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THE CLIQUE OPERATOR ON EXTENDED P_4-SPARSE GRAPHS

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Abstract

The **clique graph** of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. The iterated clique graphs $K^n(G)$ are defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$ and K is the clique operator. In this article we describe the K-behaviour of the classes of P_4-reducible, P_4-sparse and extended P_4-sparse graphs. These classes are an extension of the well known class of P_4-free graphs or cographs. Furthermore, we give some partial results for the larger class of serial (i.e. complement-disconnected) graphs.

1 Introduction

The **clique graph** of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. The **iterated clique graphs** $K^n(G)$ are defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$. We refer to [22] and [24] for the literature on iterated clique graphs. Graphs behave in a variety of ways under the iterates of the clique operator K, the main distinction being between K-convergence and K-divergence. A graph G is said to be K**-divergent** if $\lim_{n\to\infty} |V(K^n(G))| = \infty$. If G is not K-divergent, then it is K**-convergent**.

The first examples of K-divergent graphs were given by Neumann-Lara (see [4, 16]). For $n \geq 2$, define the n-dimensional octahedron $\mathcal{O}_n$ as the complement of a perfect matching on $2n$ vertices. Then $\mathcal{O}_n$ is a complete multipartite graph $K_{2,2,\ldots,2}$. Neumann-Lara showed that $K(\mathcal{O}_n) \cong \mathcal{O}_{2^{n-1}}$ and hence, for $n \geq 3$, $\mathcal{O}_n$ is K-divergent. Recently, other graphs have been found to be K-divergent [11, 12, 19].

Most of the results on convergence of iterated clique graphs are on the domain of clique-Helly graphs. In fact, clique-Helly graphs are always K-convergent [4]. In general, much less is known about K-convergence, when

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non clique-Helly graphs are considered. Some results on convergence of graphs which are not clique-Helly can be found in [1, 2, 3]. We give, in this paper, some results that guarantee K-convergence of a special class of graphs that are not clique-Helly.

The question whether the K-convergence of a graph is algorithmically decidable is an open problem. Even for restricted families of graphs very little is known. For families containing both K-convergent and K-divergent graphs, K-convergence has been characterized only for complements of cycles [16], clockwork graphs [13], regular Whitney triangulations of closed surfaces [14] and cographs [10]. However, in all these cases K-convergence can be decided in polynomial time.

In this paper, we shall study the K-behaviour of some natural extensions of the class of cographs, i.e., the graphs not containing as an induced subgraph a chordless path on four vertices.

The K-behaviour of cographs has been completely characterized in [10], where also some partial results for the larger class of serial graph were given.

Some other sufficient conditions for K-convergence and K-divergence that hold more in general for the class of serial graphs are given in Section 3. They will allow us to describe completely the K-behaviour of the following classes of graphs with few P_4's: the classes of P_4-reducible, P_4-sparse, extended P_4-reducible and extended P_4-sparse graphs. This characterization leads to a polynomial time recognition algorithm for K-convergence.

The **P_4-reducible** graphs have been defined in [8] as the class of graphs such that any vertex belongs to at most one induced P_4. The **P_4-sparse** graphs have been introduced in [7] as the graphs for which every set of five vertices induces at most one P_4. By relaxing the restriction concerning the exclusion of the chordless C_5-cycle, that is a forbidden configuration for P_4-reducible and P_4-sparse graphs, two wider classes of graphs called the class of the **extended P_4-reducible** graphs and **extended P_4-sparse** graphs have been introduced in [5]. A characterization of the K-behaviour of graphs belonging to the above classes is given in Section 4.

