Among all the better studied graph operators, K seems to be the richest one and many questions regarding it remain open.

In this work we are interested in this question: is there a graph that is a clique graph but not a clique graph of a clique graph? In other words, if $\mathcal{G}$ is the class of all graphs, is $K(\mathcal{G})$ equal to $K^2(\mathcal{G})$?

The basic result for clique graphs was given by Roberts and Spencer ([6]): a graph G is a clique graph if and only if there is a family of complete sets of

G which covers all the edges of G and satisfies the Helly property. We call an *RS family* of G a family of complete sets in G that fulfills this conditions.

Trivially, if the family of cliques of G satisfies the Helly property (or is G in the class $\mathcal{H}$) G will be a clique graph. Moreover, Escalante ([2]) proved that $K(\mathcal{H}) = \mathcal{H}$. Then it is clear that if there is a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ it must not be in $\mathcal{H}$. But to find a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$, i. e. which separates both classes seems to be very difficult.

Although we do not give an answer to our original question, we have found general results about the clique operator. Our contributions can be summarized as follows. We develop an algorithm that detects if a graph has or not an RS family, i. e. if a graph is or is not a clique graph. We also show that being a clique graph is a property that is maintained by addition of twins, but *not being* a clique graph is *not* necessarily maintained by addition of twins. In addition, we give a constructive characterization of $K^{-1}(G)$ for a fixed but generic G in terms of special RS families of G.

We obtain a necessary condition for a non-Helly graph to be in the image of K in terms of the presence of a certain subgraph A.

Using these results we present several properties that a graph must satisfy to be in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$. Beginning with graph A, which has seven vertices, we analyze all graphs derived from A by the addition of edges, and conclude that none of them is in the difference. We conclude that a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ must have at least eight vertices.

The paper is organized as follows. Section 2 contains the basic definitions used throughout. In Section 3 we give the algorithm, which will be used later to show that certain graphs are not clique graphs. Section 4 characterizes all the graphs in $K^{-1}(G)$ for every graph G and we analyze the behaviour of twin vertices. In Section 5 we study non-Helly clique graphs. Finally, Section 6 contains our concluding remarks.

2 Definitions

In this article all graphs are simple, i.e., without loops or multiple edges (except for the DAMG of Section 3.1). Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. If uv is an edge of G, we call $G-uv$ the graph such that $V(G-uv) = V(G)$ and $E(G-uv) = E(G) - \{uv\}$. u and v are twin vertices of G if $uv \in E(G)$ and $zu \in E(G)$ if and only if

$zv \in E(G)$. A set C of vertices of G is *complete* when any two vertices of C are adjacent. A maximal complete subset of $V(G)$ is called a *clique*. We denote by $\mathcal{C}(G)$ the clique family of G.

Let $\mathcal{F} = (F_i)_{i\in I}$ be a finite family of finite, nonempty sets. Its *dual family* $D\mathcal{F}$ is the family $(F(u))_{u\in U}$ where $U = \bigcup_{i\in I} F_i$ and $F(u) = \{i \in I, u \in F_i\}$. We denote by $L\mathcal{F}$ the *intersection graph* of $\mathcal{F}$, i.e., $V(L\mathcal{F}) = I$ and two vertices i and j are adjacent if and only if $F_i \cap F_j \neq \emptyset$. We also say that $\mathcal{F}$ *represents* $L\mathcal{F}$.

The *2-section* of $\mathcal{F}$, denoted by $S\mathcal{F}$, is the graph with $V(S\mathcal{F}) = \bigcup_{i\in I} F_i$ and two vertices x and y are adjacent if and only if there exists $i \in I$ such that $x, y \in F_i$. It is easy to see that $L\mathcal{F} = SD\mathcal{F}$ [1].

An intersecting family $(F_i)_{i\in I}$ is a family such that any two sets F_k, F_l with $k, l \in I$ intersect. A family $(F_i)_{i\in I}$ of arbitrary sets satisfies the *Helly property*, or is *Helly*, when for every intersecting subfamily $(F_j)_{j\in J}$ with $J \subseteq I$, we have $\bigcap_{j\in J} F_j \neq \emptyset$. A graph is *Helly* when the family of its cliques is Helly. We denote by $\mathcal{H}$ the class of Helly graphs. A family $\mathcal{F}$ is *conformal* when the cliques of $S\mathcal{F}$ are all members of $\mathcal{F}$. This amounts to saying that its dual family $D\mathcal{F}$ is Helly [1]. A family $\mathcal{F}$ is *reduced* when none of its members is contained in another member of the family.

