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NONLINEAR WAVE EQUATION WITH A NONLINEAR BOUNDARY DAMPING IN A NONCYLINDRICAL DOMAIN

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Abstract

The mixed problem for a nonlinear wave equation in a domain with a moving boundary is considered. On the moving boundary a nonlinear Neumann type condition is given. Existence, uniqueness and the exponential decay of the energy are proved.

1 Introduction

In the last years there appeared a number of papers where solvability of mixed problems for hyperbolic equations in noncylindrical domains was studied [4, 11, 16]. It must be noted that the Goursat and Darboux problems much used in the theory of mixed type equations and known more than hundred years are maybe first mixed problems studied in nonrectangular characteristic domains. There is a lot of differences between hyperbolic problems in cylindrical domains and noncylindrical ones.

A type of a mixed problem in a noncylindrical domain depends on the type of the lateral surface: it may be the mixed problem or the Cauchy problem depending on the inclination of the lateral surface, see Kozhanov, Lar'kin [9], Cooper, Medeiros [2].

To construct solutions for nonlinear hyperbolic problems in noncylindrical domains one can use either continuation of equations into a cylindrical domain or smooth hyperbolic transformations into hyperbolic problems in a cylinder. Then in a cylinder the Galerkin method or semigroup theory are available.

Continuation of hyperbolic equations into a cylinder by means of the penalty method was proposed by J-L. Lions [13] and realized by Medeiros [15], see also

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[14, 16]. Unfortunately, this continuation is not smooth and gives only weak solutions.

If the lateral surface of a noncylindrical domain is constructed by characteristic surfaces, it is possible to continue a hyperbolic operator smoothly into a cylinder and to prove the existence of strong solutions, Lar'kin [11].

On the other hand, exploiting hyperbolic diffeomorphisms, one can try to transform a domain into a cylinder taking into account that the transformed equation becomes more complex [4, 5, 6, 7, 9, 10, 12].

Most papers cited above studied the Dirichlet condition on the lateral surface. In the present work we consider a nonlinear damping that includes the normal and tangential derivatives on the moving boundary. We transform our domain into a rectangle, then we use the Galerkin method to construct a unique strong solution and, finally, prove the exponential decay of the energy. It must be noted that the diffeomorphism leads to a nonlinear hyperbolic equation with the mixed derivative. This type of equations was not studied earlier when boundary damping was used in order to prove the decay of the energy [8]. To treat this case, we exploited a double regularization of the energy that helped to prove the exponential decay of it.

2 Formulation of the problem

Let

$$\mathcal{Q}_T = \{(x,t) \in R^2; \alpha(t) < x < 1; 0 < t < T\};$$

$$\mathcal{D}_\tau = \mathcal{Q}_T \cap \{\tau = t\},$$

where $\alpha(t)$ is a given function.

We consider the following mixed problem

$$u_{tt} - k^2 u_{xx} + \Phi(u) = f(x,t) \quad \text{in} \quad \mathcal{Q}_T, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \mathcal{D}_0, \tag{2.2}$$

$$u\,|_{x=1} = 0, \quad u_x - h(u_s)\Big|_{x=\alpha(t)} = 0, \quad t > 0. \tag{2.3}$$

Here $u_s = u_t + \alpha'(t)u_x$, k is a given constant; $u_0(x), u_1(x), \Phi(u), h(u_s)$ and $f(x,t)$ are given functions.

We use standard functional spaces, see [14]; some special notations will be defined such as

$$V(\mathcal{D}_t) = \{g \in H^1(\mathcal{D}_t), \quad g(1,t) = 0, t > 0\},$$

$$(u,v)(t) = \int_{\mathcal{D}_t} u(x,t)v(x,t)dx, \quad \|u(t)\|^2 = (u,u)(t).$$

Assumptions 1

1.1 $\Phi \in C^1(R); F(u) = \int_0^u \Phi(s)ds \geq 0, \quad \Phi(0) = 0.$

1.2 $h \in C^1(R), \quad h(0) = 0, \quad |h(u_s)| \leq h_1(1 + |u_s|^{\rho+1}),$
$h'(u_s) \geq h_0(u_s + |u_s|^{\rho}), \quad \rho \geq 0.$

1.3 $\alpha \in C^3[0,\infty); \quad 0 \leq \alpha_0 \leq 1 - \alpha(t) \leq L_0 < \infty, \quad \forall t \in [0,\infty);$
$\alpha(0) = 0; \quad \sup_{t\geq 0}\{\|\alpha'(t)\| + \|\alpha''(t)\| + \|\alpha'''(t)\|\} \leq L_1 < \infty.$

1.4 $\alpha'(t) \geq 0, \quad k^2 - |\alpha'(t)|^2 \geq k_0 > 0, \qquad \forall t \geq 0,$

where $k_0, \alpha_0, L_0, L_1, h_0, h_1$ are positive constants.

