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EXISTENCE, UNIQUENESS AND CONTROLLABILITY FOR PARABOLIC EQUATIONS IN NON-CYLINDRICAL DOMAINS

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Dedicated to the memory of Jacques-Louis Lions

Abstract

We discuss several questions related to parabolic evolution equations in moving or non-cylindrical domains. We consider space-time domains such that, for all $t > 0$, their cross section at t can be transformed into a reference domain Ω, by means of a C^2-diffeomorphism $\tau_t \colon \Omega \to \Omega_t$. The reference domain is assumed to be a bounded open set of $\mathbb{R}^n$ with boundary Γ of class C^2. We also assume a C^1 dependence of the domain Ω_t with respect to time. We investigate the existence and uniqueness of strong, weak and ultra weak solutions in the sense of transposition. The problem of controllability is also discussed both in the context of approximate and null controllability.

1 Introduction

In this article we consider linear parabolic problems in domains which are moving in time. Given a time $T > 0$, the equation is assumed to be posed in an open set $\widehat{Q}$ of $\mathbb{R}^n \times (0, T) \subset \mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$ that is the union for $0 < t < T$ of open sets Ω_t of $\mathbb{R}^n$ that can be mapped into a reference domain Ω_0 by means of a C^2-diffeomorphism $\tau_t \colon \Omega \to \Omega_t$. On the other hand, the diffeomorphisms τ_t are assumed to depend on t in a C^1 way. We also assume that, without loss of generality, $\Omega = \Omega_0$ so that τ_0 is the identity map.

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To be more precise, the points in the reference domain Ω are denoted by $y = (y_1, ..., y_n)$, while those in Ω_t are denoted by $x = \tau_t(y)$: $x \in \Omega_t$ if and only if $y \in \Omega$. We shall also use the notation $\tau(y,t)$ for $\tau_t(y)$. We denote by $\widehat{Q}$ the noncylindrical domain of $\mathbb{R}^{n+1}$ defined by

$$\widehat{Q} = \bigcup_{0<t<T} \{\Omega_t \times \{t\}\}.$$

The boundary of Ω_t is represented by Γ_t and the lateral boundary of $\widehat{Q}$ is represented by $\widehat{\Sigma}$:

$$\widehat{\Sigma} = \bigcup_{0<t<T} \{\Gamma_t \times \{t\}\}.$$

Let Q be the cylinder based on the reference domain Ω, i. e. $Q = \Omega \times (0,T)$. We have a natural diffeomorphism between Q and $\widehat{Q}$

$$(y,t) \in Q \to (x,t) \in \widehat{Q}, \quad (x,t) = (\tau_t(y),t) = (\tau(y,t),t).$$

We assume that

$$\text{For all } 0 \leq t \leq T, \ \tau_t \quad \text{is a} \quad C^2 - \text{diffeomorphism from } \ \Omega \ \text{ to } \Omega_t. \tag{1}$$

We also assume that

$$\tau(y,t) \in C^1([0,T]; C^0(\bar{\Omega})). \tag{2}$$

To simplify the presentation the reference domain Ω is assumed to be bounded and of class C^2, although most of the results we present here hold when Ω is Lipschitz continuous and unbounded. The regularity assumptions on the diffeomorphism τ_t may also be weakened. However, the minimal assumptions on the reference domain Ω and the transformation τ_t will depend very much on the notion of solution and the type of control problem under consideration.

Concerning the class of domains $\widehat{Q}$ we are considering, it is important to point out that the assumptions above are not very restrictive. For instance, the condition (2) that τ_t depends in a C^1 way on time (that, in practice, can be often replaced by a Lipschitz dependence) indicates that the domain does not evolve in time too roughly but allows all kind of deformations on its shape. But, the condition that Ω_t can be mapped into the reference domain Ω, at every t, by means of a C^2 diffeomorphism, imposes the topology of Ω_t not to change as time evolves. This is the main restriction we impose on the geometry of the space-time domain $\widehat{Q}$ under consideration. In particular, we do not address here problems in which holes appear or disappear in Ω_t as time increases. This type of situation requires a separate analysis since solutions may develop singularities at those points where the topology of Ω_t changes.

In order to simplify the presentation we shall consider the constant coefficient heat equation in the domain $\widehat{Q}$, although our results extend to more general equations with variable coefficients under suitable regularity assumptions, and also to semilinear equations under natural conditions on the nonlinearity. Also, we consider Dirichlet boundary conditions, but similar results hold for other ones.

We are interested both in the existence and uniqueness of solutions and its controllability properties. We shall discuss *strong* and *weak* solutions and also the solutions defined by transposition, also referred to as *ultra weak solutions.*

Thus, this paper is devoted to the following boundary value problem for the heat equation in the non-cylindrical domain $\widehat{Q}$:

$$\left| \begin{array}{l} u' - \Delta u = f \quad \text{for} \quad (x,t) \in \widehat{Q} \\ u = 0 \quad \text{for} \quad (x,t) \in \widehat{\Sigma} \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega. \end{array} \right. \tag{3}$$

Here and in the sequel $'$ stands for the time derivative $\partial_t = \partial/\partial t$.

In the following section we discuss the problem of the existence and uniqueness of solutions, distinguishing the various notions of solutions. In section 3 we discuss the controllability problem, both in the case of approximate and null controllability. In section 4 we comment on some possible extensions of the results of this paper.

