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REPRODUCTIVE WEAK SOLUTIONS FOR GENERALIZED BOUSSINESQ MODELS IN EXTERIOR DOMAINS *

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Abstract

We established the existence of reprodutive weak solutions of a generalized Boussinesq model for thermally driven convection in an exterior domain.

1 Introduction

The Boussinesq system of hydrodynamics equations (see Joseph [7], Chandrassenkhar [1]) arise from zero order approximation to the coupling between the Navier-Stokes equation and the thermodynamic equation. Usually it is assumed that the viscosity and the thermal conductivity are positive constants. There are some physical motivations for considering fluid equations with viscosity and thermal conductivity which are temperature dependent. For instance, the experiments done by von Tippelkirch [29] confirmed these facts. A mathematical model for the case that the viscosity and heat conductivity are temperature dependent are given by Drazin and Reid [2]. Such a mathematical model reads: Find the field $\mathbf{u}:\Omega\times(0,\infty)\to I\!R^3$, the scalar functions $(\theta,p):\Omega\times(0,\infty)\to I\!R^2$ which satisfy the system of equations:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \operatorname{div}(\nu(\theta)\nabla\mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\theta\mathbf{g} + \nabla p &= 0, \text{ in } \widehat{\Omega} \\ \operatorname{div}\mathbf{u} &= 0, \text{ in } \widehat{\Omega} \qquad (1) \\ \frac{\partial\theta}{\partial t} - \operatorname{div}(k(\theta)\nabla\theta) + \mathbf{u}\cdot\nabla\theta &= 0, \text{ in } \widehat{\Omega} \end{aligned}$$

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where,

- $\widehat{\Omega} = \Omega \times (0,\infty)$, $\mathbf{u}(x,t) \in I\!R^3$ denotes the velocity of the fluid at point $x \in \Omega$ and time $t \in [0,\infty)$;
- $\theta(x,t) \in I\!R$, denotes the temperatura;
- $p(x,t) \in I\!R$ denotes the hydrostatic pressure;
- $\mathbf{g}(t,x) \in I\!R^3$ is a gravitational field;
- $\nu(\cdot)$ is the kinematic viscosity;
- $k(\cdot)$ is thermal conductivity;
- $\alpha > 0$ is a positive constant associated to the coefficient of volume expansion.

Without loss of generality, we have considered the reference temperature as zero.

The symbols ∇, Δ and div denote the gradient, Laplacian and divergence operators, respectively.We also denote $\dfrac{\partial \mathbf{u}}{\partial t}$ by $\mathbf{u}_t$. The i^{th} component of $\mathbf{u}\cdot\nabla\mathbf{u}$ is given by $[(\mathbf{u}\cdot\nabla)\mathbf{u}]_i = \sum_j u_j \dfrac{\partial u_i}{\partial x_j}$ and $\mathbf{u}\cdot\nabla\theta = \sum_j u_j \dfrac{\partial \theta}{\partial x_j}$.

The first equation in problem (1) corresponds to the balance of linear momentum; the second one says that fluid is incompressible and the third equation is the balance of temperature.

This model was considered by Lorca and Boldrini in bounded domains with Dirichlet boundary conditions. In [15] is studied the stationary model associated to (1). In [13], [14] the evolution model is considered, here they showed the existence of weak, strong and uniqueness of solutions. The argument used is the spectral Galerkin method, and the results obtained are similary to the classical Navier-Stokes equations. In [20] is given a new proof of the strong solutions for the problem (1), but an interative argument is used. In [21] are given results of the existence weak solutions for the stationary model in exterior domains. The authors used the embedding method due to Ladyzhenskaya [11] (see also Heywood [4]).

A study of problem (1) in exterior domains not has been done. Thus, our main purpose in this work is to show the existence of weak solutions in exterior domains. More precisely, we consider the following situation: The study of the dynamics of the generalized Boussinesq model in an exterior domain plays an important and useful role. We often find physical structures in which a bounded body or obstacle produces perturbation in the surrounding medium and the spatial volume of the external environment, namely the exterior domain, is extensively much larger than the obstacle. From the modeling point of view, the obstacle may be regarded as a compact domain located in all of $I\!R^3$. Let K denote this compact subset, and let Ω denote its complement in $I\!R^3$, that is, $\Omega = K^c$. We assume that the following boundary conditions holds

$$\mathbf{u}(x,t) = 0,\ x \in \Gamma,\ t \in [0,\infty) \tag{2}$$

$$\theta(x,t) = \mu > 0,\ x \in \Gamma,\ t \in [0,\infty), \tag{3}$$

where $\mu(\cdot,\cdot)$ is a given function on Γ.

To complete the system of equations, we prescribe the behaviour of the solutions at infinity. More, precisely, we consider the classical homogeneous decay

$$\lim_{|x|\to\infty} \mathbf{u}(x,t) = 0, \quad \lim_{|x|\to\infty} \theta(x,t) = 0,\ \forall\, t > 0. \tag{4}$$

Let $(\mathbf{u},\theta)$ be a weak solution of problem (1)-(4) (the precise definition will be given later on). Given $T > 0$, if there exits $(\mathbf{u}_0,\theta_0)$ such that functions $\mathbf{u}$ and θ satisfy

$$\mathbf{u}(x,T) = \mathbf{u}_0,\ \theta(x,T) = \theta_0,\ \text{a.e in } \Omega. \tag{5}$$

Then, we call $(\mathbf{u},\theta)$ a *reproductive weak solution* of the problem (1)-(4) at time T. We say that the problem (1)-(4) has the *reproductive property* if it is reproductive at every $T > 0$ (see Kaniel and Shinbrot [8] or Takeshita [27], for the case of the Navier-Stokes equations). We observe that the above property is a generalization of the notion of periodicity in the sense that any peridoc solution (in time) is a reproductive solution, but the converse is not necessary true, unless if we have the uniqueness of solution.

It is known that certain dynamical system may not have periodic solutions because there exist many orbits, or branches of bifurcations, that can be randomly reached by the solution (e.g. see [17]). However, several of these systems are still of the *reproductive type,* in the sense that there exist at leats two diferent times where the solution takes the same value. We observe that the problem of the existence of periodic (in time) solutions to (1) not has been studied. In fact, this question is not easy because the presence of the nonlinear operators $\mathrm{div}(\nu(\theta)\nabla\mathbf{u})$ and $\mathrm{div}(k(\theta)\nabla\theta)$, the arguments used in the classical Navier-Stokes equations are not can directly applied, because is necessary to guarantee the existence of periodic solution for the Galerkin system, in the case of the classical Navier-Stokes equations that is easily done, because in the ordinary differential system the operator is independent of t, see more details in [9] or in [16].

