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PERIODIC PROBLEM FOR THE EQUATION OF QUASILINEAR WAVES WITH SMALL INTERFERENCES

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Abstract

We consider the problem

$$u_{tt} - u_{xx} = \epsilon f(t,x,u), \tag{1}$$

$$\left.\begin{array}{rcl} u(0,x) & = & u(2\pi,x) \\ u_t(0,x) & = & u_t(2\pi,x) \end{array}\right\} \quad 0 < x < \pi, \tag{2}$$

$$u(t,0) = u(t,\pi) = 0, \qquad 0 \le t. \tag{3}$$

The case with prescribed initial conditions has been studied by M.S. Krol, W.T. van Horssen, A.L. Shtaras, J. Ginibre and other authors, but the periodic problem is generally open, even when f is a linear function of u with varying coefficients.

We propose a method to analyze the problem (1)–(3), leading to sufficient conditions for the existence of a solution, and a constructive method to obtain approximate solutions that converge to the exact solution in a specified norm.

1 The Initial Problem

Consider the partial differential equation with small parameter

$$u_{tt} - u_{xx} = \epsilon f(t,x,u), \tag{1}$$

satisfying the following conditions:

$$\left.\begin{array}{rcl} u(0,x) & = & u(2\pi,x) \\ u_t(0,x) & = & u_t(2\pi,x) \end{array}\right\} \quad 0 < x < \pi, \tag{2}$$

$$u(t,0) = u(t,\pi) = 0, \qquad 0 \le t. \tag{3}$$

The problem studied here is the existence of a solution to Eqs. (1)–(3). Notice that for the linear operator L such that $Lu = u_{tt} - u_{xx}$, defined on a class D of functions u that satisfy the conditions (2) and (3), the system (1)–(3) is equivalent to the equation:

$$Lu = f(t, x, u), \qquad u \in D.$$

In the case presented here, the operator L is not invertible and its kernel is countable, characterizing a resonance problem. Consequently, the operator L is not even a Fredholm operator, so classical solution techniques may not be used.

2 Transforming the initial problem

We assume that the function $f(t, x, u)$ is 2π-periodic with respect to the variable t. We will find the solution of the problem defined by Eqs. (1)– (3) on the class of functions $u(t, x)$, that may be expanded using the Fourier series:

$$u(t, x) = \sum_{k=1}^{\infty} Z_k(t) \sin(kx), \tag{4}$$

where $\sum_{k=1}^{\infty} \left\{ k^2 \left|Z_k(t)\right| + \left|\ddot{Z}_k(t)\right| \right\} < \infty$ for any $t \in [0, 2]$. We define this class as B.

Substituting Eq. (4) into Eq. (1), considering conditions (2), (3), and the following restriction with respect to $f(t, x, u)$:

$$\int_0^{2\pi} f\left(t,\ x,\ Z_k(t) \sin(kx)\right) \cos(mx)\, dx = 0 \quad \text{for all} \quad m = 1, 2, ..., \tag{5}$$

we obtain:

$$\ddot{Z}_k + k^2 Z_k = \epsilon f_k(t, Z), \tag{6}$$

$$Z_k(2\pi) - Z_k(0) = \dot{Z}_k(2\pi) - \dot{Z}_k(0) = 0, \tag{7}$$

where $Z_k = (Z_1\ , Z_2 \ldots)$, $\quad k = 1,\ 2,\ \ldots$ and

$$f_k(t,\ Z) = \frac{2}{\pi} \int_0^{2\pi} f\left(t,\ x,\ Z_k(t)\ \sin(kx)\right) \sin(mx)\, dx$$

is a 2π-periodic function at t.

The problem defined by Eqs. (6), (7) is a resonance problem, since for $\varepsilon = 0$ we have that $Z_k(t) = c_k^1 \sin(kt) + c_k^2 \cos(kt)$ is a solution of the homogeneous problem (c_k^1 and c_k^2 are arbitrary constants, and $k = 1, 2, \dots$).