To close this section we also mention some basic references on the analysis on Partial Differential Equations in non-cylindrical domains. There is an extensive literature and the following works are worth mentioning among many others: Lions [18], Cooper and Bardos [9], Medeiros [26], Inoue [17], Rabello [33], Nakao and Narazaki [31] and Cannarsa, Da Prato and Zolésio [5] for nonlinear wave equations, Acquistapace [1], Bernardi, Bonfonti and Lutteroti [2] for Schrödinger equations, Cheng-He and Ling Hsiano [7] for the Euler equation, Miranda and Limaco [30] for the Navier-Stokes equations, Chen and Frid [6] for hyperbolic systems of conservation laws. The particular case where $\tau_t(y) = K(t)y$ for $K(t), \quad t \geq 0$, has been analyzed in Miranda and Medeiros [29] and Miranda and Limaco [30] and the more general case in which

$\tau_t(y) = K(t)y + h(t)$, with $K(t)$ as above and $h(t)$ a vector of $\mathbb{R}^n$ in Bernardi, Bonfanti and Lutteroti [2].

2 Existence and uniqueness of solutions

This section is devoted to the analysis of the existence and uniqueness of solutions of system (3). We distinguish three different classes of solutions: strong, weak and ultra weak solutions defined by transposition.

2.1 Strong Solutions

A function $u = u(x,t)$ defined in $\widehat{Q}$ is said to be a strong solution for problem (3) if

$$u \in C\big([0,T]; H_0^1(\Omega_t)\big) \cap L^2(0,T; H^2 \cap H_0^1(\Omega_t)) \cap H^1(0,T; L^2(\Omega_t)), \tag{4}$$

and the three equations in (3) are satisfied almost everywhere in their corresponding domains.

We have the following result on the existence and uniqueness of strong solutions:

Theorem 2.1 *Assume that the non-cylindrical domain $\widehat{Q}$ satisfies the geometric conditions of section 2 above. Then, if $u_0 \in H_0^1(\Omega_0)$ and $f \in L^2(0,T;L^2(\Omega_t))$, problem* (3) *has a unique strong solution u in the class* (4).

Moreover, there exists a positive constant C (depending on $\widehat{Q}$ but independet of u_0 and f) such that

$$\begin{aligned} &||u||_{L^\infty(0,T;H_0^1(\Omega_t))} + ||u'||_{L^2(\widehat{Q})} + ||u||_{L^2(0,T;H^2(\Omega_t))} \\ &\qquad \leq C[||u_0||_{H_0^1(\Omega)} + ||f||_{L^2(\widehat{Q})}], \end{aligned} \tag{5}$$

and

$$\begin{aligned} &||u||_{L^\infty(0,T;H_0^1(\Omega_t))} + ||u||_{L^2(0,T;H^2(\Omega_t))} \\ &\quad \leq C[||u_0||_{H_0^1(\Omega)} + ||f||_{L^1(0,T,H_0^1(\Omega_t))}]. \end{aligned} \tag{6}$$

Proof. We use a classical idea consisting in transforming the heat equation in the non-cylindrical domain $\widehat{Q}$, into a variable coefficient parabolic equation in the reference cylinder Q by means of the diffeomosphism: $(x,t) = (\tau_t(y),t) = (\tau(y,t),t)$, for $x \in \Omega_t, y \in \Omega, 0 \leq t \leq T$, i. e. for $(x,t) \in \widehat{Q}$ and $(y,t) \in Q$.

In fact, set

$$v(y,t) = u(\tau_t(y),t) = u(\tau(y,t),t) \quad \text{for} \quad y \in \Omega,\ 0 < t < T,$$

or, equivalently,

$$u(x,t) = v(\tau_t^{-1}(x),t) = v(\rho(x,t),t) \quad \text{for} \quad x \in \Omega_t,\ 0 < t < T.$$

Here and in the sequel τ_t^{-1} denotes the inverse of τ_t, which, according to assumption (1), is a C^2-map from Ω_t to Ω, for all $0 \leq t \leq T$. This map will be denoted by ρ_t. We shall also use the notation $\rho(x,t) = \rho_t(x)$.

We have,

$$\partial u(x,t)/\partial t = u'(x,t) = \frac{\partial v}{\partial t}(\rho(x,t),t) + \nabla_y v(\rho(x,t),t) \cdot \partial(\rho(x,t))/\partial t,$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$. In other words,

$$\partial u(x,t)/\partial t = u'(x,t) = \frac{\partial v}{\partial t}(y,t) + \nabla_y v(y,t) \cdot b(y,t),$$

where b denotes the vector field

$$b(y,t) = \partial(\rho(x,t)/\partial t.$$

Note that, according to assumption (2),

$$b \in C^0(\bar{Q}).$$

On the other hand,

$$\partial_{x_i} u(x,t) = \nabla_y v(y,t) \cdot \partial_{x_i}(\rho(x,t)),$$

and

$$\partial^2 u(x,t)/\partial x_i^2 =< \mathcal{D}_y^2 v(y,t)\partial_{x_i}(\rho(x,t)), \partial_{x_i}(\rho(x,t)) > + \nabla_y v(y,t) \cdot \partial^2(\rho(x,t))/\partial {x_i}^2,$$

where $D_y^2 v$ denotes the Hessian matrix of v in the space variable y and $\langle \mathcal{D}_y^2 v \cdot, \cdot \rangle$ the corresponding bilinear form.

Taking this into account, system (3) may be rewritten in the following equivalent way:

$$\left| \begin{array}{l} v' + A_t v + b \cdot \nabla_y v = f \quad \text{for} \quad (y,t) \in Q \\ u = 0 \quad \text{for} \quad (y,t) \in \Sigma \\ v(y,0) = u_0(y) \quad \text{for} \quad y \in \Omega. \end{array} \right. \tag{7}$$

Here and in the sequel, the operator A_t is defined as follows:

$$[A_t v](y) = -\sum_{i=1}^{n}[< \mathcal{D}_y^2 v(y)\partial_{x_i}(\rho(x,t)), \partial_{x_i}(\rho(x,t)) > +\nabla_y v(y)\cdot\partial^2(\rho(x,t))/\partial {x_i}^2].$$

System (7) is a variable coefficient parabolic equation in the cylindrical domain Q.