As we said earlier, the *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. Thus, $K(G)$ is nothing else than the intersection graph of the family of all cliques of G. The graph $K(G)$ is called the *clique graph* of G.

We call $\mathcal{G}$ the class of all graphs then $K(\mathcal{G})$ will be the class of all clique graphs.

In an important paper for the theory of clique graphs, Roberts and Spencer [6] found the following characterization of $K(\mathcal{G})$:

Theorem 1 (Roberts and Spencer, 1971) *A graph G is in $K(\mathcal{G})$ if and only if there is a family $\mathcal{K}$ of complete sets in G such that:*

1. *$\mathcal{K}$ covers all the edges of G (i.e., if $xy \in E(G)$, then $\{x, y\}$ is contained in some element of $\mathcal{K}$).*

2. *$\mathcal{K}$ satisfies the Helly property.*

We call an *RS family* of G a family of complete sets in G that fulfills the hypothesis of the Roberts and Spencer theorem.

3 An Algorithm Recognizing Clique Graphs

In this section we present an algorithm that looks for RS families in an arbitrary graph G.

Let $\mathcal{A}$ and $\mathcal{F}$ be two families of complete sets of a graph G. We say that $\mathcal{A}$ is *below* $\mathcal{F}$ (notation: $\mathcal{A} \leq \mathcal{F}$) when for every member A of $\mathcal{A}$ there is a member F of $\mathcal{F}$ with $A \subseteq F$.

A family $\mathcal{A}$ of complete sets of G is *admissible* when there is a Helly family $\mathcal{F}$ such that $\mathcal{A} \leq \mathcal{F}$ and $S\mathcal{A} = S\mathcal{F}$.

A *conflict* in a family $\mathcal{A}$ of complete sets of a graph G is an intersecting subfamily without a common intersection. Thus, conflicts exist in non Helly families only. A *solution* for a conflict $\mathcal{C}$ in $\mathcal{A}$ is a vertex v such that, for every member C of $\mathcal{C}$, the set $\{v\} \cup C$ is a complete set. A conflict without solutions is *unsolvable*.

If $\mathcal{A}$ is a family of complete sets of G, $\mathcal{C}$ is a conflict in $\mathcal{A}$, and v is a solution for $\mathcal{C}$, we denote by $M(\mathcal{A}, \mathcal{C}, v)$ the family obtained from $\mathcal{A}$ by replacing every member C of $\mathcal{C}$ by $\{v\} \cup C$, and then reducing the family by removing duplicate sets and then taking only the maximal sets.

The following theorem will be useful.

Theorem 2 *Let $\mathcal{F}$ be a Helly family of complete sets of a graph G, $\mathcal{A}$ a family below $\mathcal{F}$, and $\mathcal{C}$ a conflict in $\mathcal{A}$. Then there is a solution v for $\mathcal{C}$ such that $M(\mathcal{A}, \mathcal{C}, v) \leq \mathcal{F}$.*

Proof: Since $\mathcal{C}$ is a subfamily of $\mathcal{A}$ and $\mathcal{A} \leq \mathcal{F}$, for each member C of $\mathcal{C}$ there is a member F_C of $\mathcal{F}$ with $C \subseteq F_C$. The family formed by the F_C's is intersecting, because $\mathcal{C}$ is intersecting. Since $\mathcal{F}$ is Helly, there is a vertex v common to all the F_C's. We claim that v is the solution sought. Indeed, given a member of $M(\mathcal{A}, \mathcal{C}, v)$, if this member belongs to $\mathcal{A}$, then it is already below $\mathcal{F}$. If it is of the form $\{v\} \cup C$ for some member C of the conflict, then it is contained in F_C. Thus we have $M(\mathcal{A}, \mathcal{C}, v) \leq \mathcal{F}$.