We assume that $Z_k(t) = W_k(t) + \xi_k \sin(kt) + \eta_k \cos(kt)$.

Notice then that if $W_k(t)$, ξ_k and η_k satisfy conditions

(1) $\sum\limits_{k=1}^{\infty} \left\{k^2 \, |W_k(t)| + \left|\ddot{W}_k(t)\right|\right\} < \infty$ for each $t \in [0, 2]$, and

(2) $\sum\limits_{k=1}^{\infty} \{k^2 (|\xi_k| + |\eta_k|)\} < \infty$,

the corresponding function $u(t, x)$ will belong to a class B. We say that a sequence $\{W_k(t)\}$ belongs to class B_1 if it satisfies the condition $\sum\limits_{k=1}^{\infty} \left\{k^2 \, |W_k(t)| + \left|\ddot{W}_k(t)\right|\right\} < \infty$ for each $t \in [0, 2]$. We say that the sequence $\{\xi_k, \eta_k\}$ belongs to class B_2 if it satisfies the condition $\sum\limits_{k=1}^{\infty} \{k^2 (|\xi_k| + |\eta_k|)\} < \infty$.

Rewriting the system (6)– (7), now with respect to the unknown functions $W_k(t)$ of class B_1 and to the countable vectors $\{\xi_k, \eta_k\}$ of class B_2, we have:

$$W_k(t) = \frac{\epsilon}{k} \int\limits_0^{2\pi} G_k(t, s) f\big(s, \dots, W_m(s) + \xi_m \sin(ms) + \eta_m \cos(ms), \dots\big) ds, \quad (8)$$

$$\int_0^{2\pi} f_k\big(s, \dots, W_m(s) + \xi_m \sin(ms) + \eta_m \cos(ms), \dots\big) \sin(ks) ds = 0, \quad (9)$$

$$\int_0^{2\pi} f_k\big(s, \dots, W_m(s) + \xi_m \sin(ms) + \eta_m \cos(ms), \dots\big) \cos(ks) ds = 0, \quad (10)$$

where $m = 1, 2, \dots$ and

$$G_k(t, s) = \begin{cases} \dfrac{2}{\pi} \sin(k(t - s)), & 0 \le s < t \le 2\pi \\ \left(\dfrac{s}{2\pi} - 1\right) \sin(k(t - s)), & 0 \le t < s \le 2\pi \end{cases}$$

is a generalized Green's function [2, pp. 308, 309].

It is easy to verify that

$$\max_{t \in [0, 2\pi]} \sup_k \int_0^{2\pi} |G_k(t, s)| \, ds \quad \le \pi. \quad (11)$$

Settting $\zeta_1 = \alpha_1$, $\eta_1 = \alpha_2$, $\zeta_2 = \alpha_3$, $\eta_2 = \alpha_4, \ldots, \zeta_k = \alpha_{2k-1}$, $\eta_k = \alpha_{2k}$, Eqs. (8), (9), (10) may be rewritten as:

$$W_k(t) = \frac{\epsilon}{k} \int\limits_0^{2\pi} G_k\,(t\,,\,s) \left[\begin{array}{l} f_k(s, W_1\,(s) + \alpha_1 \sin\,(s) + \\ +\alpha_2\,\cos\,(s)\,, \ldots) \end{array} \right] ds, \tag{12}$$

$$\int_0^{2\pi} \left[\begin{array}{l} f_k(s, W_1\,(s) + \alpha_1 \sin\,(s) + \alpha_2\,\cos\,(s)\,, \\ W_2\,(s) + \alpha_3 \sin\,(2s) + \alpha_4\,\cos\,(2s)\,, \ldots) \end{array} \right] \sin(ks)\,ds = 0, \tag{13}$$

$$\int_0^{2\pi} \left[\begin{array}{l} f_k(s, W_1\,(s) + \alpha_1 \sin\,(s) + \alpha_2\,\cos\,(s)\,, \\ W_2\,(s) + \alpha_3 \sin\,(2s) + \alpha_4\,\cos\,(2s)\,, \ldots) \end{array} \right] \cos(ks)\,ds = 0. \tag{14}$$