In order to apply the existent classical results, it is important to observe that the operators A_t are uniformly coercive. Indeed, let us represent by $J_t(y)$ the Jacobian of the transformation $\Omega_t = \tau_t(\Omega)$. By the assumptions (1)-(2) on τ_t it follows that there exists $\delta > 0$ such that $|J_t(y)| \geq \delta > 0$ in Q. Thus,

$$\int_\Omega [A_t v](y)v(y)dy \geq c\int_\Omega [A_t v](y)v(y)|J_t(y)|dy$$

$$\geq c\int_{\Omega_t} -\Delta u(x)u(x)dx = \int_{\Omega_t} |\nabla u(x)|^2 dx \geq c'\int_\Omega |\nabla_y v|^2 dy$$

for all $v \in H^2 \cap H_0^1(\Omega)$ (which is equivalent to saying that $u \in H^2 \cap H_0^1(\Omega_t)$, since τ_t is a C^2-diffeomorphism). Moreover, the coefficients in the principal part of the elliptic operator A_t are in C^1 and those of the first order term involving $\nabla_y v$ are continuous.

Then, classical results on parabolic equations (see Lions and Magenes [23]) guarantee that system (7) admits a unique strong solution $v \in C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$. Undoing the change of variables $x \to y$, we deduce the existence of an unique strong solution u of system (3). At this point we underline that, under assumptions (1)-(2), the transformation $y \to x$ does indeed map the space of functions $C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$ into $C([0,T]; H_0^1(\Omega_t)) \cap L^2(0,T; H^2(\Omega_t)) \cap H^1(0,T; L^2(\Omega_t))$.

In order to prove the estimate (5), we first establish the classical *energy estimate*. Multiplying in (3) by u and integrating with respect to $x \in \Omega_t$ and t, we get

$$\begin{aligned} ||u(t)||^2_{L^2(\Omega_t)} + \int_0^t ||\nabla u(s)||^2_{L^2(\Omega_s)}\, ds &= ||u_0||^2_{L^2(\Omega)} \\ + \int_0^t \int_{\Omega_t} fu\, dxds &\leq ||u_0||^2_{L^2(\Omega)} \\ + C_\varepsilon ||f||^2_{L^2(0,T,H^{-1}(\Omega_t))} &+ \varepsilon ||u||^2_{L^2(0,T,H_0^1(\Omega_t))}, \end{aligned} \tag{8}$$

for any $\varepsilon > 0$ for a suitable constant C_ε independent of the solution.

Here we have used in a fundamental way the fact that, since u vanishes on the lateral boundary $\widehat{\Sigma}$ of $\widehat{Q}$, we have (see Duvaut [11], p. 26)

$$\int_0^t \int_{\Omega_t} uu' dxdt = \frac{1}{2} \int_0^t \int_{\Omega_t} \frac{\partial |u|^2}{\partial t} dxdt = \frac{1}{2} \Big[\int_{\Omega_t} u^2(x,t)dx - \int_{\Omega} u_0^2(x)dx \Big].$$

We now use the fact that Poincaré's inequality is satisfied uniformly in the domains Ω_t for all $0 \leq t \leq T$. This is again a consequence of assumptions (1)-(2). Then, in view of (8) we have

$$\begin{aligned} ||u(t)||^2_{L^2(\Omega_t)} + \frac{1}{2} \int_0^t ||\nabla u(s)||^2_{L^2(\Omega_s)}\, ds \leq ||u_0||^2_{L^2(\Omega)} \\ + C||f||^2_{L^2(0,T,H^{-1}(\Omega_t))}, \end{aligned} \tag{9}$$

for a suitable $C > 0$. In particular, strong solutions satisfy the *energy estimate*

$$\begin{aligned} ||u||^2_{L^\infty(0,T;L^2(\Omega_t))} + ||u||^2_{L^2(0,T;H_0^1(\Omega_t))} \\ \leq C[||u_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,T,H^{-1}(\Omega_t))}], \end{aligned} \tag{10}$$

with $C > 0$ independent of the solution.

We now multiply (3) by $-\Delta u$. We then have

$$-\int_{\Omega_t} u' \Delta u dx + \int_{\Omega_t} |\Delta u|^2 dx = -\int_{\Omega_t} f \Delta u dx = \int_{\Omega_t} \nabla f \cdot \nabla u dx. \tag{11}$$

Moreover,

$$-\int_{\Omega_t} u' \Delta u dx = \int_{\Omega_t} \nabla u \cdot \nabla u' dx = \frac{d}{dt} \int_{\Omega_t} |\nabla u|^2 dx - \int_{\Gamma_t} |\nabla u|^2 w \cdot n_t d\sigma, \tag{12}$$

where n_t denotes the unit outward normal vector to Ω_t and w is the velocity field $w = [\partial_t \tau](\rho(x,t))$ (see Duvaut [11], p. 26). Note that, according to (1)-(2), by uniform (with respect to t) elliptic regularity, classical trace results and interpolation, we have

$$|\int_{\Gamma_t} |\nabla u|^2 w \cdot n_t \, d\sigma| \leq C \int_{\Gamma_t} |\nabla u|^2 d\sigma \leq C_\alpha [\int_{\Omega_t} |\Delta u|^2 dx]^\alpha [\int_{\Omega_t} |\nabla u|^2 dx]^{1-\alpha} \tag{13}$$

for all $\alpha \geq 1/2$.