2 Preliminaries and definitions

We consider simple, undirected, finite graphs. The sets $V(G)$ and $E(G)$ are the vertex and edge sets of a graph G. For any vertex v in $V(G)$, the **neigh-**

bourhood of v is the set $N(v) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$. A **trivial** graph is a graph with a single vertex. The symbol $\overline{G}$ represents the **complement** of G. A **complete** is a set of pairwise adjacent vertices in G and a **stable set** is formed by pairwise non adjacent vertices of G. A **clique** of G is a complete not properly contained in any other complete. A **subgraph** of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the **subgraph induced by** X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X. If v is a vertex of a subgraph H of G adjacent to every other vertex of H, then we say that v is **universal** in H. Let X be a subset of $V(G)$ and x any vertex of X. The **quotient graph** G/X is defined as $V(G/X) = (V(G) - X) \cup \{x\}$ and $E(G/X) = E(G[V(G)-X]) \cup \{\{x,v\} \mid \{u,v\} \in E(G),\ u \in X,\ v \in V(G)-X\}$.

Let H and H' be vertex disjoint graphs. The **union** or **parallel composition** of H and H' is the graph $G = H \cup H'$ defined as $V(G) = V(H) \cup V(H')$ and $E(G) = E(H) \cup E(H')$. The **join**, **sum**, or **serial composition** of H and H' is the graph $G = H + H'$ defined as $V(G) = V(H) \cup V(H')$ and $E(G) = E(H) \cup E(H') \cup \{\{x,y\} \mid x \in V(H),\ y \in V(H')\}$. The **product** $G \times G'$ of two graphs G and G' is given by $V(G \times G') = V(G) \times V(G')$ and $E(G \times G') = \{\{(u,u'),(v,v')\} : \{u,v\} \in E(G),\ \{u',v'\} \in E(G')\}$. We will also use the fact that $\overline{K}(G_1 + G_2) = \overline{K}(G_1) \times \overline{K}(G_2)$ (see [16, 24]).

One promising paradigm for studying properties of a class of graphs involves partitioning the set of vertices of a graph into subsets called modules, and the decomposition process is called modular decomposition.

A **module** of G is a set of vertices M of $V(G)$ such that all the vertices of M have the same neighbours outside of M, that is, each vertex in $V(G) - M$ is either adjacent to all vertices of M, or to none. For instance, every singleton vertex as well as the whole $V(G)$ are modules. We say that M is a **strong** module if for any other module A the intersection $M \cap A$ is empty or equals either M or A. For non-trivial G, the family $\{M_1, M_2, \ldots, M_p\}$ of all maximal (proper) strong modules is a partition of $V(G)$ and $p \geq 2$. This partition is the **modular decomposition of** G. We will often identify the modules M_i with the induced subgraphs $G_i = G[M_i]$.

For disconnected G, the maximal strong modules are the connected components. In this case $G = G_1 \cup G_2 \cup \cdots \cup G_p$ is called **parallel**.

If $\overline{G}$ is disconnected, the maximal strong modules of G are the connected components of $\overline{G}$. In this case $G = G_1 + G_2 + \cdots + G_p$ is called **serial**.

If both G and $\overline{G}$ are connected, then G is called **neighbourhood**.

The modular decomposition of a non-trivial graph G is used recursively in order to define its unique **modular decomposition tree** $T(G)$. The root of $T(G)$ is G, the first-level vertices of $T(G)$ are the maximal strong modules of G, and so on. The leaves of $T(G)$ are the vertices of G and the internal nodes of $T(G)$ are modules labeled with P, S or N (for parallel, serial, or neighbourhood module, respectively). A linear time algorithm that produces the modular decomposition tree is given in [15].

If G is a serial graph and each G_i has a modular decomposition of the form

$$G_i = \cup_{j=1}^{p_i} G_{ij},\ p_i \geq 2,$$

we say that G is a **parallel-decomposable serial graph**.

In order to study K-convergence, an important role has the class of clique-Helly graphs that have been introduced in [4, 6] and studied in [20, 21], among others. A graph is **clique-Helly** if its cliques satisfy the Helly property: each family of mutually intersecting cliques has non-trivial intersection. The Theorem 1 characterizes clique-Helly graphs [23].

Let T be a triangle of a graph G. The **extended triangle of G, relative to T,** is the subgraph $\widehat{T}$ of G induced by the vertices which form a triangle with at least one edge of T.