Using complex notation, Eqs. (12), (13) take the following form:

$$T_k\,(s) = \int_0^{2\pi} \left[\begin{array}{l} f_k(s, W_1\,(s) + \alpha_1 \sin\,(s) + \\ +\alpha_2\,\cos\,(s)\,, W_2\,(s) + \\ +\alpha_3 \sin\,(2s) + \alpha_4 \cos\,(2s)\,, \ldots) \end{array} \right] e^{i\,k\,s}\,ds = 0. \tag{15}$$

In the next Section, we present auxiliary results from [3] that will be used to study the system (12)–(15), regarding sufficient conditions for the existence of a solution and an iterative method for its construction.

3 Auxiliary results

Consider the system

$$u = \varepsilon F(u, \alpha), \tag{16}$$

$$\langle D_k\;(u, \alpha)\,, \psi_k \rangle = 0 \qquad k = 1, 2, \ldots, \tag{17}$$

with respect to $u \in E_1$ and to $\alpha \in E$ where
$F\colon E_1 \times E \to E_1$ and $D_k\colon E_1 \times E \to E_2$ $(k = 1, 2, \ldots)$ are boundary operators, E_1, E_2 are Banach spaces with norms $\|\cdot\|_{E_1}$, $\|\cdot\|_{E_2}$ respectively, E is a countable Banach space with the following base:

$$\{e_k\}_1^{\infty}\,, \quad e_1 = (1,0,0,\ldots)\,, \quad e_2 = (0,1,0,\ldots)\,, \quad e_3 = (0,0,1,\ldots)\,, \ldots$$

and norm $\|\cdot\|_E = \sup_k |\,\alpha_k|$, where $\alpha = \sum\limits_{k=1}^{\infty} \alpha_k e_k$.

We denote $\langle\ .\ ,\psi_k\rangle$ the values of the functional Ψ_k . They are uniformly bounded for $k = 1, 2, ...$

Let us denote $\widetilde{T_k}(\alpha) = \langle D_k\ (0\,,\alpha)\,,\psi_k\rangle$, $\quad \widetilde{T}(\alpha) = (\widetilde{T_1}(\alpha), \widetilde{T_2}(\alpha), \dots)$. Then we have the following theorem:

Theorem 1 *Consider the operators F and D_i (where $i = 1, 2, \dots$) that are continuously Fréchet-differentiable with respect to u and α at each point (u, α) of their domain of definition. Assume that the equation $\widetilde{T}(\alpha) = 0$ has a solution α^*, and that the following conditions are valid in a certain neighborhood $||\alpha - \alpha^*||_E \le R$, where $R > 0$:*

The Jacobian matrix $\widetilde{T}(\alpha)$ is continuously invertible, where the norm of the "countable matrix" A with elements a_{ij} is defined by $|||A||| = \sup\limits_i \sum_j |a_{ij}|$. Moreover, the estimate $|||\,[\widetilde{T}(\alpha)]^{-1}|||\le M$ is valid.

For each α, η at the indicated neighborhood, for each u, v in a certain neighborhood of finite ratio r : $||u|| \le r$, and for all $i = 1, 2, 3, \dots$, the following conditions are valid:

a) $|||F_u(u, \alpha) - F_u(v, \eta)||| \le M(||u - v|| + ||\alpha - \eta||)$,

$|||D_{i_u}(u, \alpha) - D_{i_u}(v, \eta)||| \le N(||u - v|| + ||\alpha - \eta||)$,

b) $|||F_\alpha(u, \alpha) - F_\alpha(v, \eta)||| \le l(||u - v|| + ||\alpha - \eta||)$,

$|||D_{i_{\alpha_j}}(u, \alpha) - D_{i_{\alpha_j}}(v, \eta)||| \le \widetilde{l}_{ij}(||u - v|| + ||\alpha - \eta||)$,