Combining (12)-(13) and the Cauchy-Schwarz' inequality we deduce that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} |\nabla u|^2 dx \leq -\frac{1}{2} \int_{\Omega_t} |\Delta u|^2 dx + ||\nabla f||_{L^2(\Omega_t)} ||\nabla u||_{L^2(\Omega_t)} \\ + c||\nabla u||^2_{L^2(\Omega_t)}. \end{aligned} \tag{14}$$

Solving this differential inequality we deduce the existence of a constant C that only depends on $\widehat{Q}$ such that

$$\begin{aligned} ||u||^2_{L^\infty(0,T;H^1_0(\Omega_t))} + ||u||^2_{L^2(0,T;H^2(\Omega_t))} \le C[||u_0||^2_{H^1_0(\Omega)} \\ + ||f||^2_{L^1(0,T,H^1_0(\Omega_t))}]. \end{aligned} \tag{15}$$

A slight variation in this argument allows also to get

$$\begin{aligned} ||u||^2_{L^\infty(0,T;H^1_0(\Omega_t))} + ||u||^2_{L^2(0,T;H^2(\Omega_t))} \le C[||u_0||^2_{H^1_0(\Omega)} \\ + ||f||^2_{L^2(\widehat{Q})}]. \end{aligned} \tag{16}$$

Indeed, for getting (16) instead of (15) it is sufficient to estimate the term $\int_{\Omega_t} f\Delta u dx$ as follows

$$\int_{\Omega_t} f\Delta u dx \le \frac{1}{2}[\int_{\Omega_t} (|f|^2 + |\Delta u|^2)dx].$$

This completes the proof of the estimates (5) and (6) of the Theorem.

□

Remark. Note that we could also have obtained these estimates using the existing results for the variable coefficient parabolic equation satisfied by v and then undoing the change of variable $x \to y$. But we have prefered to work directly on system (3) to see how the non-cylindrical structure of the domain afects the estimates.

□

2.2 Weak Solutions

We say that u is a weak solution of (3) if

$$u \in C^0([0,T]; L^2(\Omega_t)) \cap L^2([0,T]; H^1_0(\Omega_t)) \tag{17}$$

and

$$\begin{aligned} -\int_0^T \int_{\Omega_t} u\,\varphi'\,dxdt - \int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T \int_{\Omega_t} \nabla_x u \cdot \nabla_x \varphi\,dxdt \\ = \int_0^T \int_{\Omega_t} f\,\varphi\,dxdt \end{aligned} \tag{18}$$

for all $\varphi \in L^2(0,T; H^1_0(\Omega_t)) \cap C^1([0,T]; L^2(\Omega_t))$ such that $\varphi(T) = 0$.

The following holds:

Theorem 2.2 *Given $u_0 \in L^2(\Omega_0)$ and $f \in L^2(0,T;H^{-1}(\Omega_t))$ there exists a unique weak solution of* (3).

Moreover, there exists a constant C (dependending on $\widehat{Q}$ but independent of the data u_0 and f) such that

$$||u||_{L^\infty(0,T;L^2(\Omega_t))} + ||\nabla u||_{L^2(\widehat{Q})} \leq C[||u_0||_{L^2(\Omega)} + ||f||_{L^2(0,T;H^{-1}(\Omega_t))}]. \tag{19}$$

A similar argument allows to replace the assumption $f \in L^2(0,T;H^{-1}(\Omega_t))$ by $f \in L^1(0,T;L^2(\Omega_t))$ and to obtain the estimate

$$||u||_{L^\infty(0,T;L^2(\Omega_t))} + ||\nabla u||_{L^2(\widehat{Q})} \leq C[||u_0||_{L^2(\Omega)} + ||f||_{L^1(0,T;L^2(\Omega_t))}]. \tag{20}$$

Proof: We proceed in two steps.
Step 1. Existence. Let $u_{0m} \in H_0^1(\Omega_0)$ and $f_m \in L^2(\widehat{Q})$ be a sequence of regularized initial data and right hand side terms respectively such that

$$u_{0m} \to u_0 \quad \text{strongly in} \quad L^2(\Omega_0), \quad f_m \to f \quad \text{strongly in} \quad L^2(0,T;H^{-1}(\Omega_t)).$$

Then, for each $m \in \mathbb{N}$, let us consider the unique strong solution u_m of (3) with initial data u_{0m} and right hand side f_m, i.e. of system

$$\left| \begin{array}{l} u_m' - \Delta u_m = f \quad \text{a.e. in} \quad \widehat{Q} \\ u_m = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u_m(0) = u_{0m} \quad \text{in} \quad \Omega. \end{array} \right. \tag{21}$$

Thus, for any $n,k \in \mathbb{N}$, we have

$$\left| \begin{array}{l} (u_n - u_k)' - \Delta(u_n - u_k) = f_n - f_k \quad \text{a.e. in} \quad \widehat{Q} \\ u_n - u_k = 0 \quad \text{on} \quad \widehat{\Sigma} \\ (u_n - u_k)(0) = u_{0n} - u_{0k} \quad \text{in} \quad \Omega. \end{array} \right. \tag{22}$$

Using the energy estimate (10) we obtain that u_m is a Cauchy sequence in the space $C^0([0,T];L^2(\Omega_t)) \cap L^2([0,T];H_0^1(\Omega_t))$. Thus, it converges, as $n \to \infty$, to a limit $u \in C^0([0,T];L^2(\Omega_t)) \cap L^2([0,T];H_0^1(\Omega_t))$.

It is easy to see that the limit u is a weak solution of (3) satisfying (18) and the estimate (19). Indeed, for every n, u_n is a strong solution. Multiplying the equation satisfied by u_n by a test function φ and integrating by parts we deduce that u_n satisfies also the weak formulation (18). The convergence in the space $C^0([0,T];L^2(\Omega_t)) \cap L^2([0,T];H_0^1(\Omega_t))$ of u_n towards u allows to pass to the limit in the weak formulation to deduce that u satisfies (18), too.