Theorem 1 *A graph G is clique-Helly if and only if each of its extended triangles has a universal vertex.*

In order to study K-divergence the following results are useful tools. We recall them from [16, 17] for the reader's convenience.

Let G, H be graphs. A **morphism** $\alpha : G \to H$ is a vertex-function $\alpha : V(G) \to V(H)$ such that the images under α of adjacent vertices of G either coincide or are adjacent in H. A **retraction** is a morphism α from a graph G to a subgraph H of itself such that the restriction $\alpha|H$ of α to $V(H)$ is the identity. In this case, H is a **retract** of G. Notice that, if v is a vertex of G, there is always a **total retraction** from G to v. If H is a retract of G, then $K(H)$ is a retract of $K(G)$.

The following theorem describes the relationship between retracts and K-divergence [16].

Theorem 2 *If G has a K-divergent retract H, then G is K-divergent.*

Other useful results that guarantee K-divergence relate to coaffine graphs.

A **coaffination** in a graph G is an automorphism σ of G such that for all $u \in V(G)$, $u \neq \sigma(u)$ and $\{u, \sigma(u)\} \notin E(G)$. A graph G with a fixed coaffine automorphism is called a **coaffine graph**.

Let G and H be coaffine graphs and σ_G and σ_H their coaffinations, respectively. A morphism $\alpha : G \longrightarrow H$ is **admissible** if $\alpha\sigma_G = \sigma_H\alpha$. The coaffine graphs together with admissible morphisms form a category. A subgraph H of a coaffine graph G is a **coaffine subgraph** of G if the inclusion morphism $\alpha : H \longrightarrow G$ is admissible. admissible morphism $\alpha : G \longrightarrow H$.

If G is a coaffine graph, then $K(G)$ is also a coaffine graph with a coaffination $\sigma_K : V(K(G)) \longrightarrow V(K(G))$ defined by $\sigma_K(Q) = \sigma(Q)$, where $\sigma(Q)$ is the image of Q under σ.

A coaffine graph G is **expansive** when there exists a sequence $n_1, n_2, \ldots$ of natural numbers, $n_i \longrightarrow \infty$, and a sequence $H_1, H_2, \ldots$ of coaffine graphs where H_i contains an increasing number of joined coaffine terms when $i \longrightarrow \infty$ and H_i is a coaffine subgraph of $K^{n_i}(G)$.

Note that if G is an expansive graph, then G is a K-divergent graph.

For coaffine graphs the following theorems hold [17].

Theorem 3 *Let G and $H \neq \emptyset$ be coaffine graphs. The graph G is expansive, if $K(G)$ contains $G + H$ as an induced coaffine subgraph.*

Theorem 4 *Let G be a coaffine graph and H an induced coaffine subgraph of G. If H is expansive, then also is G.*

Given a modular decomposition of a graph G the following lemmas, proved in [10], are useful for finding a retraction of G.

Lemma 5 *Let G be a graph and M a module of G. Let R be a retract of $G[M]$. Then any retraction $\rho : G[M] \to R$ can be extended to a retraction $\rho' : G \to (G - G[M]) \cup R$.*

Lemma 6 *Let G be a graph and M a module of G. Then the quotient graph G/M is a retract of G.*

Lemma 7 *Let G be a graph. If $P = S_1 \cup S_2 \cup \cdots \cup S_q$ is a parallel module of G and some S_i is a single vertex v, then $G - v$ is a retract of G.*

Finally, we recall the following result given in [10].

Theorem 8 *Let $G = G_1 + G_2 + \cdots + G_p$ be a serial graph. Then G is clique-Helly if and only if it satisfies one of the following conditions:*

1. *G has a universal vertex, or*
2. *$p = 2$ and all the connected components of G_1 and G_2 have a universal vertex.*

3 Some results about the K-behaviour of serial graphs

In Theorem 8 the clique-Helly serial graphs have been characterized. In this section we shall give some results that guarantee K-convergence of serial graphs that are not clique-Helly.

If a graph is not clique-Helly, one might wonder whether its iterated clique graph could become clique-Helly. For a graph G, define **Helly defect** of G as the smallest value i, such that $K^i(G)$ is clique-Helly.