We would like to say that the arguments that we will use in the proof work for bounded domains and also to prove the existence of weak solutions of the initial boundary value problem in exterior domains.

When ν and κ are positive constants, the reduced model was discussed by many authors, see for instance Korenev [10], Rojas-Medar and Lorca [25], [26] (in a bounded domain), Morimoto [19], Hishida [6], $\overline{O}$eda [22], [23], [24] (in exterior domain), Moretti et al [18] (in unbounded domains).

This paper is organized as follows. In Section 2, we gave the preliminaries results used throughout the paper and we stablished the main result of this work. In Section3, we studied an auxiliary problem. Finally, in Section 4, we gave the proof of our main result.

2 Preliminaries

The functions in this paper are either $I\!R$ or $I\!R^3$ valued, and we will not distinguish these two situations in our notation, since that will be clear from the context. The extending domain method was introduced by Ladyzhenskaya [11] to study the Navier-Stokes equations in unbounded domains. As observed by Heywood [4] the method is useful in certain class of unbounded domains. Certainly, our domain is in this class. The basic idea is the following: The exterior domain Ω can be approximated by bounded domains $\Omega_m = B_m \cap \Omega$, where B_m is a ball with radius m and center at 0, as $m \to \infty$. In each bounded domain Ω_m, we will prove the existence of a weak solution, by using the Galerkin method together with the Brouwer's fixed point theorem as in Heywood [4]. Next, by using the estimates given in Ladyzhenskaya's book [11] together with diagonal argument and Rellich's compactness theorem, we obtain a desirable weak solution to problem (1) satisfying conditions (2) through (5).

Let D denote Ω or Ω_m, $\widehat{D} = D \times [0,T]$ and $\widehat{D\cup\Gamma} = (D\cup\Gamma)\times[0,T]$. And, consider the following notation

$$\begin{aligned}
W^{r,p}(D) &= \{\mathbf{u}\ ;\ D^\alpha \mathbf{u} \in L^p(D), |\alpha| \le r\},\\
W_0^{r,p}(D) &= \text{Completion of } C_0^k(D) \text{ in } W^{r,p}(D),\\
C_{0,\sigma}^\infty(D) &= \{\mathbf{v} \in C_0^\infty(D)\ ;\ \operatorname{div} \mathbf{v} = 0\},\\
J(D) &= \text{Completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\nabla\phi\|,\\
H(D) &= \text{Completion of } C_{0,\sigma}^\infty(D) \text{ in norm } \|\phi\|,\\
\widehat{W}_\sigma(\widehat{D}) &= \{\varphi \in C_0^\infty(\widehat{D}); \operatorname{div}\ \varphi = 0\},\\
\widehat{W}(\widehat{D}) &= \{\psi \in C_0^\infty(\widehat{D\cup\Gamma});\ \varphi(\Gamma) = 0\},\\
\widehat{W_{\sigma,\pi}}(\widehat{D}) &= \{\varphi \in C_{0,\sigma}^\infty(\widehat{D});\ \varphi(x,T) = \varphi(x,0)\},\\
\widehat{W}_\pi(\widehat{D}) &= \{\psi \in \widehat{W}(\widehat{D});\ \psi(x,T) = \psi(x,0)\},\\
L_\pi^p(0,T;J(\Omega_k)) &= \{\mathbf{u} \in L_\pi^p(0,T;J(\Omega_k));\ \mathbf{u}(x,T) = \mathbf{u}(x,0)\ x \in \Omega_k \text{ a.e.}\},\\
L_\pi^p(0,T;H_0^1(\Omega_k)) &= \{\mathbf{w} \in L_\pi^p(0,T;H_0^1(\Omega_k));\ \mathbf{w}(x,T) = \mathbf{w}(x,0)\ x \in \Omega_k \text{ a.e.}\},\\
L_\pi^p(0,T;L^6(\Omega_k)) &= \{\mathbf{f} \in L_\pi^p(0,T;L^6(\Omega_k));\ \mathbf{f}(x,T) = \mathbf{f}(x,0)\ x \in \Omega_k \text{ a.e.}\}.
\end{aligned}$$

The norm $\|\cdot\|$ is the L^2-norm and $\|\cdot\|_p$ denotes the L^p-norm for $1 \le p \le \infty$. We observe that $J(D)$ is equivalent to

$$\{\phi \in W^{1,2}(D)\ ;\ \phi|_{\partial\Omega} = 0,\ \operatorname{div}\phi = 0\},$$

as was proved by Heywood[5]. When $p = 2$, as it usual, we denote $W^{r,p}(D) \equiv H^r(D)$ and $W_0^{r,p}(D) \equiv H_0^r(D)$. We make use of some inequalities with constants that depend only on the dimension and are independent of the domain (see [11] chapter I).

Lemma 1 *Suppose the space dimension is 3, with D bounded or unbounded. Then*

(a) For $\mathbf{u} \in W_0^{1,2}(D)$ (or $J(D)$ or $H_0^1(D)$), we have

$$\|\mathbf{u}\|_{L^6(D)} \le C_L \|\nabla \mathbf{u}\|_{L^2(D)}$$

where $C_L = (48)^{1/6}$.

(b) (Hölder's inequality). If each integral makes sense. Then we have

$$|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})| \le 3^{\frac{1}{p}+\frac{1}{r}} \|\mathbf{u}\|_{L^p(D)} \|\nabla\mathbf{v}\|_{L^q(D)} \|\mathbf{w}\|_{L^r(D)}$$

where $p, q, r0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Lemma 2 *Suppose that D is a bounded domain in $I\!R^n$ and its boundary $\partial\Omega$ is of class C^2. Let us take an orthonormal basis $\{\omega^j\}_{j=1}^{\infty}$ of $L^2(D)$. Then for any $\varepsilon > 0$, there exists a number N_ε such that*

$$\|\mathbf{u}\|^2_{L^2(D)} \le \sum_{j=1}^{N_\varepsilon} (\mathbf{u}, \omega^j)^2 + \varepsilon \|\mathbf{u}\|^2_{W^{1,m}} \text{ for all } \mathbf{u} \in W_0^{1,m}(D), \tag{6}$$

where $m\frac{2n}{n+2}$ $(n \ge 2)$, $m \ge 1$ $(n = 1)$ and N_ε is independent of $\mathbf{u}$. The following assumptions will be needed throughout this paper.