Step 2. Uniqueness. Uniqueness can be proved by a classical argument. Assume that system (3) admits two weak solutions u and v satisfying (18). Introduce $w = u - v$. Then, $w \in C^0([0,T]; L^2(\Omega_t)) \cap L^2([0,T]; H_0^1(\Omega_t))$ satisfies

$$\int_0^T \int_{\Omega_t} w\,\varphi'\,dxdt - \int_0^T \int_{\Omega_t} \nabla_x w \cdot \nabla_x \varphi\,dxdt = 0,$$

for all test function φ. In order to conclude that $w \equiv 0$ it is sufficient to use $\varphi = w$ as test function. Of course, one can not do it directly, since w does not belong to the admissible class of test functions but this choice may be justified using a classical regularization and cut-off argument.

In this way we end up getting the energy estimate for w that guarantess that

$$|w(t)|^2_{L^2(\Omega_t)} + \int_0^t ||\nabla w(s)||^2_{L^2(\Omega_s)} ds \leq 0.$$

Obviously, this implies that $w \equiv 0$.

Step 3. Estimate (20). Estimate (20) can be proved easily. It suffices to change the way of estimaing the term $\int_{\Omega_t} fu dx$ in the classical energy estimate leading to (10). More precisely, it suffices to use the upper bound

$$\left| \int_0^T \int_{\Omega_t} fu\,dxdt \right| \leq ||f||_{L^1(0,T;L^2(\Omega_t))} ||u||_{L^\infty(0,T;L^2(\Omega_t))}.$$

□

2.3 Ultra Weak Solutions by the Transposition Method

In this section we address the question of finding solutions u of

$$\left| \begin{array}{l} u' - \Delta u = 0 \quad \text{in} \quad \Omega_t \quad \text{for} \quad 0 \leq t < T \\ u = 0 \quad \text{on} \quad \Gamma_t \quad \text{for} \quad 0 < t < T \\ u(0) = u_0 \quad \text{in} \quad \Omega \end{array} \right. \tag{23}$$

where u_0 is given in $H^{-1}(\Omega)$.

When the domain where the equation holds is cylindrical, i. e. when $\Omega_t = \Omega$ for all $0 \leq t \leq T$, the problem can be easily reduced to that of weak or strong solutions. Indeed, in that case, u solves (23) if and only if $v = (-\Delta)^{-s} u$ satisfies the same equation with initial data $v_0 = (-\Delta)^{-s} u_0$. Here $-\Delta$ denotes the Dirichlet laplacian in Ω. By taking $s = 1/2$ we then have $v_0 \in L^2(\Omega)$ and v turns out to be a weak solution. When $s = 1$, $v_0 \in H_0^1(\Omega)$ and v is then a strong solution. In any case, when the domain is cylindrical, there is a one to one correspondence between weak, strong and ultra weak solutions.

In the present case, where the domain Ω_t does depend on time, this argument does not apply directly. Then it is better to use the *method of transposition* (see Lions and Magenes [23]).

A function $u = u(x,t)$ is said to be an *ultra weak solution* of (23) or solution by transposition if

$$u \in C([0,T]; H^{-1}(\Omega_t)) \cap L^2(0,T; L^2(\Omega_t)) \tag{24}$$

and

$$\int_0^T \int_{\Omega_t} u(x,t) f(x,t)\, dxdt = \langle u_0, \varphi(0)\rangle, \quad \forall f \in L^2(\widehat{Q}),$$

where φ is the unique solution of the adjoint system

$$\left|\begin{array}{l} -\varphi' - \Delta\varphi = f \quad \text{in} \quad \Omega_t \quad \text{for} \quad 0 < t < T \\ v = 0 \quad \text{on} \quad \Gamma_t \quad \text{for} \quad 0 < t < T \\ \varphi(x,T) = 0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{25}$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

According to Theorem 2.1 system (25) admits an unique strong solution φ. Thus, this definition makes sense.

In order to prove the existence and uniqueness of a ultra weak solution u it is sufficient to observe that there exists a constant $C > 0$, which is independent of f, such that the strong solution φ satisfies the following estimates

$$||\varphi||_{L^\infty(0,T;H_0^1(\Omega_t))} \leq C||f||_{L^2(\widehat{Q})}, \tag{26}$$

and

$$||\varphi||_{L^\infty(0,T;H_0^1(\Omega_t))} \leq C||f||_{L^1(0,T;H_0^1(\Omega_t))}, \tag{27}$$

These estimates were proved in Theorem 2.1. Indeed, it is sufficient to make the change of variable $t \to T - t$ to reduce system (25) to (3).

Then, by duality or, more precisely, as a consequence Riesz-Frechet theorem, we deduce that there exists a unique ultra weak solution in the class (24). To be more precise, in view of (26), we deduce the existence of an unique solution $u \in L^2(\widehat{Q})$ and the second estimate (27) provides the additional regularity $u \in L^\infty(0,T; H^{-1}(\Omega_t))$. Moreover, one deduces the existence of a constant, independent of u_0, such that

$$||u||_{L^\infty(0,T;H^{-1}(\Omega_t))} + ||u||_{L^2(\widehat{Q})} \leq C||u_0||_{H^{-1}(\Omega)}. \tag{28}$$

In order to show that $u \in C([0,T]; H^{-1}(\Omega_t))$ we use a classical density argument. When u_0 is smooth enough, u is a weak (or even strong) solution, and therefore u is indeed continuous with respect to time with values in $H^{-1}(\Omega_t)$. According to (28), by density, we deduce that $u \in C([0,T]; H^{-1}(\Omega_t))$ whenever $u_0 \in H^{-1}(\Omega)$.