Theorem 9 *Let $G = G_1 + \cdots + G_p$ be a serial graph. If $K(G_i)$, for some i ($1 \leq i \leq p$) has a universal vertex, then $K(G)$ has a universal vertex.*

Proof: Let Q_i^u be a universal vertex of $K(G_i)$.

Any clique Q of G is of the form: $Q = Q_1 + \cdots + Q_p$, where Q_i is any clique of G_i. Since for any Q_i belonging to G_i we have $Q_i \cap {Q_i}^u \neq \emptyset$, then we also have that $Q \cap {Q_i}^u \neq \emptyset$, for any clique Q of G. Hence any clique of the form $Q = Q_1 + \cdots + Q_p$ with $Q_i = Q_i^u$ is a universal vertex of $K(G)$.

□

Theorem 10 *Let $G = G_1 + G_2$ be a parallel-decomposable serial graph. If every $K(G_{ij})$, for any i, j ($1 \leq i \leq 2$, $1 \leq j \leq p_i$) has a universal vertex, then $K(G)$ is a clique-Helly graph.*

Proof: The cliques of G are formed by the sum of two cliques: some Q_{1j} from G_{1j} and some Q_{2l} from G_{2l}, where $1 \leq j \leq p_1$, $1 \leq l \leq p_2$. In this situation in which $Q = Q_{1j} + Q_{2l}$, let us write $j(Q) = j$ and $l(Q) = l$. Notice that $Q \cap Q' \neq \emptyset$ implies $j(Q) = j(Q')$ or $l(Q) = l(Q')$.

By hypothesis, each $K(G_{ij})$ has a universal vertex Q^u_{ij}. Then the special cliques Q^u_{ij} of G_{ij} intersects any other clique of G_{ij}. Therefore, for each pair j, l as above, the special clique $Q^u_{1j} + Q^u_{2l}$ of G intersects any clique Q of G for which $j(Q) = j$ or $l(Q) = l$.

Let $T = \{a, b, c\}$ be a triangle of $K(G)$. Then a, b, c are three pairwise intersecting cliques of G, so we have either $j(a) = j(b) = j(c)$ or $l(a) = l(b) = l(c)$. Without loss of generality, we consider $j(a) = j(b) = j(c) = j'$.

By Theorem 1 we have to show that the extended triangle $\widehat{T}$ has a universal vertex. In fact, if $l(a)$, $l(b)$ and $l(c)$ are all different, then any special clique $Q^u_{1j'} + Q^u_{2l}$ of G $(1 \leq l \leq p_2)$ is a universal vertex of $\widehat{T}$. If at least two of these indexes are equal, say $l(a) = l(b) = l'$, then the special clique $Q^u_{1j'} + Q^u_{2l'}$ is a universal vertex of $\widehat{T}$.

□

Corollary 11 *Let G be a graph satisfying the hypothesis of Theorem 9 or Theorem 10, then G is K-convergent with Helly defect at most 1.*

Let $G = G_1 + \cdots + G_p$, $p \geq 2$, be a serial graph. If some G_i is trivial, then G has a universal vertex and, by Theorem 8, G is K-convergent. Now, in Theorem 12, we give sufficient conditions to a serial graph without universal vertex to be K-divergent.

Theorem 12 *Let $G = G_1 + \cdots + G_p$, $p \geq 2$, be a serial graph without a universal vertex.*

1. *If $p \geq 3$ and G_i are parallel or C_5, then G is K-divergent.*
2. *If $p = 2$, G_1 is C_5 and G_2 is C_5 or parallel, then G is K-divergent.*

Proof: Let $G = G_1 + \cdots + G_p$, $p \geq 2$, be a serial graph without a universal vertex. In [10], it is proved that G is K-divergent if $p \geq 3$ and G_i are parallel. Thus, it is sufficient to consider $p \geq 2$ and G with at least one G_i isomorphic to C_5.