(A1) $w_0 \subset K$ (w_0 is a neighborhood of the origin 0) and $K \subseteq B = B(0, d)$ which is a ball with radius d and center at 0.

(A2) $\partial\Omega = \Gamma = \partial K \in C^2$.

(A3) $\mathbf{g}(x)$ is a bounded and continuous vector function in $I\!R^3 \backslash w_0$. Moreover $\mathbf{g} \in L^p(\Omega)$ for $p \ge 6/5$.

(A4) $\mu \in C^2(\Gamma \times [0, \infty))$ and is periodic with respect to t with period T.

We assume that the functions $\nu(\cdot)$ and $\kappa(\cdot)$ satisfy

$$\begin{aligned} 0 &< \nu_0 \le \nu(\tau) \le \nu_1, \\ 0 &< \kappa_0 \le \kappa(\tau) \le \kappa_1 \end{aligned}$$

for all $\tau \in I\!R$, where ν_0, ν_1, κ_0 and κ_1 are constants, and $\nu(\cdot)$ and $\kappa(\cdot)$ are continuous functions. To transform the boundary conditions on θ to a homogeneous boundary condition, we introduce an auxiliar function S (see Gilbarg and Trudinger [3] pp. 137).

Lemma 3 *There exists a function S which satisfies the following properties*

(i) $S = \mu$ on $\Gamma \times [0, \infty)$,

(ii) $S \in C_0^2(I\!R_x^2)$ for any fixed t and S, S_t are continuous for $t \in [0, T]$,

(iii) S is periodic in t with period T,

(iv) $\forall\ \varepsilon > 0$ and $p > 1$, we can redefine S, if necessary, such that $\sup_{t\in[0,T]} ||S(t)||_{L^p} < \varepsilon$.

Now, making $\varphi = \theta - S$ we obtain

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \operatorname{div}(\nu(\varphi + S)\nabla \mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\varphi\mathbf{g} - \alpha S\mathbf{g} + \nabla p &= 0,\\
\operatorname{div}\,\mathbf{u} &= 0, \\
\frac{\partial \varphi}{\partial t} - \operatorname{div}(\kappa(\varphi + S)\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi - \operatorname{div}(\kappa(\varphi+S)\nabla S) + \\
\mathbf{u}\cdot\nabla S + \frac{\partial S}{\partial t} &= 0,
\end{aligned} \tag{7}$$

in Ω, with boundary conditions

$$\begin{aligned}
&\mathbf{u} = 0 \text{ and } \varphi = 0 \text{ on } \partial\Omega,\\
&\lim_{|x|\to\infty} \mathbf{u}(t,x) = 0\,; \quad \lim_{|x|\to\infty} \varphi(t,x) = 0.
\end{aligned} \tag{8}$$

In what follows, we will concentrate our analysis on (7)-(8), instead (1)-(4).

Now, we can define precisely the notion of a reproductive weak solution for the whole system (7)-(8).

Definition 1 *Let $T > 0$. We say that the dupla of functions $(\mathbf{u},\varphi)$, defined on $\Omega \times (0,T)$, is a reproductive weak solution of (7)-(8) at time T if and only if there exists $(\mathbf{u}_0,\varphi_0) \in H(\Omega) \times L^2(\Omega)$ such that*

i) $\mathbf{u}(x,T) = \mathbf{u}_0(x), \qquad \varphi(x,T) = \varphi_0(x) \qquad$ a.e. in Ω,
ii) $\mathbf{u} \in L^2(0,T;J(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega))$,
iii) $\varphi \in L^2(0,T;H^1_0(\Omega)) \cap L^2_\pi(0,T;L^6(\Omega))$,
iv) $\mathbf{u}$ and φ satisfy the variational equations:

$$\begin{aligned}
\int_0^T \{(\mathbf{u},\mathbf{v}_t) + (\nu(\varphi+S)\nabla\mathbf{u},\nabla\mathbf{v}) + B(\mathbf{u},\mathbf{v},\mathbf{u}) - \\
\alpha(\varphi\mathbf{g},\mathbf{v}) - \alpha(S\mathbf{g},\mathbf{v})\}dt &= 0,\\
\int_0^T \{(\varphi,\psi_t) + (\kappa(\varphi+S)\nabla\varphi,\nabla\psi) + b(\mathbf{u},\psi,\varphi) + \\
(\kappa(\varphi+S)\nabla S,\nabla\psi) + b(\mathbf{u},\psi,S) + (\frac{\partial S}{\partial t},\psi)\}dt &= 0
\end{aligned}$$

for all $v \in \widehat{D}_{\sigma,\pi}(\widehat{\Omega})$ and all $\psi \in \widehat{D}_{\pi}(\widehat{\Omega})$. Where

$$B(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) = \int_\Omega \sum_{i,j=1}^{3} u_j(t,x)(\partial v_i/\partial x_j)(t,x)w_i(t,x)\,dx\,,$$

$$b(\mathbf{u},\varphi,\psi) = (\mathbf{u}\cdot\nabla\varphi,\psi) = \int_\Omega \sum_{i,j=1}^{3} u_j(t,x)(\partial \varphi_i/\partial x_j)(t,x)\psi_i(t,x)\,dx\,.$$

Remark 1 *It is important to note that:*

i) We obtain in the proof of the theorem more regularity of the solution. In fact, we prove that
$(\mathbf{u},\varphi) \in L^\infty_\pi(0,T;H(\Omega)) \times L^\infty_\pi(0,T;L^2(\Omega))$.
ii)As $\mathbf{u}(\cdot,t) \in J(\Omega)$ and $\varphi(\cdot,t) \in H^1_0(\Omega)$ a.e. in $(0,T)$, we have

$$\mathbf{u}|_{\partial\Omega} = 0\ ;\ \varphi|_{\partial\Omega} = 0\ ,\ \text{a.e in } (0,T).$$

iii) By part (a) of Lemma 1,

$$\lim_{|x|\to\infty} \mathbf{u}(x,t) = 0, \quad \lim_{|x|\to\infty} \varphi(x,t) = 0 \text{ a.e. in } (0,T).$$

iv) We also see that the pressure is recovered by a standar application of De Rham's Theorem.