□

A *solution tree* T for a graph G is a finite tree with the following characteristics.

1. Each node of T is labeled with a pair $(\mathcal{A}, \mathcal{C})$, where $\mathcal{A}$ is a reduced family of complete sets of G and $\mathcal{C}$ is a conflict in $\mathcal{A}$ if $\mathcal{A}$ is not Helly.

2. The root of T is labeled with a pair $(\mathcal{A}, \mathcal{C})$ where $\mathcal{A}$ is the family of edges of G.

3. If $(\mathcal{A}, \mathcal{C})$ is the label of a leaf of T, then either $\mathcal{A}$ is a Helly family, or $\mathcal{C}$ is an unsolvable conflict. In the former case we say that the leaf is a *good* leaf, and in the latter case, a *bad* leaf.

4. Each internal node labeled with $(\mathcal{A}, \mathcal{C})$ has exactly one child per solution v of the conflict $\mathcal{C}$, and the child corresponding to v has label $(\mathcal{A}', \mathcal{C}')$, where $\mathcal{A}' = M(\mathcal{A}, \mathcal{C}, v)$.

The following theorem is essential to guarantee that the algorithm works for all graphs.

Theorem 3 *Every graph G admits a solution tree.*

Proof: Consider pre-solution trees, where all the conditions for a solution tree are met except that leaves may be labeled with only the first family of the pair. Such leaves will be called *pending* leaves. Start with a tree with just the root node labeled with the family of all edges of G. Perform the following steps until no more pending leaves remain:

> choose a pending leaf p, and let $\mathcal{A}$ be its label
> if $\mathcal{A}$ is Helly, make $(\mathcal{A}, \mathcal{A})$ the label of p
> if $\mathcal{A}$ is non Helly, choose a conflict $\mathcal{C}$ in $\mathcal{A}$ and for each solution v of
> $\mathcal{C}$ create a child of p with label $M(\mathcal{A}, \mathcal{C}, v)$.

This procedure terminates, since each time an edge is created the families involved grow, all families are reduced, and the ultimate limit is $C(G)$, the family of all cliques of G. There is a finite number of reduced families below $C(G)$. When it terminates, there are no more pending leaves and the tree is a solution tree.

□

The following result shows that a solution tree actually answers the question: is G a clique graph?

Theorem 4 *Let G be a graph and T any solution tree for G. Then $G \in K(\mathcal{G})$ if and only if there is a good leaf in T.*

Proof: Suppose that $G \in K(\mathcal{G})$ and let $\mathcal{F}$ be an RS family of G. Consider a solution tree T for G. We will show that, for every $n \geq 0$, either there is a path of length $k \leq n$ from the root to a good leaf, or there is a path of length n from the root to an internal node labeled with $(\mathcal{A}, \mathcal{C})$ where $\mathcal{A} \leq \mathcal{F}$. Since the tree is finite, this will imply that there is a good leaf in T.

To simplify the proof, we will use the phrase "path to a family $\mathcal{A}$" meaning "path from the root to a node labeled with $(\mathcal{A}, \mathcal{C})$". The proof will be by induction on n. For $n = 0$, take the path containing just the root. It satisfies the stated conditions since the family of the edges is below $\mathcal{F}$. For $n > 0$, assume the result true for $n-1$, hence, either there is a path of length $k \leq n-1$ to a good leaf or there is a path of length $n-1$ to a family $\mathcal{A} \leq \mathcal{F}$. In the first case, we are done. In the second case, if $\mathcal{A}$ is Helly we are done as well. If $\mathcal{A}$ is not Helly, the path ends in a node p that has also a conflict $\mathcal{C}$ in its label. Since $\mathcal{C} < \mathcal{F}$, Theorem 2 tells us that there is a solution v for $\mathcal{C}$ such that $M(\mathcal{A}, \mathcal{C}, v) \leq \mathcal{F}$. Augmenting the path with the child of p corresponding to v, we prove the induction step. This concludes the "only if" part of the theorem.