By Lemmas 5 and 6 we can retract each connected component of every parallel module G_i to a single vertex. By repeating the application of Lemma 7, we can retract each parallel G_i to $\overline{K_2}$. Then $C_5 + \cdots + \overline{K_2} + \cdots + \overline{K_2}$ is a retract of G.

Notice that $C_5+\cdots+C_5$ and $C_5+\cdots+\overline{K_2}+\cdots+\overline{K_2}$ are coaffine graphs. These graphs contain an induced coaffine subgraph C_5+C_5 and $C_5+\overline{K_2}$, respectively. If C_5+C_5 and $C_5+\overline{K_2}$ are expansive graphs then, by Theorem 4, $C_5+\cdots+C_5$ and $C_5+\cdots+\overline{K_2}+\cdots+\overline{K_2}$ are expansive graphs too. Follows that they are K-divergent and, by Theorem 2, so is G.

Now we show that $G = C_5 + C_5$ and $G = C_5 + \overline{K_2}$ are expansive graphs. Then, they are K-divergent. Let us notice that G is, in both cases, a coaffine graph. Then $K(G)$ is also coaffine.

If $G = C_5 + \overline{K}_2$, then $K(G) = \overline{C}_{10}$. In [18] it is proved that for $n \geq 8$, $\overline{C}_n$ is expansive. For the convenience of the reader, we rewrite here the proof for $n = 10$.

Let us number by $0, \ldots, 9$ the cyclic sequence of vertices of C_{10}. Let us consider the following cliques of $\overline{C}_{10}$: $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$ and the 10 cliques obtained applying the coaffine automorphism $(0, 1, 2, 3, 4$ $5, 6, 7, 8, 9)$ of $\overline{C}_{10}$ at the clique $\{0, 2, 4, 7\}$. In $K(\overline{C}_{10})$ the vertices corresponding to the above cliques induce a coaffine subgraph isomorphic to $\overline{K}_2 + \overline{C}_{10}$ and, therefore, $K(G)$ is expansive by Theorem 3. Then, G is expansive too.

Let us now consider the case $G = C_5 + C_5$. Since $\overline{K}(G) = \overline{K}(C_5) \times \overline{K}(C_5)$ and $\overline{K}(C_5) \cong C_5$, then $\overline{K}(G)$ is a regular graph of degree 4 isomorphic to graph depicted in Figure 1.

Let us denote by ij, $i = 0, \ldots, 4$, $j = 0, \ldots, 4$ the vertex set of $K(G)$. The adjacent vertices of each vertex ij of $\overline{K}(G)$ are $i(j \pm 1)$ and $(i \pm 1)j$, where from now on all the sums are taken modulo 5.

Let us consider the following sets of vertices of $K(G)$.

$$V_i = \{(i + j)(2j),\ j = 0, \ldots, 4\},\ i = 0, \ldots, 4.$$

It is easy to see that each V_i is a stable set of $\overline{K}(G)$ and the sets V_i, $i = 0, \ldots, 4$ form a partition of $V(K(G))$. Furthermore any other vertex of $V(K(G)) \setminus V_i$ is adjacent to exactly one vertex of V_i and, therefore, V_i is a maximal stable set of $\overline{K}(G)$. Hence, each V_i is a clique in $K(G)$. Such cliques do not intersect, then the graph induced by the vertices V_i of $K^2(G)$ is isomorphic

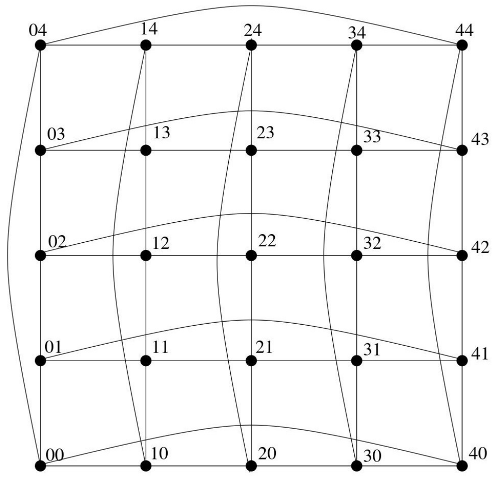

Figure 1: The graph $\overline{K}(G)$

to $\overline{K}_5$. Notice that $\overline{K}_5$ is coaffine, since any cyclic permutation is a coaffination of $\overline{K}_5$.