□

Remark. It is easy to see that, when the datum u_0 is more smooth so that weak and/or strong solutions exist, they coincide with the ultra weak solution. For, it is sufficient to integrate by parts in the strong formulation of system (3) or in its weak formulation, depending on wether we are dealing with strong or weak solutions.

□

3 Controllability

In this section we discuss the controllability properties of solutions of the parabolic equation (3) in the non-cylindrical domain $\widehat{Q}$.

We denote by $\widehat{q}$ an open, non-empty subset of $\widehat{Q}$. We also denote by ω_t its cross section at any $0 < t < T$ and by $\chi_{\widehat{q}}$ the characteristic set of $\widehat{q}$.

We consider the heat equation

$$\left| \begin{array}{l} u' - \Delta u = h\,\chi_{\widehat{q}} \quad \text{in} \quad \widehat{Q} \\ u = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{29}$$

Here the function $h = h(x,t)$ plays the role of the control that acts on the system through the subset $\widehat{q}$.

The initial data, to fix ideas, is taken to be in $L^2(\Omega)$. The control function h is assumed to be in $L^2(0,T; L^2(\Omega_t))$. Under these conditions, from Section 2 we know that problem (29) has a unique weak solution u with the regularity

$$u \in C([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t)).$$

The control problem we address is the following: *Can we describe the set of reachable states at time $t = T$ when h varies in $L^2(\widehat{Q})$? and, more precisely, is this set dense in $L^2(\Omega_T)$? Does it contain the equilibrium state $u \equiv 0$?*

The density of the range of solutions of (29) at time $t = T$ is referred to as the *approximate controllability* property, while the property of guaranteeing that the zero equilibrium state is reachable is referred to as *null controllability*. These two properties are analyzed in the following two sections.

3.1 Approximate controllability

The property of approximate controllability may be formulated as follows: Given $u_0 \in L^2(\Omega_0)$, $u_1 \in L^2(\Omega_T)$ and $\varepsilon > 0$ to find a control $h \in L^2(\widehat{q})$ such that the solution of (29) satisfies

$$||u(T) - u_1||_{L^2(\Omega_T)} \leq \varepsilon. \tag{30}$$

In [35] the following (apparently) stronger notion of controllability was introduced, the so called *approximate-finite controllability* property: Given $u_0 \in L^2(\Omega)$, $u_1 \in L^2(\Omega_T)$, $\varepsilon > 0$ and a finite-dimensional subspace E of $L^2(\Omega_T)$, to find a control $h \in L^2(\widehat{Q})$, such that the solution u of (29) satisfies

$$\left| \begin{array}{l} |u(T) - u_1|_{L^2(\Omega_T)} \leq \varepsilon \\ \pi_E\, u(T) = \pi_E(u_1). \end{array} \right. \tag{31}$$

Here and in the sequel $\pi_E \colon L^2(\Omega_T) \to E$ denotes the orthogonal projection from $L^2(\Omega_T)$ into E.

In [35] this property of approximate-finite controllability was analyzed for the semilinear heat equation in a cylinder. It was proved that, essentially, the methods introduced in [18] and developed in [12] suffice to show that, when the nonlinearity is globally Lipschitz, approximate-finite controllability holds. Later, in [25], it was proved that, in the context of general linear systems, approximate controllability implies approximate-finite controllability.

The following result shows that this property is still true for the heat equation in the non-cylindrical domain $\widehat{Q}$.

Theorem 3.1 *For any $T > 0$, $u_0 \in L^2(\Omega), u_1 \in L^2(\Omega_T)$, $\varepsilon > 0$ and E, finite-dimensional subspace of $L^2(\Omega_T)$, there exists a control $h \in L^2(\hat{q})$ such that the corresponding solution of* (29) *satisfies* (31).

In other words, system (29) *is approximate-finite controllable in any time* $T > 0$.

Proof: Taking into account that the equation (29) is linear, it is sufficient to consider the case where $u_0 \equiv 0$. Thus, in the sequel, we shall assume that $u_0 \equiv 0$.

According to the abstract result in [25], the problem of approximate-finite controllability can be reduced to that of approximate controllability. However, for the sake of completeness, we proceed as in [35], where a constructive approach for building the approximate-finite control is presented.

Let us consider the adjoint system:

$$\left| \begin{array}{l} -\varphi' - \Delta\varphi = 0 \quad \text{in} \quad \widehat{Q} \\ \varphi = 0 \quad \text{on} \quad \widehat{\Sigma} \\ \varphi(T) = \varphi^0 \quad \text{in} \quad \Omega_T \end{array} \right. \tag{32}$$

for $\varphi^0 \in L^2(\Omega_T)$. From Theorem 2.2 (it is sufficient to apply it after reversing the sense of time) the solution φ of (32) has the regularity $\varphi \in C([0,T]; L^2(\Omega_t)) \cap L^2(0,T; H_0^1(\Omega_t))$. We define the functional

$$J(\varphi^0) = \frac{1}{2} \int_{\widehat{q}} \varphi^2 \, dxdt + \varepsilon ||(I - \pi_E)\varphi^0||_{L^2(\Omega_T)} - \int_{\Omega_T} u_1 \, \varphi^0 \, dx, \tag{33}$$

where φ is the solution of (32) corresponding to φ^0.

It is easy to see that the functional J satisfies the conditions:

(i) J is continuous in $L^2(\Omega_T)$,

(ii) J is coercive,

(iii) J is strictly convex.

Indeed:

(i) The continuity of J is a direct consequence of the well-posedness properties of the adjoint system (32) and, more precisely, of the continuous dependence of weak solutions of (32) with respect to the initial data φ^0 in $L^2(\Omega_T)$.