For the "if" part, assume there is a good leaf in T, a solution tree for G. Observe that all labels $(\mathcal{A}, \mathcal{C})$ of nodes in T are such that $\mathcal{A}$ is a family of complete sets that covers the edges. If there is a good leaf, the family $\mathcal{A}$ is Helly, that is, it is an RS family of G. Thus, $G \in K(\mathcal{G})$.

□

3.1 Solution DAMGs

A solution tree is a concrete object that can prove that a graph is not a clique graph. To simplify those proofs, a more compact object, called a solution directed acyclic multigraph or solution DAMG, can be used.

Given a graph G, a *solution DAMG* for a graph G is a finite, directed, acyclic multigraph D with a single source node with the following characteristics.

1. Each node of D is labeled with a pair $(\mathcal{A}, \mathcal{C})$, where $\mathcal{A}$ is a reduced family of complete sets of G and $\mathcal{C}$ is a conflict in $\mathcal{A}$ if $\mathcal{A}$ is not Helly.

2. The unique source of D is labeled with a pair $(\mathcal{A}, \mathcal{C})$ where $\mathcal{A}$ is the family of edges of G.

3. If $(\mathcal{A}, \mathcal{C})$ is the label of a sink of D, then either $\mathcal{A}$ is a Helly family, or $\mathcal{C}$ is an unsolvable conflict. In the former case we say that the sink is a *good* sink, and in the latter case, a *bad* sink.

4. Each non sink node labeled with $(\mathcal{A}, \mathcal{C})$ has exactly one outcoming arc per solution of the conflict $\mathcal{C}$, and this outcoming arc is labeled with the corresponding solution.

5. Each non source node p with incoming arcs from $p_1, p_2, \ldots, p_k$ with labels $v_1, v_2, \ldots, v_k$ has label $(\mathcal{A}, \mathcal{C})$ satisfying $\mathcal{A} \leq M(\mathcal{A}_i, \mathcal{C}_i, v_i)$ for every i such that $1 \leq i \leq k$, where $(\mathcal{A}_i, \mathcal{C}_i)$ is the label of node p_i.

Theorem 5 *Let G be a graph and D any solution DAMG for G. Then $G \in K(\mathcal{G})$ if and only if there is a good sink in D.*

Proof: Suppose that $G \in K(\mathcal{G})$ and let $\mathcal{F}$ be an RS family of G. Consider a solution DAMG D for G. We will show that, for every $n \geq 0$, either there is a path of length $k \leq n$ from the source to a good sink, or there is a path of length n from the source to a node labeled with $(\mathcal{A}, \mathcal{C})$ where $\mathcal{A} \leq \mathcal{F}$. Since the DAMG is finite, this will imply that there is a good sink in D.

To simplify the proof, we will use the phrase "path to a family $\mathcal{A}$" meaning "path from the source to a node labeled with $(\mathcal{A}, \mathcal{C})$". The proof will be by induction on n. For $n = 0$, take the path containing just the source. It satisfies the stated conditions since the family of the edges is below $\mathcal{F}$. For $n > 0$, assume the result true for $n-1$, hence either there is a path of length $k \leq n-1$ to a good sink or there is a path of length $n-1$ to a family $\mathcal{A} \leq \mathcal{F}$. In the first case, we are done. In the second case, if $\mathcal{A}$ is Helly we are done as well. If $\mathcal{A}$ is not Helly, the path ends in a node p that has also a conflict $\mathcal{C}$ in its label. Since $\mathcal{C} < \mathcal{F}$, Theorem 2 tells us that there is a solution v for $\mathcal{C}$ such that $M(\mathcal{A}, \mathcal{C}, v) \leq \mathcal{F}$. Augmenting the path with the outcoming arc of p corresponding to v, we prove the induction step. This concludes the "only if" part of the theorem.

For the "if" part, assume there is a good sink in D, a solution tree for G. Observe that all labels $(\mathcal{A}, \mathcal{C})$ of nodes in D are such that $\mathcal{A}$ is a family of complete sets that covers the edges. If there is a good sink, the family $\mathcal{A}$ is

Helly, that is, it is an RS family of G. Thus, $G \in K(\mathcal{G})$.