Let us denote by V_{ij} the set of vertices obtained from V_i by substituting the vertex $(i+j)(2j)$ of V_i by its adjacents in $\overline{K}(G)$. The correspondence between the vertices $uv = (i+j)(2j)$ of $K(G)$ and the sets V_{ij} is a bijection. In fact, for any pair of distinct vertices u and v of $K(G)$, we have $N_{\overline{K}(G)}(u) \neq N_{\overline{K}(G)}(v)$. Moreover, by definition of V_{ij}, $|V(C_5 \times C_5)| = |\cup_{i,j} V_{ij}|$.

It is easy to see that each set V_{ij} is a clique of $K(G)$. Let us consider the subgraph H of $K^2(G)$ induced by the set of vertices V_{ij}, $i = 0, \ldots, 4$ and $j = 0, \ldots, 4$. The one to one correspondence defined above is an isomorphism from $K(G)$ onto H. In fact, by construction, $\{(i{+}j)(2j), (k{+}l)(2l)\} \in E(\overline{K}(G))$ if and only if $V_{ij} \cap V_{kl} = \emptyset$. Therefore H is isomorphic to $K(G)$.

Furthermore in $K(G)$ every clique V_{ij} intersects all the cliques $\{V_i\}_{i=0,\ldots,4}$ and, therefore, $K^2(G)$ contains an induced subgraph isomorphic to $\overline{K}_5 + K(G)$.

Recall that $K(G)$ and $\overline{K}_5$ are coaffine graphs. Then so is $\overline{K}_5 + K(G)$. Hence, by Theorem 3, $K(G)$ is expansive and so is G. Therefore, the proof is complete.

□

Theorem 13 *Let $G = G_1 + G_2$ be a parallel-decomposable serial graph. If at* *least one G_{ij} is a C_5 or a serial graph whose modules are either C_5's or parallel modules, then G is K-divergent.*

Proof: Without loss of generality, let us assume that G_{11} satisfies the hypothesis.

By Lemma 6 we can retract each $G_{ij} \neq G_{11}$ to a single vertex.

By eventually repeating the application of Lemma 7 we can retract G_1 to G_{11} and G_2 to $\overline{K}_2$. Therefore, by Lemma 5, $G_{11} + \overline{K}_2$ is a retract of G. Hence G is K-divergent by Theorems 12 and 2.

□

4 P_4-reducible, P_4-sparse, extended P_4–reducible and extended P_4-sparse graphs

The purpose of this section is to characterize the K-behaviour of graphs belonging to the classes of P_4-reducible, P_4-sparse, extended P_4–reducible and extended P_4-sparse graphs. The class of P_4-sparse graphs properly contains the class of P_4-reducible graphs. The graph featured in Figure 2 is P_4-sparse graph, but not P_4-reducible.

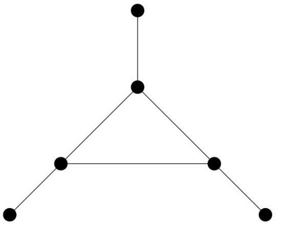

Figure 2: The graph G is P_4-sparse graph, but not P_4-reducible.

We first recall from [8] the following characterization of a P_4-reducible graph:

Theorem 14 *A graph G is a P_4-reducible if and only if for every induced subgraph H of G exactly one of the following conditions is satisfied:*

1. *H is disconnected;*

2. $\overline{H}$ *is disconnected;*

3. there exists a unique $P_4 = abcd$ *in* H *such that every vertex of* H *outside* $\{a, b, c, d\}$ *is adjacent to both* b *and* c *and non-adjacent to both* a *and* d*.*

A characterization of a P_4-sparse graph is given in [9] and it is based on a special class of graphs, the **spiders**, whose definition is as follows:

A graph G is a **spider** if the vertex set $V(G)$ admits a partition into sets S, Q and R such that:

1. S is a stable set, Q is a complete and $|S| = |Q| \geq 2$;

2. Every vertex in R is adjacent to all vertices in Q and non-adjacent to all vertices in S;

3. There exists a bijection f between S and Q such that either $N(x) = \{f(x)\}$ for $x \in S$ or $N(x) = Q - \{f(x)\}$ for $x \in S$.