(ii) To prove the coercivity it is sufficient to show that

$$\liminf_{||\varphi^0||_{L^2(\Omega_T)} \to \infty} \frac{J(\varphi^0)}{||\varphi^0||_{L^2(\Omega_T)}} \geq \varepsilon, \qquad \varepsilon > 0. \tag{34}$$

We argue as in [12] and [35].

Let $\varphi_j^0 \in L^2(\Omega_T)$ be such that $||\varphi_j^0||_{L^2(\Omega_T)} \to \infty$ and φ_j be the corresponding solutions of (32). We introduce the normalized data $\hat{\varphi}_j^0 = \varphi_j^0 / ||\varphi_j^0||_{L^2(\Omega_T)}$ and the corresponding solutions $\hat{\varphi}_j = \varphi_j / ||\varphi_j^0||_{L^2(\Omega_T)}$. Then

$$\begin{aligned} \frac{J(\varphi_j^0)}{||\varphi_j^0||_{L^2(\Omega_T)}} &= \frac{1}{2} ||\varphi_j^0||_{L^2(\Omega_T)} \int_{\widehat{q}} |\hat{\varphi}_j|^2_{L^2(\Omega_T)} \, dxdt + \varepsilon ||(I - \pi_E)\hat{\varphi}_j^0||_{L^2(\Omega_T)} - \\ &\quad - \int_{\Omega_T} u_1 \, \hat{\varphi}_j^0 \, dx. \end{aligned} \tag{35}$$

We have to distinguish two cases:

$$\liminf_{||\varphi^0||_{L^2(\Omega_T)}\to\infty} \int_{\widehat{q}} |\hat{\varphi}_j|^2 \, dxdt > 0$$

or

$$\liminf_{||\varphi^0||_{L^2(\Omega_T)}\to\infty} \int_{\widehat{q}} |\hat{\varphi}_j|^2 \, dxdt = 0.$$

In the first case property (34) is obvious since the first term on the right hand side of (35) tends to infinity.

Let us now consider the second case. Then, there exists a subsequence of $(\hat{\varphi}_j)$ (still represented by $(\hat{\varphi}_j)$) such that

$$\int_{\widehat{q}} |\hat{\varphi}_j|^2 \, dxdt \to 0. \tag{36}$$

Taking into account that $||\hat{\varphi}_j^0||_{L^2(\Omega_T)} = 1$, we also have (for another suitable subsequence)

$$\hat{\varphi}_j^0 \rightharpoonup \hat{\varphi}^0 \quad \text{weakly} \quad L^2(\Omega_T), \quad \pi_E\hat{\varphi}_j^0 \to \pi_E\hat{\varphi}^0 \quad \text{strongly} \quad L^2(\Omega_T). \tag{37}$$

Using the well-posedness properties of weak solutions of (32) we deduce that

$$\hat{\varphi}_j \rightharpoonup \hat{\varphi} \quad \text{weakly in} \quad L^2(\widehat{Q}), \tag{38}$$

where $\hat{\varphi}$ is the solution of (32) corresponding to $\hat{\varphi}^0$. According to (36) we have

$$\hat{\varphi} = 0 \quad \text{a.e. in} \quad \widehat{q}.$$

By Holmgren's Uniqueness Theorem we deduce that

$$\hat{\varphi} = 0 \quad \text{a.e. in the horizontal component of } \widehat{q}.$$

By backward uniqueness (see [24], [15] and [13]) we deduce that $\hat{\varphi}(T) = \hat{\varphi}^0 = 0$ and therefore

$$\hat{\varphi} \equiv 0. \tag{39}$$

Consequently, from (38)-(39) we obtain

$$\hat{\varphi}_j^0 \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega_T) \tag{40}$$

and

$$\pi_E\hat{\varphi}_j(T) \to 0 \quad \text{strongly in} \quad L^2(\Omega_T). \tag{41}$$

Taking into account that $||\hat{\varphi}_j^0||_{L^2(\Omega_T)} = 1$, we deduce that

$$||(I - \pi_E)\hat{\varphi}_j^0||_{L^2(\Omega_T)} \to 1. \tag{42}$$

According to (40) we deduce that $\int_{\Omega_T} u_1 \,\hat{\varphi}_j^0 \,dx \to 0$ as $j \to \infty$. And this fact, combined with (41), implies that (34) holds along the sequence φ_j.

This completes the proof of the coercivity.

(iii) The strict convexity of J is easy to prove taking into account its structure and the unique continuation property, consequence of Holmgren's Uniqueness Theorem and backward uniqueness, guaranteeing that if $\varphi = 0$ in $\widehat{q}$, then $\varphi \equiv 0$ everywhere.

In view of properties (i)-(iii) that the functional J satisfies, it follows that there exists a unique $\hat{\varphi}^0 \in L^2(\Omega_T)$ minimizing J in $L^2(\Omega_T)$. Set $\hat{\varphi}$ the solution of (32) corresponding to $\hat{\varphi}^0$.

It is then easy to see, following the arguments in [12] and [35], that $h = \hat{\varphi}$ is the control we are looking for such that (31) holds.

3.2 Null controllability

In the previous section we have proved that the set of reachable states for system (29) is dense in $L^2(\Omega_T)$. But this property does not guarantee any particular element of $L^2(\Omega_T)$ to belong to the set of reachable states.

In this section we study a different notion of controllability, the so called property of null controllability in which one wonders whether the zero state belongs to the set of reachable states. In other words: *Given $u_0 \in L^2(\Omega)$ and $T > 0$, is there $h \in L^2(\widehat{Q})$ such that the solution of* (29) *satisfies $u(T) \equiv 0$?*

This property, because of the backward uniqueness property, is stronger that the approximate controllability one.