Theorem 1 *(Existence) Under Assumptions (A1), (A2), (A3) and (A4), there exists a weak reproductive solution for problem (7) and (8).*

3 Auxiliary problem.

Following the extending domain method, we first present a lemma which ensures the existence of weak solutions for interior problems in domains $\Omega_m = B_m \cap \Omega$. A interior problem, P_m, is stated as follows:

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} - \text{div}\,(\nu(\eta+S)\nabla\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v} - \alpha\eta\mathbf{g} - \alpha S\mathbf{g} + \nabla p &= 0,\\
\text{div}\,\mathbf{v} &= 0,\\
\frac{\partial \eta}{\partial t} - \text{div}\,(k(\eta+S)\nabla\eta) + \mathbf{v}\cdot\nabla\eta - \text{div}\,(k(\eta+S)\nabla S) + \mathbf{v}\cdot\nabla S + \frac{\partial S}{\partial t} &= 0,
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} &= 0, \text{ on } \partial\Omega_m = \partial\Omega\cap\partial B_m,\\
\eta &= 0 \text{ on } \partial\Omega_m = \partial\Omega\cap\partial B_m,
\end{aligned}$$

$$\mathbf{v}(\cdot,T) = \mathbf{v}(\cdot,0),\ \eta(\cdot,T) = \eta(\cdot,0).$$

Definition 2 $(\mathbf{v},\eta) \in (L^2(0,T;J(\Omega_m)) \cap L^2_\pi(0,T;L^6(\Omega))) \times (L^2(0,T;H^1_0(\Omega_m)) \times L^2_\pi(0,T;L^6(\Omega)))$ *is called a reproductive weak solution for* (P_m) *if it satisfies*

$$\begin{aligned} \int_0^T \{(\mathbf{v},\mathbf{w}_t) + (\nu(\eta+S)\nabla\mathbf{v},\nabla\mathbf{w}) + B(\mathbf{v},\mathbf{w},\mathbf{v}) - & \\ \alpha(\eta\mathbf{g},\mathbf{w}) - \alpha(S\mathbf{g},\mathbf{w})\}dt &= 0, \\ \int_0^T \{(\eta,\psi_t) + (\kappa(\eta+S)\nabla\eta,\nabla\psi) + b(\mathbf{v},\psi,\eta) + (\kappa(\eta+S)\nabla S,\nabla\psi) + & \\ b(\mathbf{v},S,\psi) + (S,\psi_t)\}dt &= 0 \end{aligned}$$

for all $\mathbf{w} \in \widehat{D}_{\sigma,\pi}(\widehat{\Omega}_m)$*, and for all* $\psi \in \widehat{D}_\pi(\widehat{\Omega}_m)$.

Lemma 4 *Under Assumptions (A1), (A2), and (A3) there exists a weak solution* $(\overline{\mathbf{u}}^m,\overline{\eta}^m)$ *of* (P_m).

To prove the existence of reproductive weak solutions for the system (P_m) we use the Galerkin method together with Brouwer's fixed point theorem as in Lions [12](see also Heywood [4]). First, we prove the a priori estimates for weak solutions of (P_m).

Lemma 5 *Let* $(\mathbf{v}^m,\eta^m)$ *a weak solution of* (P_m)*. Then, they satisfy the following estimate*

$$\frac{d}{dt}(\|\mathbf{v}^m\|^2 + \|\eta^m\|^2) + \frac{\nu_0}{2}(\gamma - \frac{9C_L^2}{k_0}\|S\|_3^2)\|\nabla\mathbf{v}^m\|^2 + \frac{k_0}{2}\gamma\|\nabla\eta^m\|^2 \le f(t), \quad (9)$$

where $\gamma = 1 - 3\alpha C_L^2/\sqrt{k_0\nu_0}\|\mathbf{g}\|^2_{\frac{3}{2}}$ *and* $f(t) = 9C_L^2/k_0\|S_t\|^2 + k_1^2/k_0\|\nabla S\|^2 + 9\alpha^2C_L^2/2\nu_0\|\mathbf{g}\|^2\|S\|_3^2$.

Proof. Multiplying $(P_m)_i$ and $(P_m)_{iii}$ by $\mathbf{v}^m$ and η^m, respectively, after on integrate on Ω_m, we get

$$\begin{aligned} \frac{d}{dt}\|\mathbf{v}^m\|^2 + (\nu(\eta^m+S)\nabla\mathbf{v}^m,\nabla\mathbf{v}^m) &= (\alpha\eta^m\mathbf{g},\mathbf{v}^m) + (\alpha S\mathbf{g},\mathbf{v}^m), \\ \frac{d}{dt}\|\eta^m\|^2 + (k(\eta^m+S)\nabla\eta^m,\nabla\eta^m) &= -(\mathbf{v}^m\cdot\nabla S,\eta^m) - (S_t,\eta^m) - \\ & \quad (k(\eta^m+S)\nabla S,\nabla\eta^m). \end{aligned} \quad (10)$$

Now, we estimate the right-hand sides of the above equalities by using the Lemma 1

$$\begin{aligned}(\alpha\eta^m\mathbf{g},\mathbf{v}^m) &\leq 3\alpha\|\mathbf{g}\|_{\frac{3}{2}}\|\eta^m\|_6\|\mathbf{v}^m\|_6,\\ (\alpha S\mathbf{g},\mathbf{v}^m) &\leq 3\alpha\|\mathbf{g}\|\|S\|_3\|\mathbf{v}^m\|_6,\\ (\mathbf{v}^m\cdot\nabla S,\eta^m) &= (\mathbf{v}^m\cdot\nabla\eta^m,S)\leq 3\|\mathbf{v}^m\|_6\|\nabla\eta^m\|\|S\|_3,\\ (S_t,\eta^m) &\leq 3\|S_t\|_{\frac{6}{5}}\|\eta^m\|_6,\\ (k(\eta^m+S)\nabla S,\nabla\eta^m) &\leq k_1\|\nabla\eta^m\|\|\nabla S\|.\end{aligned}\tag{11}$$