Theorem 15 *A graph* G *is* P_4*-sparse if and only if for every induced subgraph* H *of* G *with at least two vertices exactly one of the following conditions are satisfied:*

1. H *is disconnected;*

2. $\overline{H}$ *is disconnected;*

3. H *is isomorphic to a spider.*

By relaxing the restriction concerning the exclusion of the chordless C_5 cycle, that is a forbidden configuration for P_4-sparse and P_4-reducible graphs, two wider classes of graphs called extended P_4-sparse and extended P_4-reducible graphs are obtained [5]. Extended P_4-sparse graphs (extended P_4-reducible graphs) differ from the P_4-sparse graphs (P_4-reducible graphs) by the presence of modules that are C_5. Hence the condition 3 of Theorem 15 can be changed by condition 3': H is isomorphic to a spider or a C_5, when G is a connected extended P_4-sparse graph.

Theorem 16 *Let* G *be a connected extended* P_4*-sparse graph with no universal vertex. Then* G *is* K*-divergent if and only if* G *is a serial graph and it satisfies one of the following conditions:*

1. *$p \geq 3$ and each G_i is a parallel module or a C_5.*

2. *$p = 2$, G_1 is a C_5 and G_2 is a C_5 or a parallel module.*

3. *$p = 2$, G is a parallel-decomposable serial graph and at least one G_{ij} is a C_5 or a serial graph whose modules are either C_5's or parallel modules.*

Proof: Sufficiency follows by Theorems 12 and 13.

To prove necessity we show that in all the other cases G is K-convergent. Let G be a connected extended P_4-sparse graph with no universal vertex.

If $\overline{G}$ is also a connected graph, then by condition 3', G is a spider or a C_5. In the first case, $K(G)$ has a universal vertex. In the second case, $K(C_5) = C_5$. In both cases G is K-convergent.

If $\overline{G}$ is a disconnected graph, then G is a serial graph. If G is not parallel-decomposable serial graph and none of its modules is a C_5, then at least one G_i is either a trivial graph or a spider. Then, by Theorem 9, we have in both cases that $K(G)$ has a universal vertex and, by Corollary 11, G is K-convergent.

The only remaining case is when G is a parallel-decomposable serial graph with $p = 2$ and no G_{ij} is either a C_5 or a serial graph whose modules are either C_5's or parallel modules. Hence each G_{ij} is either a trivial graph or a spider or a serial graph with at least one module that is a spider or it has a universal vertex. In all cases $K(G_{ij})$ contains a universal vertex and $K(G)$ is a clique-Helly graph, by Theorem 10, and by Corollary 11, G is K-convergent.

□

Since extended P_4-reducible graphs are extended P_4-sparse graphs, the Theorem 16 is also true for connected extended P_4-reducible graphs.

By excluding the presence of C_5 modules we obtain the following characterization of the K-behaviour of P_4-sparse graphs.

Corollary 17 *Let G be a connected P_4-sparse graph. Then G is K-divergent if and only if G is a serial graph and it satisfies one of the following conditions:*

1. *$p \geq 3$ and G is a parallel-decomposable serial graph.*

2. *$p = 2$, G is a parallel-decomposable serial graph and at least one G_{ij} is a parallel-decomposable serial graph.*

It easy to see that Corollary 17 is also true for P_4-reducible graphs. In fact, a P_4-reducible graph is a P_4-sparse graph with $|S| = |Q| = 2$ (see Theorems 14 and 15).

Corollary 18 *A connected extended P_4-sparse graph is K-convergent if and only if the Helly-defect of G is at most 1.*

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