$|||D_{i_u}||| \le N_1, \qquad N_1 < \infty$,

$\sup\limits_i \sum\limits_{j=1}^{\infty} |\widetilde{l}_{ij}| < \widetilde{l} < \infty.$

Then, the problem defined by system (16)–(17) has a solution for ε sufficiently small, and the following system formed by a sequence of equations with respect to vector α_k, $k = 1, 2, \dots$:

$$\begin{array}{l} \langle D_i\ (u^k\,,\alpha^k)\,,\psi_i\rangle = 0, \\ u^1 = 0, \quad u^{k+1} = \varepsilon\, F(u^k\,,\alpha^k), \\ \textit{where} \quad i = 1, 2, 3, \dots, \end{array} \tag{18}$$

has a solution for each k, such that the sequence $(\widehat{u}^k, \widehat{\alpha}^k) = (u^k(\widehat{\alpha}^k), \widehat{\alpha}^k)$ converges under the previously defined norm, to the exact solution $(\widehat{u}, \widehat{\alpha})$ of system (16), (17), and $||\widehat{u}^k - \widehat{u}||_{E_1} \to 0, \qquad ||\widehat{\alpha}^k - \widehat{\alpha}||_E \to 0$ where $k \to \infty$.

The proof of this theorem uses functional analysis techniques and results obtained by the author in [3].

4 The Solution

Evidently the system (12)–(15) can be rewritten in the form (16)–(17), where $w = (w_1, w_2, \ldots)$, and

$$\begin{aligned} &F(w, \alpha) = (F_1(w,), F_2(w,), \ldots), \\ &F_k(w, \alpha) = \frac{1}{k}\int_0^{2\pi} G_k(t, s) \begin{bmatrix} f_k(s, W_1(s) + \alpha_1 \sin(s) + \\ +\alpha_2 \cos(s), \ldots) \end{bmatrix} ds, \\ &\langle D_k(u, \alpha), \psi_k\rangle = \int_0^{2\pi} \begin{bmatrix} f_k(s, W_1(s) + \alpha_1 \sin(s) + \\ +\alpha_2 \cos(s), \ldots) \end{bmatrix} e^{iks}\, ds = 0. \end{aligned}$$

Then Theorem 1 offers sufficient conditions for the existence of the solution of the problem (12)–(15), and an iterative convergent method for the construction of the solution.

5 Example

In the system (1)–(3) we consider the function

$f(t, x, u) = (\gamma \cos x + 1)u + h(t, x)$, where

$$h(t, x) = \sum_{k=1}^{\infty} h_k(t) \sin(kx), |h_k(t)| \leq \frac{1}{k^\nu}, (\nu > 3), t \in [0, 2], \text{ for } \gamma = \text{ const.}, \gamma \leq 2.$$

In this case, Eqs. (12) and (15) may be written as:

$$W_k(t) = \frac{\varepsilon}{k}\int_0^{2\pi} G_k(t, s) \begin{bmatrix} \frac{1}{2}\gamma(w_{k-1} + w_{k+1}) + \alpha_{2k-1} \sin(k-1)s \\ +\alpha_{2k-2} \cos(k-1)s + \\ +\alpha_{2k+1} \sin(k+1)s + w_k + \\ +\alpha_{2k+2} \cos(k+1)s) + \\ +\alpha_{2k-1} \sin(k)s + \\ +\alpha_{2k} \cos(k)s + h(s) \end{bmatrix} ds, \quad (19)$$

$$\int_0^{2\pi} \left[\begin{array}{l} \frac{1}{2}\gamma(w_{k-1} + \alpha_{2k-1}\sin(k-1)s \\ +\alpha_{2k-2}\cos(k-1)s + w_{k+1}) + w_k \\ +\alpha_{2k+1}\sin(k+1)s + \alpha_{2k+2}\cos(k+1)s) \\ +\alpha_{2k-1}\sin(k)s + \alpha_{2k}\cos(k)s + h(s) \end{array} \right] e^{iks}\, ds = 0, \tag{20}$$

where we will assume that $\alpha_{_{-1}} = \alpha_{_0} := 0,\ k = 1, 2, \ldots,\ W_0 := 0$.