Observe that

$$\begin{aligned}(\nu(\eta^m+S)\nabla\mathbf{v}^m,\nabla\mathbf{v}^m) &\geq \nu_0\|\nabla\mathbf{v}^m\|^2,\\ (k(\eta^m+S)\nabla\eta^m,\nabla\eta^m) &\geq k_0\|\nabla\eta^m\|^2,\end{aligned}\tag{12}$$

the estimates (10) and (11) plus the inequalities (12) imply

$$\begin{aligned}\frac{d}{dt}\|\mathbf{v}^m\|^2+\nu_0\|\nabla\mathbf{v}^m\|^2 &\leq 3\alpha\|\mathbf{g}\|_{\frac{3}{2}}\|\eta^m\|_6\|\mathbf{v}^m\|_6+3\alpha\|\mathbf{g}\|\|S\|_3\|\mathbf{v}^m\|_6,\\ \frac{d}{dt}\|\eta^m\|^2+k_0\|\nabla\eta^m\|^2 &\leq 3\|\mathbf{v}^m\|_6\|\nabla\eta^m\|\|S\|_3+3\|S_t\|_{\frac{6}{5}}\|\eta^m\|_6+k_1\|\nabla\eta^m\|\|\nabla S\|.\end{aligned}$$

The estimate (a) in Lemma 1 implies

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{v}^m\|^2+\|\eta^m\|^2)+\nu_0\|\nabla\mathbf{v}^m\|^2+k_0\|\nabla\eta^m\|^2\\ \leq\ &\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}(\frac{k_0}{2}\|\nabla\eta^m\|^2+\frac{\nu_0}{2}\|\nabla\mathbf{v}^m\|^2)+\frac{9\alpha^2C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2\\ &+\frac{\nu_0}{2}\|\nabla\mathbf{v}^m\|^2+\frac{9C_L^2}{k_0}\|S\|_3^2\|\nabla\mathbf{v}^m\|^2+\frac{3k_0}{4}\|\nabla\eta^m\|^2\\ &+\frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}}+\frac{k_1^2}{k_0}\|\nabla S\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}(\|\mathbf{v}^m\|^2+\|\eta^m\|^2)+\frac{\nu_0}{2}(1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}-\frac{9C_L^2}{k_0}\|S\|_3^2)\|\nabla\mathbf{v}^m\|^2+\\ &+\frac{k_0}{2}(1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}})\|\nabla\eta^m\|^2\\ \leq\ &\frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}}+\frac{k_1^2}{k_0}\|\nabla S\|^2+\frac{9\alpha^2C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2.\end{aligned}$$

We put $\gamma=1-\frac{3\alpha C_L^2}{\sqrt{k_0\nu_0}}\|\mathbf{g}\|_{\frac{3}{2}}$ and $f(t)=\frac{9C_L^2}{k_0}\|S_t\|_{\frac{6}{5}}+\frac{k_1^2}{k_0}\|\nabla S\|^2+\frac{9\alpha^2C_L^2}{2\nu_0}\|\mathbf{g}\|^2\|S\|_3^2$. This proves Lemma 5.

Proof of the Lemma 5 Now, we prove the existence of the solution $(\mathbf{v}^m, \eta^m)$ for (P_m). Let m be arbitrarily fixed. Let $\{e^i(x)\}_{i=1}^{\infty} \subseteq C_{0,\sigma}^{\infty}(\Omega_m)$ (respec. $\{\phi^i(x)\}_{i=1}^{\infty}$ $C_0^{\infty}(\Omega_m)$) be a sequence of functions orthonormal in $L^2(\Omega_m)$ and total in $J(\Omega_m)$ (respec. $H_0^1(\Omega_m)$). As k^{th} approximate solution of (P_m), we choose the functions

$$\mathbf{v}^k(t,x) = \sum_{j=1}^{k} c_{kj}(t) e^j(x)\,, \quad \eta^k(t,x) = \sum_{j=1}^{k} d_{kj}(t)\phi^j(x)\,.$$

which satisfy the equations

$$\begin{aligned} (\mathbf{v}_t^k, \varphi^j) + (\nu(\eta^k + S)\nabla \mathbf{v}^k, \nabla \varphi^j) + & \\ + B(\mathbf{v}^k, \mathbf{v}^k, \varphi^j) - \alpha(\eta^k \mathbf{g}, \varphi^j) - \alpha(S\mathbf{g}, \varphi^j) &= 0 \\ (\eta_t^k, \phi^j) + (\kappa(\eta^k + S)\nabla \eta^k, \nabla \phi^j) + b(\mathbf{v}^k, \eta^k, \phi^j) + & \\ + (\kappa(\eta^k + S)\nabla S, \nabla \phi^j) + b(\mathbf{v}^k, S, \phi^k) &= 0\,, \end{aligned}$$

for $1 \le j \le k$. Note that the solutions $(\mathbf{v}^k, \eta^k)$ must satisfy the estimate (9). Thus, we have

$$\frac{d}{dt}(\|\mathbf{v}^k\|^2 + \|\eta^k\|^2) + M(\|\nabla \mathbf{v}^k\|^2 + \|\nabla \eta^k\|^2) \le f(t),$$

where

$$M = \min\{\frac{\nu_0}{2}(\gamma - \frac{9C_L^2}{k_0}\|S\|_3^2), \frac{k_0}{2}\gamma).$$

Let d_m be the diameter of Ω_m. Making use of Poincaré inequality, we obtain

$$\frac{d}{dt}(\|\mathbf{v}^k\|^2 + \|\eta^k\|^2) + \lambda_m(\|\mathbf{v}^k\|^2 + \|\eta^k\|^2) \le f(t)$$

where $\lambda_m = \frac{2M}{d_m^2}$. Or equivalently,

$$\frac{d}{dt} e^{\lambda_m t}(\|\mathbf{v}^k\|^2 + \|\eta^k\|^2) \le e^{\lambda_m t} f(t).$$

Integrating from 0 to T, we get

$$e^{\lambda_m T}(\|\mathbf{v}^k(T)\|^2 + \|\eta^k(T)\|^2) \le \|\mathbf{v}^k(0)\|^2 + \|\eta^k(0)\|^2 + \int_0^T e^{\lambda_m t} f(t)dt. \tag{13}$$