$\square$

To simplify the arguments ever further, we will place the edges of G plus the cliques of size three in the source, which are necessarily present in every RS family. In addition, we will show only the conflicts and the arc labels; the family $\mathcal{A}_p$ of each node p will be implicitly defined as

$$\bigwedge_{q \to p} M(\mathcal{A}_q, \mathcal{C}_q, v_{q,p}),$$

where $\wedge$ denotes the (reduced) infimum with respect to $\leq$, that is,

$$\bigwedge_{i=1}^{k} \mathcal{F}_i = \mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \ldots \wedge \mathcal{F}_k = \{A \mid \{A\} \leq \mathcal{F}_i \text{ for every } i \text{ such that } 1 \leq i \leq k \text{ and } A \text{ is maximal with this property } \}$$

and $v_{q,p}$ is the label of the arc from q to p.

4 The inverse image of a fixed G

Given a graph G in $K(\mathcal{G})$, we characterize the class of graphs whose image under K is G. Remember that an RS family of G is a family that satisfies the conditions of the Roberts-Spencer Theorem. If in addition, the RS family has a reduced dual, then we said that it is an *ERS family* of G. We show in this section that each ERS family of G corresponds to a graph H in $K^{-1}(G)$, and vice versa.

Before giving the characterization, let us recall a couple of results on intersection graphs. For the proofs, see the work of Gutierrez [3]

Lemma 1 *[3] If $\mathcal{F}$ is a family of complete sets of G which covers all edges of G, then G is the intersection graph of the family $D\mathcal{F}$.*

Lemma 2 *[3] If G is the intersection graph of a family $\mathcal{F}$ then $D\mathcal{F}$ is a family of complete sets of G which covers all edges of G.*

The following result is a consequence.

Theorem 6 *[4] Let G and H be two graphs. Then $K(H) = G$ if and only if H is the intersection graph of an ERS family of G.*

4.1 Twin Vertices

We have seen that the graphs in the inverse image $K^{-1}(G)$ are in one-to-one correspondence with the ERS families of G. There is an infinite number of these families. It is easy to see that if a given complete set of G appears in an ERS family two or more times, this produces twin vertices in the corresponding graph $H \in K^{-1}(G)$. We would like to simplify the study of $K^{-1}(G)$ by taking only ERS families with no repeated elements, because there is a finite number of such families. For instance, to test whether a given graph G belongs to $K^2(\mathcal{G})$ we could take all reduced (i.e., without twins) graphs in $K^{-1}(G)$ and check each one for pertinence in $K(\mathcal{G})$. Unfortunately, this result is only partially true:

Theorem 7 *[4] Let G be a graph and u, v twin vertices in G. If $G - u \in K(\mathcal{G})$ then $G \in K(\mathcal{G})$.*

The converse of Theorem 7 does not hold. Denote by P the first graph depicted in Figure 1, which is not a clique graph. The graph $P*$ in the same figure is obtained from P by adding a twin to one of the central vertices. However, $P*$ does belong to $K(\mathcal{G})$, because the complete sets

$$\begin{aligned}
C_1 &= \{1,4,2\},\\
C_2 &= \{1,3,6\},\\
C_3 &= \{2,3',5\},\\
C_4 &= \{1,2,3,3'\},\\
C_5 &= \{3,3',5\},\\
C_6 &= \{3,3',6\},
\end{aligned}$$

form a reduced RS family of the graph in question. In fact, there are only two reduced RS families of $P*$. Replacing C_2 and C_3 by $C_2' = \{1,3',6\}$ and $C_3' = \{2,3,5\}$ we obtain the other one.

5 Non-Helly Graphs in $K(\mathcal{G})$

We know that $K^2(\mathcal{G}) \subseteq K(\mathcal{G})$. If these two classes are distinct, there must be a clique graph which is not a clique graph of a clique graph. Helly graphs are known to have inverse images under the clique operator which are also Helly graphs, and therefore are in $K^2(\mathcal{G})$. Thus, we must study the non Helly graphs in $K(\mathcal{G})$.