The following holds:

Theorem 3.2 *Under assumptions* (1)-(2)*, for any $T > 0$ and any open non-empty subset $\widehat{q}$ of $\widehat{Q}$, system* (29) *is null-controllable.*

Proof. It is sufficient to show that the following observability inequality holds for the adjoint system (32): There exists a positive constant $C > 0$ such that

$$||\varphi(0)||^2_{L^2(\Omega)} \leq \int_{\widehat{q}} |\varphi|^2 dxdt \tag{43}$$

for every solution of the adjoint system (32).

Inequality (43) is a direct consequence of the results in [16]. Indeed, making the change of variables $x \to y$, the adjoint system (32) may be transformed into a variable coefficient equation of the form

$$\left| \begin{array}{l} -\psi' + A_t\psi - b \cdot \nabla\psi = 0 \quad \text{in} \quad Q \\ \psi = 0 \quad \text{on} \quad \Sigma \\ \psi(T) = \varphi^0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{44}$$

The operators A_t and the coefficient b are as in section 2. Thus, in particular, the coefficients in the principal part of A_t, according to (1) and (2) are of class C^1 and b is bounded. Thus, the observability inequalities in [16] guarantee that, for every $T > 0$ and every open subset q of Q, there exists a constant $C > 0$ such that

$$||\psi(0)||^2_{L^2(\Omega)} \leq C \int_q |\psi|^2 dxdt. \tag{45}$$

This applies in particular to q obtained from $\widehat{q}$ by means of the change of variables $x \to y$. Estimate (43) can be easily obtained from (45) by undoing this change of variables.

Once the observability inequality (31) holds the null controllability of Theorem 3.2 is easy to prove. Indeed, given any u_0 in $L^2(\Omega)$ and $\varepsilon > 0$ we consider the functional

$$\widehat{J}(\varphi^0) = \frac{1}{2}\int_{\widehat{q}} \varphi^2 \, dxdt + \varepsilon||\varphi^0||_{L^2(\Omega_T)} + \int_\Omega u_0 \, \varphi(0) \, dx, \tag{46}$$

where φ is the solution of (32) corresponding to φ^0.

Following the arguments of the previous section it is easy to see that $\widehat{J}$ has a unique minimizer φ^0_ε and that, by taking $h_\varepsilon = \varphi_\varepsilon$ where φ_ε is the solution of the adjoint system (32) with datum φ^0_ε, then the solution of (29) satisfies

$$||u(T)||_{L^2(\Omega_T} \leq \varepsilon. \tag{47}$$

Moreover, the observability inequality (43) allows to show that the functional $\widehat{J}_\varepsilon$ are uniformly coercive as ε tends to zero. This allows to show that the controls h_ε are uniformly bounded. By letting $\varepsilon \to 0$ we obtain, as weak limit of h_ε, a control h such that, according to (47), the corresponding state u satisfies $u(T) \equiv 0$.

This completes the proof of the Theorem.

4 Further comments

4.1 On the assumptions (1)-(2)

The assumptions (1)-(2) are not needed for all the results in this paper. Indeed, for instance, the C^2 regularity assumption of the domain and of the diffeomorphism τ_t is necessary to establish the existence of strong solutions in the class (4) but less regularity is required to prove the existence of weak solutions. For that, it is sufficient the domain Ω and the diffeomorphism τ_t to be of class C^1. The same can be said about the approximate controllability property. It is sufficient to have enough regularity on the domain $\widehat{Q}$ for the parabolic equation to be well-posed to have also, immediately, the approximate controllability property. Indeed, the approximate controllability property turns out to be equivalent to the uniqueness or unique continuation property for the adjoint system and this is true without regularity assumptions on $\widehat{Q}$ since the equation has constant coefficients and Holmgren's uniqueness Theorem applies. The situation is different in what concerns the null-controllability property. Indeed, note that, in that case, we have applied the observability inequalities in [16] to the adjoint equation after the coordinate transformation $x \to y$ and this requires C^1 or Lipschitz coefficients in the principal part. Thus, the geometric assumptions (1)-(2) are almost necessary in this case to apply the existing results.

Note however that, as far as we know, there is no evidence in the literature of parabolic equation with bounded and coercive coefficients for which the null-controllability property fails. Thus, one can not exclude the null-controllability to hold under much weaker assumptions than (1)-(2).

4.2 On the boundedness assumption on the domain Ω

All along this paper, to simplify the presentation, we have assumed the reference domain Ω to be bounded. But this assumption is only necessary when dealing with the null-controllability property. Indeed, as shown in [27], [28], the heat equation in an unbounded domain may fail to be null-controllable if the set which is left without control is also unbounded. However, whether the domain Ω is bounded or not is irrelevant for approximate controllability.

The results in [27], [28] indicate that, in order for the null-controllability property to hold it is natural to assume that the subdomain that is left without control is bounded. In that situation, in the context of cylindrical domains, the null-controllability property was proved to be true in [4]. When $\widehat{Q}$ is unbounded, combining the techniques of this paper and those in [4], under geometric assumptions of the form (1)-(2), system (29) may indeed be proved to be null-controllable if $\widehat{Q} \setminus \widehat{q}$ is bounded.

4.3 Semilinear equations

The results of this paper extend easily to semilinear heat equations. Indeed, in what concerns the well-posedness, it is sufficient to combine the results in this paper with the classical fixed point arguments for semilinear evolution equations. In what concerns controllability the results of this paper extend to semilinear equations in the following situations:

a) Approximate-finite controllability for globally Lipschitz nonlinearities following the methods developed in [12], [14] and [35].

b) Null-controllability for nonlinearities of the form $f = f(u)$ with f such that

$$f(s)/|s|\log^{3/2}(|s|) \to 0 \quad \text{as} \quad |s| \to \infty$$

by the techniques in [14].

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