We denote by $z^k(t)$ the vector $(\mathbf{v}^k, \eta^k)$ and $\|z^k(t)\|^2 = \|\mathbf{v}^k(t)\|^2 + \|\eta^k(t)\|^2$. With this notation, the above inequality is rewritten as

$$e^{\lambda_m T}\|z^k(T)\|^2 \le \|z^k(0)\|^2 + \int_0^T e^{\lambda_m t} f(t)dt.$$

Now, let us define the mapping $L^k : [0,T] \to I\!R^{2k}$ as

$$L^k(t) = (c_{1k}(t), ..., c_{kk}(t), d_{1k}(t), ..., d_{kk}(t))$$

where $c_{ik}(t), d_{ik}(t)$, $i = 1, ..., k$ are respectively the coefficient of the expansion of $\mathbf{v}^k(t)$ and $\eta^k(t)$, as defined before.

Keeping on mind that

$$\|L^k(t)\|_{R^{2k}} = \|z^k(t)\|, \tag{14}$$

since we have chosen the basis $\{e^i(x)\}_{i=1}^{\infty}$ and $\{\phi^i(x)\}_{i=1}^{\infty}$ to be orthonormal in $(L^2(\Omega_m))^n$.

Now, we define the mapping $\Phi^k : I\!R^{2k} \to I\!R^{2k}$ as follows: given $L_0 \in I\!R^{2k}$ and define $\Phi^k(L_0) = L^k(T)$, where $L^k(t)$ corresponds to the solution of problem finito-dimensional (Galerkin system) with initial value corresponding to L_0. It is easy to see that Φ^k is continuous. We want to prove that Φ^k has a fixed point. As a consequence of fixed point theorem of Brouwer, it is enough to prove that for any $\lambda \in [0,1]$, a possible solution of the equation

$$L_0^k(\lambda) = \lambda \Phi^k(L_0^k(\lambda)) \tag{15}$$

is bounded independent by λ. Since $L_0^k(0) = 0$, by (15), it is enough to prove this fact for $\lambda \in (0,1]$. In this case, (15) is equivalent to $\Phi^k(L_0^k(\lambda)) = L_0^k(\lambda)/\lambda$. By definition of Φ^k and condition (14), inequality (13) implies that

$$e^{\lambda_m T}\|L_0^k(\lambda)/\lambda\|^2_{I\!R^{2k}} \le \|L_0^k(\lambda)\|^2_{I\!R^{2k}} + \int_0^T e^{\lambda_m t} f(t)dt,$$

which yields

$$\|L_0^k(\lambda)\|^2_{I\!R^{2k}} \le \frac{\int_0^T e^{\lambda_m t}(f(t)dt}{e^{\lambda_m T} - 1} = N, \tag{16}$$

since $\lambda \in (0,1]$. This bound is independent of $\lambda \in [0,1]$ and, therefore, Φ^k has a fixed point $L_0^k(1)$ satisfying the same bound as (16). This corresponds to the existence of a solution $\mathbf{v}^k(t), \eta^k(t))$ of (P_1) satisfying $\mathbf{v}^k(0) = \mathbf{v}^k(T)$, and $\eta^k(0) = \eta^k(T)$, that is a reproductive approximated solution.

Moreover, $\|\mathbf{v}^k(0)\| + \|\eta^k(0)\|^2 = \|L_0^k(1)\|^2_{I\!R^{2k}} \le N$, which is also independent of k . On the other hand, from (9) we have

$$\begin{aligned} &\|\mathbf{v}^k(t)\|^2 + \|\eta^k(t)\|^2 + M\int_0^t ((\nabla \mathbf{v}^k, \nabla \mathbf{v}^k) + a(\nabla \eta^k, \nabla \eta^k))ds \\ \le\ & \int_0^T f(t)dt + \|\mathbf{v}^k(0)\|^2 + \|\eta^k(0)\|^2 \\ \le\ & N(f) + N, \end{aligned} \tag{17}$$

for $k \geq 1$, where $N(f) = \int_0^T f(t)dt$. Moreover, the sequence $(\mathbf{v}^k, \eta^k)$ is bounded in $L^2(0,T;J(\Omega_m)) \times L^2(0,T;H_0^1(\Omega_m))$ and in $L_\pi^\infty(0,T;H(\Omega_m) \times L_\pi^\infty(0,T;L^2(\Omega_m))$. Since $J(\Omega_m)$ (respectively $H_0^1(\Omega_m)$) is compactly embedded in $H(\Omega_m)$ (respectively $L^2(\Omega_m)$) we can choose subsequences, which we again denote by $(\mathbf{v}^k, \eta^k)$, and elements $\overline{\mathbf{u}}^m \in L^2(0,T;J(\Omega_m))$, $\overline{\eta}^m \in L^2(0,T;H_0^1(\Omega_m))$ such that

$$\begin{aligned} \mathbf{v}^k &\to \overline{\mathbf{u}}^m \text{ weakly in } L^2(0,T;J(\Omega_m)) \text{ and weakly* in } L^\infty(0,T;H(\Omega_m)), \\ \eta^k &\to \overline{\eta}^m \text{ weakly in } L^2(0,T;H_0^1(\Omega_m)) \text{ and weakly* in } L^\infty(0,T;L^2(\Omega_m)). \end{aligned}$$

Furthermore, by using the Lemma 2 and (17) we see that

$$\begin{aligned} \mathbf{v}^k &\to \overline{\mathbf{u}}^m \text{ strongly in } L^2(0,T;H(\Omega_m)), \\ \eta^k &\to \overline{\eta}^m \text{ strongly in } L^2(0,T;L^2(\Omega_m)). \end{aligned}$$

Now, it is enough to take the limit $k \to \infty$ in (P_m). Therefore, $(\overline{\mathbf{u}}^m, \overline{\eta}^m)$ is a required weak solution to problem (P_m).