Since $\int_o^{2\pi} \sin(ms) e^{iks} ds = \left\{ \begin{array}{l} 0, m \neq k \\ i\pi, m = k \end{array} \right.$, and

$\int_o^{2\pi} \cos(ms) e^{iks} ds = \left\{ \begin{array}{l} 0, m \neq k \\ \pi, m = k \end{array} \right.$,

equation (20) may be rewritten as:

$$\int_0^{2\pi} \left[\begin{array}{l} \frac{1}{2}\gamma\,(w_{k-1} + w_{k+1}) + \\ +w_k + h_k(s) \end{array} \right] e^{iks} ds + \pi\,(\alpha_{2k} + i\alpha_{2k-1}) = 0. \tag{21}$$

The auxiliary operator $\widetilde{T}(\alpha)$ for $W = 0$ is

$$\widetilde{T}(\alpha) = \int_0^{2\pi} h_k(s)\, e^{iks} ds + \pi\,(\alpha_{2k} + i\alpha_{2k-1}) \,.$$

It is easy to verify that the system $\widetilde{T}(\alpha) = 0$ has a solution $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots)$, where

$$\alpha_{2k} = -\frac{1}{\pi} \int_0^{2\pi} h_k(s) \cos(ks)\, ds, \quad \alpha_{2k-1} = -\frac{1}{\pi} \int_0^{2\pi} h_k(s) \sin(ks)\, ds, \\ k = 1, 2, \ldots. \tag{22}$$

Moreover, since $|\alpha_k^*| \leq \dfrac{2}{k^\nu}$ for $\nu > 3, k = 1, 2, \ldots,$ we have that $\sum k^2 |\alpha_k^*| \leq \sum\limits_{k=1}^{\infty} \dfrac{2}{k^{\nu-2}} < \infty$; therefore, the solution α^* belongs to class B_2 .

The integral operator corresponding to equation (19) maps the elements of the class $B = \big\{(W, \alpha) : W \in B_1,\ \alpha \in B_2\big\}$ into elements of the class B_1 .

Thus, for a complete demonstration of the theorem we only need to verify conditions (a) and (b) of Theorem 1:

$$\widetilde{T}'(\alpha) = \begin{bmatrix} i\pi & \pi & 0 & 0 & 0 & \dots \\ 0 & 0 & i\pi & \pi & 0 & \dots \\ 0 & 0 & 0 & 0 & i\pi & \dots \\ . & . & . & . & . & \dots \end{bmatrix},$$

$$\widetilde{T}'(\alpha) = \begin{bmatrix} -\frac{1}{2\pi} & 0 & 0 & 0 & \dots \\ +\frac{1}{2\pi} & 0 & 0 & 0 & \dots \\ 0 & -\frac{1}{2\pi} & 0 & 0 & \dots \\ 0 & +\frac{1}{2\pi} & 0 & 0 & \dots \\ . & . & . & . & \dots \end{bmatrix},$$

$$\left|\left|\left|\left[\widetilde{T}'(\alpha)\right]^{-1}\right|\right|\right| = \frac{1}{2\pi}.$$

From (19) and (21), we have that

$$F_k(w,\alpha) = \frac{1}{k}\int_0^{2\pi} G_k(t,s) \begin{bmatrix} \frac{1}{2}\gamma(w_{k-1} + w_{k+1} + \alpha_{2k-1}\sin(k-1)s \\ +\alpha_{2k-2}\cos(k-1)s + \\ w_k + \alpha_{2k+1}\sin(k+1)s + \\ \alpha_{2k+2}\cos(k+1)s) + \\ +\alpha_{2k-1}\sin(k)s + \\ +\alpha_{2k}\cos(k)s + h(s) \end{bmatrix} ds$$

$$D_k(w,\alpha) = \int_0^{2\pi} \begin{bmatrix} \frac{1}{2}\gamma(w_{k-1} + w_{k+1}) + w_k + \\ h_k(s) + \alpha_{2k-1}\sin(ks) + \alpha_{2k}\cos(ks) \end{bmatrix} ds,$$

where $k = 1, 2, \dots$, $W_0 = 0$, $\alpha_{_{-1}} = \alpha_{_0} = 0$.