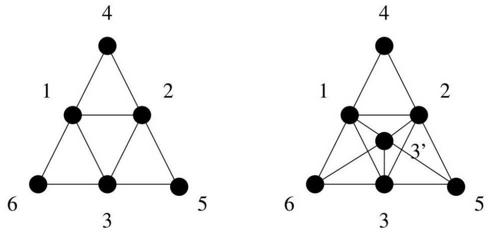

Figure 1: Graphs P (left) and $P*$. The latter is in $K(\mathcal{G})$, and is obtained by addition of a twin to P, which does not belong to $K(\mathcal{G})$.

5.1 The graphs A and B

The graph P depicted in Figure 1 is the smallest non Helly graph. We say that a graph G *has* P when G has three mutually adjacent vertices 1, 2, and 3, and three other vertices 4, 5, and 6 such that 4 is adjacent to 1 and 2 but not to 3, 5 is adjacent to 2 and 3 but not to 1, 6 is adjacent to 1 and 3 but not to 2.

Notice that this is different from saying that G has P as an induced subgraph, and it is also different from saying that G has a subgraph (not necessarily induced) isomorphic to P. However, this concept is important because of the following fact. Define a graph to be *Helly hereditary* when it is Helly and all of its induced subgraphs are Helly as well. Prisner [5] showed that G is Helly hereditary if and only if G does not have P in the sense defined above.

The following result tells us more about the structure of a graph in $K(\mathcal{G})$ that has P.

Theorem 8 *If $G \in K(\mathcal{G})$ and G has P then G has a subgraph isomorphic to A (see Figure 2).*

Proof: In [4] we have proved that G has a subgraph isomorphic to A or B (see Figure 2). Observe that A is a subgraph of B (take vertices 1, 2, 4, and all unnamed vertices in Figure 2 and all edges between them). Therefore, in this case again G has a subgraph isomorphic to A.

$\square$

A graph can have P and also subgraphs isomorphic to A without belonging to $K(\mathcal{G})$ (use the algorithm to see that this graph is not a clique graph), as the example in Figure 3 shows.

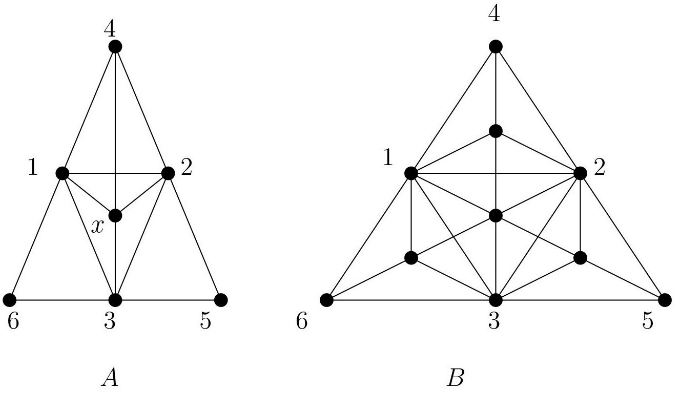

Figure 2: Graphs A and B.

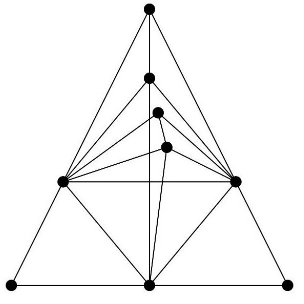

Figure 3: Graph with two A's but not a clique graph.

The following corollary easily follows.

Corollary 1 *If $G \in K(\mathcal{G})$, then G is Helly hereditary if and only if G does not have subgraphs isomorphic to A.*

Proof: Since G is Helly hereditary if and only if G does not have P [5], the result follows.

$\square$

Graph A is therefore a good starting point in the search for graphs that "separate" the Helly hereditary ones inside $K(\mathcal{G})$.

5.2 Graphs derived from A

As a step towards characterizing $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ we ask ourselves what is the minimum number of vertices a graph in this class must have. Since A is smaller than B, we concentrate our efforts on graphs derived from A.

For the graph A itself, we have that it actually belongs to $K^2(\mathcal{G})$ (see Figure 4).

In the remainder of this section we will analyze all graphs with seven vertices derived from A by the addition of extra edges, and show that none of them belongs to $K(\mathcal{G}) \setminus K^2(\mathcal{G})$. The next result will be then that at least 8 vertices are necessary to separate $K(\mathcal{G})$ from $K^2(\mathcal{G})$.