Lemma 6 *Let $(\overline{\mathbf{u}}^m, \overline{\eta}^m)$ be a weak solution for (P_m) obtained in Lemma 5. Put*

$$\mathbf{u}^m(t,x) = \begin{cases} \overline{\mathbf{u}}^m(t,x) & \text{if } x \in \Omega_m, \\ 0 & \text{if } x \in \Omega \setminus \Omega_m, \end{cases}$$

$$\varphi^m(t,x) = \begin{cases} \overline{\eta}^m(t,x) & \text{if } x \in \Omega_m, \\ 0 & \text{if } x \in \Omega \setminus \Omega_m. \end{cases}$$

Then it follows that

$$\mathbf{u}^m \in L^2(0,T;J(\Omega)) \cap L_\pi^2(0,T;L^6(\Omega)),$$

$$\varphi^m \in L^2(0,T;H_0^1(\Omega)) \cap L_\pi^2(0,T;L^6(\Omega))$$

and,

$$\int_0^T \|\nabla \mathbf{u}^m\|^2 \leq \ell_1, \quad \int_0^T \|\nabla \varphi^m\|^2 \leq \ell_2,$$

$$\int_0^T \|\mathbf{u}^m\|_{L^6(\Omega)}^2 \leq \ell_1, \quad \int_0^T \|\varphi^m\|_{L^6(\Omega)}^2 \leq \ell_2,$$

where ℓ_1, ℓ_2 are taken uniformly in m.

Proof. From (9), we have, integrating in $[0,T]$

$$M \int_0^T (\|\nabla \mathbf{v}^k(t)\|^2 + \|\nabla \eta^k(t)\|^2)dt \leq N(f), \tag{18}$$

since $\mathbf{v}^k(t)$, $\eta^k(t)$ are reproductive with period T. Consequently, if we take $k \to \infty$ in (18),then we obtain by the lower semicontinuity of the norm with respect to the weak convergence

$$M \int_0^T (\|\nabla \overline{\mathbf{u}}^m(t)\|^2 + \|\nabla \overline{\eta}^m(t)\|^2) dt \leq N(f). \tag{19}$$

On the other hand, the equality $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ in $L^2(\Omega_m)$ implies $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ for a.e. $x \in \Omega_m$ and by using the Lemma 1 we obtain $\overline{\mathbf{u}}^m(t) \in L^6(\Omega_m)$, therefore we find $\overline{\mathbf{u}}^m(T) = \overline{\mathbf{u}}^m(0)$ as elements of $L^6(\Omega_m)$. Thus, we obtain $\overline{\mathbf{u}}^m \in L^2_\pi(0,T; L^6(\Omega_m))$. Analogously, we show that $\overline{\eta}^m \in L^2_\pi(0,T; L^6(\Omega_m))$.

From this and (19) , it follows that for all $m \geq 1$,

$$\overline{\mathbf{u}}^m \in L^2(0,T; J(\Omega)) \cap L^2_\pi(0,T; L^6(\Omega)),$$

$$\overline{\eta}^m \in L^2(0,T; H^1_0(\Omega)) \cap L^2_\pi(0,T; L^6(\Omega)),$$

and

$$\begin{aligned} &\frac{1}{C_L} \int_0^T (\|\overline{\mathbf{u}}^m(t)\|^2_{L^6(\Omega)} + \|\overline{\eta}^m(t)\|^2_{L^6(\Omega)}) dt \\ \leq &\int_0^T (\|\nabla \overline{\mathbf{u}}^m(t)\|^2 + \|\nabla \overline{\eta}^m(t)\|^2) dt \\ \leq &\frac{1}{C_0} N(f). \end{aligned} \tag{20}$$

4 Proof of Theorem 4

According to the uniform estimate (20), we can choose subsequences $\mathbf{u}^{m'}$ and $\varphi^{m'}$ and $\mathbf{u} \in L^2(0,T; J(\Omega)) \cap L^2_\pi(0,T; L^6(\Omega))$ and $\varphi \in L^2(0,T; H^1_0(\Omega)) \cap L^2_\pi(0,T; L^6(\Omega))$ such that

$$\begin{aligned} \mathbf{u}^{m'} &\to \mathbf{u} \text{ weakly in } L^2(0,T; J(\Omega)) \text{ and weakly in } L^2_\pi(0,T; L^6(\Omega)), \\ \varphi^{m'} &\to \varphi \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ and weakly in } L^2_\pi(0,T; L^6(\Omega)) \\ &\quad \text{as } m' \longrightarrow +\infty. \end{aligned}$$

Now, we claim that there exist subsequences $\mathbf{u}^{m'}$ and $\varphi^{m'}$ such that for any bounded $\Omega' \subset \Omega$

$$\begin{aligned} \mathbf{u}^{m'} &\to \mathbf{u} \text{ strongly in } L^2(0,T; L^2(\Omega')), \\ \varphi^{m'} &\to \varphi \text{ strongly in } L^2(0,T; L^2(\Omega')). \end{aligned} \tag{21}$$

We put $K_j = \overline{\Omega_j}$, then $\{K_j\}_{j=1}^{\infty}$ a sequence of compact sets such that $K_1 \subseteq K_2 \subseteq \ldots \to \Omega$ $(j \to \infty)$. Here, for each K_j we take $\alpha_j(x) \in C_0^{\infty}(\Omega)$ with the property $0 \le \alpha \le 1$, $\alpha_j|_{K_j} \equiv 1$, and supp $\alpha_j \subset \Omega_{j+1}$. We note that $K_j \subset$ supp α_j. Here and from now on, let us denote $\|.\|_{\Omega_j} \equiv \|.\|_{L^2(\Omega_j)}$ and d_j = diameter of Ω_j. Then we construct the desired $\{\mathbf{u}^{m'}\}$ as follows. First we consider a sequence $\{\alpha_j(x)\mathbf{u}^m(x)\}_{m=1}^{\infty}$; this is a uniformly bounded sequence of $L^2(0,T;H_0^1(\Omega_2))$. Indeed, noting that $\mathbf{u}^m(\Gamma) = 0$ and using Poincaré's inequality on Ω_2, we see that $\|\alpha_1\mathbf{u}^m\|_{\Omega_2} \le \|\mathbf{u}^m\|_{\Omega_2} \le \frac{d_2}{2}\|\nabla\mathbf{u}^m\|_{\Omega_2}$. Hence we have by (20)

$$
\begin{aligned}
\int_0^T \|\alpha_1\mathbf{u}^m(t)\|_{\Omega_2}^2 dt &\le \frac{d_2^2}{2}\int_0^T \|\nabla\mathbf{u}^m(t)\|^2 dt \\
&\le \frac{d_2^2}{2C_0}N(f).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\|\nabla(\alpha_1\mathbf{u}^m)\|_{\Omega_2} \le \|(\nabla\alpha_1)\mathbf{u}^m\|_{\Omega_2} + \|\alpha_1(\nabla\mathbf{u}^m)\|_{\Omega_2} \\
\le\ & (\frac{d_2}{2}\|\nabla\alpha_1\|_{L^{\infty}(\Omega_2)} + \|\alpha_1\|_{L^{\infty}(\Omega_2)})\|\nabla\mathbf{u}^m\|_{\Omega_2}.
\end{aligned}
$$