Notice that the operators F and D_k $(k = 1, 2, 3, \dots)$ transform $C_{([0,2\pi],E)} \times E$ into $C_{([0,2\pi],E)} \times E$ and $C_{([0,2\pi],\,R)}$ respectively, considering in $C_{([0,2\pi],E)}$ the norm $||w||_{C_E} = \max_{t\in[0,2\pi]} \sup_k |w_k(t)|$ and in $C_{([0,2\pi],\,R)}$ the norm $||g||_{C_R} = \max_{t\in[0,2\pi]} |g(t)|$. The values of functional Ψ_j of $C_{([0,2\pi],E)}$ have been defined for the expression $\langle D_k(w,\alpha), \Psi_k\rangle = \int_0^{2\pi} D_k(w,\alpha) e^{iks} ds$, and for all k $(k = 1, 2, \dots)$: $|||\Psi_k||| \leq 2\pi$.

In this case we also have:

$|||F_\alpha(w,\alpha)||| = (2+\gamma)\pi \leq 4\pi$, $|||F_w(w,\alpha)||| = \left(1 + \frac{\gamma}{2}\right)\pi \leq 2\pi$.

For $D_k(w,\alpha)$ $(k=1,2,\dots)$, we have

$$D_{1_w}(w,\alpha)=\left(1,\frac{1}{2}\gamma,0,\ 0,\ 0,...\right),$$
$$D_{2_w}(w,\alpha)=\left(\frac{1}{2}\gamma,1,\frac{1}{2}\gamma,0,0,...\right),$$
$$D_{3_w}(w,\alpha)=\left(0\ ,\frac{1}{2}\gamma,1,\frac{1}{2}\gamma,0,...\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Moreover,

$$D_{k_{\alpha_j}}(w,\alpha)=\begin{cases}\sin ks, & j=2k-1,\\ \cos ks, & j=2k,\\ 0\quad , & j\neq(2k-1,2k).\end{cases}$$

For W in a certain neighborhood $||w||_{C_E}\leq r$ and for $||\alpha-\alpha^*||_E\leq R$, $(r,R>0)$, where α^* is a solution of the system $\widetilde{T}(\alpha)=0$, it is obvious that the operators F,D_k are continuously Fréchet-differentiable, for each point (W,α) of the previously defined set, and the conditions of Theorem 1 for the problem (19)–(21) are satisfied. Then, we obtain the existence of the exact solution $(\widehat{w},\widehat{\alpha})$ in the class B previously indicated for values of ϵ sufficiently small.

Moreover, the following system formed by a sequence of equations with respect to a vector α^n $(n=1,2,3,\dots)$:

$$\int_0^{2\pi}\left[\frac{1}{2}\gamma(w_{k-1}^n+w_{k+1}^n)+w_k^n+h_k(s)\right]e^{iks}ds+\pi\left(\alpha_{2k}^n+i\alpha_{2k-1}^n\right)=0,$$

$$w^1=w^1(t)=\left(w_1^1(t),w_2^1(t),w_3^1(t),\dots\right)=(0,0,0,\dots),$$

$$w_k^{n+1}(t)=\frac{1}{k}\int_0^{2\pi}G_k(t,s)\left[\begin{array}{l}\frac{1}{2}\gamma(w_{k-1}+\alpha_{2k-1}\sin(k-1)s\\ +\alpha_{2k-2}\cos(k-1)s+w_{k+1})\\ +\alpha_{2k+1}\sin(k+1)s+\alpha_{2k+2}\cos(k+1)s)\\ +w_k+\alpha_{2k-1}\sin(k)s+\alpha_{2k}\cos(k)s+h(s)\end{array}\right]ds,$$

for each step n has a solution $\widehat{\alpha}^n$ such that the sequence $(\widehat{w}^n,\widehat{\alpha}^n)=(\widehat{w}^n(t,\widehat{\alpha}^n),\widehat{\alpha}^n)$ satisfies:

$$|||\widehat{w}^n-\widehat{w}|||_{C_E}\to 0 \text{ and } |||\widehat{\alpha}^n-\widehat{\alpha}|||_E\to 0 \text{ for } n\to\infty.$$