Graph A has 13 edges. The complete graph on seven vertices has 21 edges. The eight edges not present in A must be considered. However, edges 15, 26, and 34 are not present by our initial hypothesis that the graph is not Helly hereditary. Therefore, five edges remain: 45, 46, 56, $5x$, $6x$. Considering all the possibilities of presence or absence of these edges, we can build a total of 32 graphs. Of those, graph A, as we saw, is in $K^2(\mathcal{G})$.

What follows is an analysis of the other 31 graphs. Two important facts for the analysis are stated below. To simplify the notation, we will write sets without the surrounding curly braces and without commas separating the elements. For instance, set $\{1, 2, x\}$ will be denoted simply by $12x$.

Fact I. The only reduced RS family for A is : 14x, 24x, 123x, 136, 235.

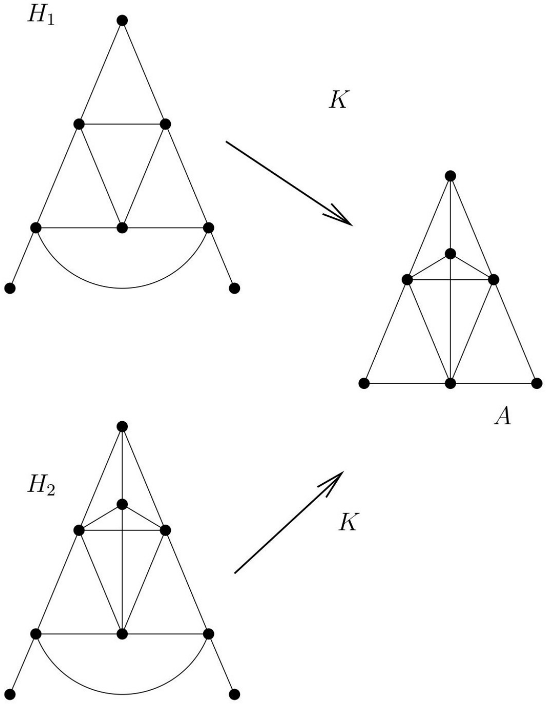

Figure 4: Graph A and two graphs H_1 and H_2 in $K^{-1}(A)$. Notice that H_2 is in $K(\mathcal{G})$, showing that $A \in K^2(\mathcal{G})$.

Fact II. There are only two reduced RS families for $P*$ and both have the central K_4 : 1233'; and the two K_3 that have 33' (see Section 4.1).

The following theorem is useful to reduce the number of graphs analyzed.

Theorem 9 *Let G be a graph, $C = \{a, b, c\}$ a clique of G, and $\mathcal{F}$ a reduced RS family of G. In addition, let $G' = G - ab$ and $\mathcal{F}' = \mathcal{F} - \{C\}$. Then:*

1. *C is a member of $\mathcal{F}$.*
2. *If $\{a, c\}$ is not a clique of G' then $\mathcal{F}'$ covers ac.*
3. *If $\{a, c\}$ and $\{b, c\}$ are not cliques of G' then $\mathcal{F}'$ is an RS family of G'.*

Proof: 1) Is immediate.

2) If $\{a, c\}$ is not a clique of G', there is a vertex $y \neq b$ adjacent to a and c in G and G'. Notice that y is not adjacent to b, otherwise $\{a, b, c\}$ would not be maximal. Let F and F' be members of $\mathcal{F}$ that cover ay and cy respectively. Hence F, F' and $\{a, b, c\}$ form an intersecting subfamily of $\mathcal{F}$. Since $\mathcal{F}$ has the Helly property there is a vertex $z \in F \cap F' \cap \{a, b, c\}$. We have $z \neq b$ because b is not adjacent to y. If $z = a$ or $z = c$ there is a complete set, F or F', different from $\{a, b, c\}$ that covers ac.

3) Is trivial from 2).

$\square$

The analysis of the 31 remaining graphs can be divided in three cases, as follows.

Case 1: If both edges $5x$ and $6x$ are present, the graph is Helly and is in $K^2(\mathcal{G})$.