Therefore, we have

$$
\int_0^T \|\nabla(\alpha_1\mathbf{u}^m)(t)\|_{\Omega_2}^2 dt \le (\frac{d_2}{\sqrt{2}}\|\nabla\alpha_1\|_{L^{\infty}(\Omega_2)} + \|\alpha_1\|_{L^{\infty}(\Omega_2)})^2\frac{d_2^2}{2C_0}N(\mathbf{f})T.
$$

These estimates imply that $\{\alpha_1\mathbf{u}^m\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega_2))$. Consequently, there exists a subsequence $\{\alpha_1\mathbf{u}^{1p}\}_{p=1}^{\infty}$ which converges weakly in $L^2(0,T;H_0^1(\Omega_2))$. Furthermore, according to Lemma 2, we get

$$
\begin{aligned}
&\int_0^T \|\alpha_1\mathbf{u}^{1p} - \alpha_1\mathbf{u}^{1q}\|_{\Omega_2}^2 dt \le \sum_{n=1}^{l_\varepsilon}\int_0^T (\alpha_1\mathbf{u}^{1p} - \alpha_1\mathbf{u}^{1q}, e^n)_{\Omega_2}^2 \\
&+\varepsilon\int_0^T \|\alpha_1\mathbf{u}^{1p} - \alpha_1\mathbf{u}^{1q}\|_{W^{1,2}(\Omega_2)}^2 dt \qquad (22)\\
\le\ & \sum_{n=1}^{l_\varepsilon}\int_0^T (\alpha_1\mathbf{u}^{1p} - \alpha_1\mathbf{u}^{1q}, e^n)_{\Omega_2}^2 + 4\varepsilon C_{\alpha_1}N(\mathbf{f})
\end{aligned}
$$

where C_{α_1} depends on $.\|\alpha_1\|_{\infty}, \|\nabla\alpha_1\|_{\infty}$ and is independent of p and q. Consequently, if $p,q \to \infty$, we have in (22), since ε is arbitrary, the sequence $\{\alpha_1\mathbf{u}^{1p}\}_{p=1}^{\infty}$ converges strongly in $L^2(0,T;L^2(\Omega_2))$. This implies that $\{\mathbf{u}^{1p}\}_{p=1}^{\infty}$ converges strongly in $L^2(0,T;L^2(K_1))$. Using the same reasoning as before, we obtain $\{\mathbf{u}^{jp}\}_{p=1}^{\infty}$ $(j = 1, 2, ...)$. We choose diagonal components and denote them by $\{\mathbf{u}^{m'}\}_{m'=1}^{\infty}$, then it converges on all K_j in $L^2(0,T;L^2(K_j))$ sense. The proof for $\{\varphi^{m'}\}_{m'=1}^{\infty}$,can be done in a similar way. Once we obtain these convergence and limit results, we can show that $(\mathbf{u}, \varphi)$ is the desired reproductive weak solution for (7)-(8). Indeed, let $(\mathbf{v}, \psi)$ be any arbitrary test function. Then we find a bounded domain Ω' and k_0 such that supp $\mathbf{v}$, supp $\psi \subseteq \Omega' \subseteq \Omega_{k_0} \subseteq \Omega_k$, for all $k \geq k_0$. Moreover, by Lemma 1 and (20)

$$\begin{aligned}
&\int_0^T (\mathbf{u}^k \cdot \nabla\mathbf{v}, \mathbf{u}^k) - (\mathbf{u} \cdot \nabla\mathbf{v}, \mathbf{u})dt \\
\leq\ & \int_0^T \{3\|\mathbf{u}^k - \mathbf{u}^k\|^2_{L^2(\Omega')}\|\mathbf{u}^k\|_{L^6(\Omega)}\|\nabla\mathbf{v}\|_{L^3(\Omega')} \\
&+3\|\mathbf{u}^k - \mathbf{u}^k\|^2_{L^2(\Omega')}\|\mathbf{u}\|_{L^6(\Omega)}\|\nabla\mathbf{v}\|_{L^3(\Omega')}\}dt \\
\leq\ & 9(\int_0^T \|\mathbf{u}^k - \mathbf{u}\|^2_{L^2(\Omega')}dt)^{1/2}(\int_0^T \|\mathbf{u}^k\|^2_{L^6(\Omega)}dt)^{1/2} \sup \|\nabla\mathbf{v}\|_{L^3(\Omega')} \\
&+9(\int_0^T \|\mathbf{u}^k - \mathbf{u}\|^2_{L^2(\Omega')}dt)^{1/2}(\int_0^T \|\mathbf{u}\|^2_{L^6(\Omega)}dt)^{1/2} \sup \|\nabla\mathbf{v}\|_{L^3(\Omega')}.
\end{aligned}$$

Using convergences (21) and the above estimate, we get

$$\int_0^T (\mathbf{u}^k \cdot \nabla\mathbf{v}, \mathbf{u}^k) - (\mathbf{u} \cdot \nabla\mathbf{v}, \mathbf{u})dt \to 0,$$

as $k \to \infty$. The other convergences are in the same way established. Thus, $(\mathbf{u},\varphi)$ is a reproductive weak solution for problem (7)-(8).

Remark 2 *We would like say that, we obtain the same level de results as in Lorca and Boldrini [13](for the weak solutions). The uniqueness of solution to weak solutions is difficult, to more strong solution (for instance to the case of initial boundary problem studied by Lorca and Boldrini, it is necessary that* $\mathbf{u}_0 \in J_1(\Omega)$ *and* $\varphi_0 \in H^2(\Omega)$ *for* $n = 2$ *or* 3, *see Pag 462 in [13]).*

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