Notice that the functions

$$u\,(t,x,\epsilon)=\sum_{k=1}^{\infty}[\widehat{w}_k(t,\epsilon)+\widehat{\xi}_k\sin kt+\widehat{\eta}_k\cos kt]\sin kx \qquad \text{and}$$

$$u^n\,(t,x,\epsilon)=\sum_{k=1}^{\infty}[\widehat{w}_k^n(t,\epsilon)+\widehat{\xi}_k^n\sin kt+\widehat{\eta}_k^n\cos kt]\sin kx, \quad \text{for} \quad n=1,2,3,\ldots,$$

belong to a class B and that they are exact and approximate classical solutions, respectively, of the problem (1)–(3) when the right hand side of Eq. (1) has the form $f(t,x,u)=(\gamma\cos x+1)u+h(t,x)$, as in the example presented here.

6 Final considerations

The techniques and theorems presented in this article can also be applied to more general problems, for example Eq. (1) with the following conditions:

$$\begin{aligned}
&U\,(0,x)=U\,(T,x)\,,\\
&\frac{\partial u}{\partial t}\,(0,x)=\frac{\partial u}{\partial t}\,(T,x)\,,\\
&U\,(t,0)=U\,(t,L)\,,
\end{aligned}$$

whenever $\dfrac{L}{T}$ is a rational number. The irrational case is not a resonance problem.

Other conditions can be:

$$U_x\,(0,x)=U_x\,(T,x)=0,$$

or the periodic type:

$$U\,(t,0)=U\,(t,L)\,,\quad U_x\,(t,0)=U_x\,(t,L)\,.$$

Another very interesting case occurs when the right hand side of equation (1) is $f(t,x,u)=h(t,x)+u^3$, where $h(t,x)$ can be represented by $h(t,x)=\sum\limits_{k=3}^{\infty}h\,(t)\sin(kx)$. This problem is still open (see [2], [5], [6]).

The method can be applied to a countable system of differential equations and to a certain type of integro-differential equations.

Appendix

We consider the nonlinear operator $T : E_1 \to E_2$, E_1, E_2 - Banach spaces with norms $||\cdot||_1$ and $||\cdot||_2$ respectively. Denote by $\tilde{T} : E_1 \to E_2$ a certain auxiliary operator, by $T'(y)$ and $\tilde{T}'(y)$ the corresponding Fréchet-differential operators and by $|||\cdot|||$ the norm in $\mathcal{L}(E_1, E_2)$ problem of the resolution of equation $T(y) = 0$.

Theorem (of Comparison). Suppose that an equation $\tilde{T}(y) = 0$ has solution y^* and that there is a real number $R > 0$ such that for each $y : ||y - y^*||_1 \leq R$ the following conditions are valid:

(a) $T'(y), \tilde{T}'(y)$ are continuous, operator $\tilde{T}'(y)$ is continuously invertible and

(b) $\left|\left|\left|[\tilde{T}'(y)]^{-1}\right|\right|\right| < \delta, \quad \delta > 0;$

(c) $\left|\left|T(y) - \tilde{T}(y)\right|\right|_2 \leq \alpha;$

(d) $\left|\left|\left|T'(y) - \tilde{T}'(y)\right|\right|\right| \leq \beta;$

(e) $w\beta\,\delta < 1.$

For each x, y in the indicated neighborhood the following inequality holds:

$$|||T'(y) - T'(x)||| \leq l\ ||y - x||_1 ,$$

where l is constant.