Case 2: If edges $5x$ and $6x$ are both not present:

1. $A + 45 \cong A + 46$ is not in $K(\mathcal{G})$ because of Fact I and Theorem 9.
2. $A + 56$ is in $K^2(\mathcal{G})$, since it suffices to add a vertex in H_2 of Figure 4 corresponding to clique 356.
3. $A + 45 + 46$ is not in $K(\mathcal{G})$ because $A + 45$ is not in $K(\mathcal{G})$ and by Theorem 9.
4. $A + 45 + 56 \cong A + 46 + 56$ are not in $K(\mathcal{G})$ by the same reason.

5. $A+45+56+46$ because $A+45+56$ is not in $K(\mathcal{G})$ and by Theorem 9.

Case 3: If exactly one of the edges $5x$, $6x$ is in the graph. It suffices to analyze the cases in which $5x$ appears, because the other cases produce isomorphic graphs.

1. $A+5x \cong P*$ is in $K^2(\mathcal{G})$. Take the following graph in $K^{-1}(P*)$:

$$S(\{af, abe, abc, ebdd'i, bcdd'j, cdjg, ed'hi\}).$$

This graph admits the following RS family:

$$\{af, abe, abc, ebdd'i, bcdd'j, cjg, gjd, ehi, ihd'\}.$$

2. $A+5x+45$ is in $K^2(\mathcal{G})$. Take the following graph in $K^{-1}(P*)$:

$$S(\{af, abe, abc, ebdi, bcdj, cdjg, edhi\}).$$

This graph admits the following RS family:

$$\{af, abe, abc, ebdi, bcdj, cjg, gjd, ehi, ihd\}.$$

3. $A+5x+56 \cong A+5x+46$ is not in $K(\mathcal{G})$ because of Fact II and Theorem 9.
4. $A+5x+56+46$ because $A+5x+56$ is not in $K(\mathcal{G})$ and by Theorem 9.
5. $A+5x+45+56 \cong A+5x+45+46$ is not in $K(\mathcal{G})$ because of Theorem 5 and the solution DAMG in Figure 5.
6. $A+5x+45+56+46$ is not in $K(\mathcal{G})$ because $A+5x+45+56$ is not in $K(\mathcal{G})$ and by Theorem 9.

6 Conclusions

We studied the question "is $K(\mathcal{G})$ the same as $K^2(\mathcal{G})$?", where $\mathcal{G}$ is the class of all graphs, and concluded that a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ must have at least eight vertices.

Other partial results include the following. We obtain a necessary condition for a graph to be in the image of K in terms of the presence of certain subgraph

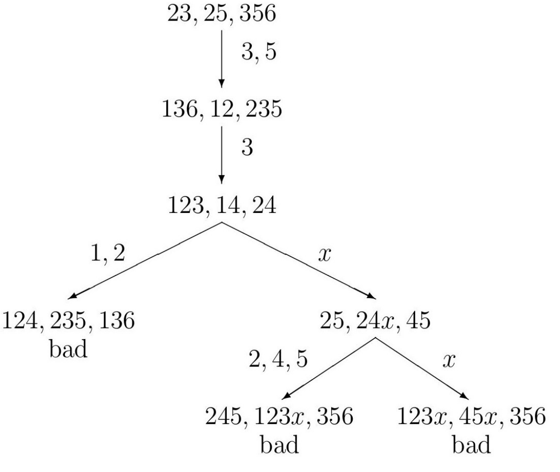

Figure 5: A solution DAMG for graph $A + 5x + 45 + 56$, showing it is not a clique graph. Only conflicts are shown at nodes. The family $\mathcal{A}$ at each node is determined implicitly as indicated in the end of Section 3.1. Arcs leaving a node are labeled with the possible solutions for its conflict. Multiple labels over an arc should be interpreted as multiple arcs parallel to the one shown, each one with a different label. The source node contains all edges and all cliques of size three of G.

A. We also show that being a clique graph is a property that is maintained by addition of twins, but *not being* a clique graph in *not* necessarily maintained by addition of twins. In addition, we give an algorithm for the recognition of clique graphs. Its complexity is unknown, probably exponential, but it is useful in producing proofs that graphs are not clique graphs.

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