Moreover, if

$$q\frac{1}{2}\left(\frac{\delta}{1-w}\right)^2 l\alpha < 1, \qquad r\left(\frac{\delta}{1-w}\right)\alpha\sum_{k0}^{\infty} q^{2^k-1} < R,$$

then the equation $T(y) = 0$ has a solution $\hat{y}$ in the neighborhood $||y - y^*||_1 \leq r.$

Proof: From conditions (a), (c), (d) we obtain, for each $y : ||y - y^*||_1 \leq R$, the estimate $\left|\left|\left|\left[T'(y) - \tilde{T}'(y)\right]\left[\tilde{T}'(y)\right]^{-1}\right|\right|\right| \leq w < 1$ and consequently ([1, pp.141]) the existence of the inverse operator $\left[T'(y)\right]^{-1}$, satisfying the inequality $\left|\left|\left|\left[\tilde{T}'(y)\right]^{-1}\right|\right|\right| \leq \dfrac{\delta}{1-w}$. From condition (b), we obtain that $||T(y^*)||_2 \leq \alpha.$

From condition (e) we have that $T'(y)$ satisfies a Liptchitz condition with constant l. Then, from the Generalized Newton's Method for Banach Spaces [4], [2] results the existence of the solution $\hat{y}$ of the equation $T(y) = 0$ in the neighborhood $\|y - y^*\|_1 \leq r$.

Corollary. Consider $E_1 = E_2 = E$, and the nonlinear operator $T : E \to E$, $E-$ Countable Banach - Space with base $\{e_i\}_{i1}^{\infty}$, $e_1(1,0,0,...)$, $e_2(0,1,0,...)$, $e_3(0,0,1,...)$, ... and norm $\|y\|_E \sup_i |y_i|$, $i = 1,2,3,...$, $y = y_1e_1 + y_2e_2 + ...$ Suppose that an equation $\tilde{T}(y) = 0$ has solution y^* and that there is a real number $R > 0$ such that for each $y : \|y - y^*\|_1 \leq R$ the following conditions are valid:

(a) $T'(y), \tilde{T}'(y)$ are continuous, operator $\tilde{T}'(y)$ is continuously invertible and

(b) $\left|\left|\left|[\tilde{T}'(y)]^{-1}\right|\right|\right| < \delta, \ \ \delta > 0;$

(c) $\left|T(y) - \tilde{T}(y)\right|_E \leq \alpha;$

(d) $\left|\left|\left|T'(y) - \tilde{T}'(y)\right|\right|\right| \leq \beta;$

(e) $w\beta\,\delta < 1;$

For each x, y in the indicated neighborhood the following inequality holds:

$$\left|\left|\left|\frac{\partial T_i}{\partial y_j}(y) - \frac{\partial T_i}{\partial y_j}(x)\right|\right|\right| \leq l_{ij}\,\|y - x\|_E\,, \quad \text{for} \quad j = 1,2,...,$$

where the terms l_{ij} satisfy the inequality $\sup_i \sum_{j1}^{\infty} \leq l$, for constant l.

Moreover, if

$$q\frac{1}{2}\left(\frac{\delta}{1-w}\right)^2 l\alpha < 1, \qquad r\left(\frac{\delta}{1-w}\right)\alpha \sum_{k0}^{\infty} q^{2^k - 1} < R,$$

equation $T(y) = 0$ has a solution $\hat{y}$ in the neighborhood $\|y - y^*\|_E \leq r$.

Proof: The proof is obtained directly from the Comparison Theorem considering that $\sup_i \sum_{j1}^{\infty} \left|\frac{\partial T_i}{\partial y_j}(y) - \frac{\partial T_i}{\partial y_j}(x)\right| \leq \sup_i \sum_{j1}^{\infty} l_{ij}\,\|y - x\|_E \leq l\,\|y - x\|_E$, that is to say, the operator $T'(y)$ satisfies a Lipschitz condition with constant l.

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