Definition 1 *A function $u(x,t)$:*

$$u \in L^\infty(0,T; H^2(\mathcal{D}_t) \cap V(\mathcal{D}_t)),$$

$$u_t \in L^\infty(0,T; V(\mathcal{D}_t)), \quad u_{tt} \in L^\infty(0,T; L^2(\mathcal{D}_t))$$

is called a strong solution to (2.1)-(2.3) if for any finite T it satisfies equation (2.1), *initial data* (2.2) *and boundary conditions (2.3).*

To prove the existence of a strong solution we transform the problem (2.1)–(2.3) into a problem in the rectangle $(0,1) \times (0,T) = \mathcal{Q}$.

2.1 Transformed problem

It is easy to see that due to Assumptions 1.3, 1.4 the diffeomorphism $(x,t) \Leftrightarrow (y,\tau)$ defined by

$$y = \frac{x - \alpha(t)}{1 - \alpha(t)}, \quad \tau = t, \quad v(y,\tau) = u(x(y,\tau),\tau) \tag{2.4}$$

with the consequential change from τ to t reduces (2.1)-(2.3) to the following hyperbolic problem

$$v_{tt} - a_1(y,t)v_{yy} - 2a_2(y,t)v_{yt} + b_1(y,t)v_y + \Phi(v) = f_1(y,t) \quad \text{in} \quad \mathcal{Q}, \tag{2.5}$$

$$\begin{aligned} &v(y,0) = u_0(y) = v_0(y), \\ &v_t(y,0) = u_1(y) + u_0'(y)(1-y)\alpha'(0) = v_1(y) \quad \text{in} \quad (0,1); \end{aligned} \tag{2.6}$$

$$v(1,t) = 0, \quad v_y - \gamma(t)h(v_t)\Big|_{y=0} = 0, \quad t > 0. \tag{2.7}$$

Here,

$$\begin{aligned} &\gamma(t) = 1 - \alpha(t); \\ &a_1(y,t) = \frac{k^2 - (1-y)^2|\alpha'(t)|^2}{\gamma^2(t)}; \\ &a_2(y,t) = \frac{(1-y)\alpha'(t)}{\gamma^2(t)}; \\ &b_1(y,t) = -\frac{\partial a_2}{\partial t}; \qquad f_1(y,t) = f(x(y,t),t). \end{aligned} \tag{2.8}$$

From (2.4) and Definition 1, it follows that if $u(x,t)$ is a strong solution to (2.1)-(2.3), then

$$v(y,t) = u(x(y,t),t);$$

$$v \in L^\infty(0,T;H^2(0,1) \cap V(0,1));$$

$$v_t \in L^\infty(0,T;V(0,1)), \quad v_{tt} \in L^\infty(0,T;L^2(0,1))$$

is a strong solution to (2.5)-(2.7), and vice versa.

We can aplly the Galerkin method in order to prove solvability of (2.5)-(2.7). To simplity calculations and to spare the time and space, we start from the homogeneous initial data.

3 Existence and uniqueness results

Theorem 1 *Let $v_0(y) \equiv v_1(y) \equiv 0$ and Assumptions 1 hold. Then for any finite T and each $f_1 \in H^1(0,T;L^2(0,1))$ there exists a unique strong solution to (2.5)-(2.7).*

Proof. Let $\{w_j(y)\}$ be a basis in $V(0,1)$ orthonormal in $L^2(0,1)$. We construct approximate solutions to (2.5)-(2.7) in the form

$$v^N(y,t) = \sum_{j=1}^{N} g_j^N(t)w_j(y),$$

where unknown $g_j^N(t)$ will be sought as solutions to the following Cauchy problem

$$\Big(v_{tt}^N, w_j\Big)(t)+\Big(a_{1y}v_y^N, w_j\Big)(t)+\Big(a_1 v_y^N, w_{jy}\Big)(t)-2\Big(a_2 v_{yt}^N, w_j\Big)(t)+\Big(b_1 v_y^N, w_j\Big)(t)+$$

$$+\Big(\Phi(v^N), w_j\Big)(t)+a_1(0,t)\gamma(t)h(v_t^N(0,t))w_j(0)=\Big(f_1, w_j\Big)(t), \qquad (3.1)$$

$$g_j^N(0)=g_{jt}^N(0)=0, \quad j=1,\dots,N. \qquad (3.2)$$

By Caratheodory's theorem, (3.1)-(3.2) has solutions on some interval $(0,T_N)$, and we need an appropriate a priori estimate in order to prolong these solutions to any $(0,T)$ and to pass to the limit as $N\longrightarrow\infty$.

3.1 Estimate I

Replacing in (3.1) w_j by $2v_t^N$ and dropping the index N, we come to the equality

$$\frac{d}{dt}E(t)+2\Big([a_{1y}+b_1]v_y, v_t\Big)(t)+2\Big(a_{2y}, v_t^2\Big)(t)-\Big(a_{1t}, v_y^2\Big)(t)+$$

$$2\frac{k^2-\alpha'^2(t)}{\gamma(t)}h(v_t(0,t))v_t(0,t)+2\frac{\alpha'(t)}{\gamma^2(t)}v_t^2(0,t)=2\Big(f_1, v_t\Big)(t), \qquad (3.3)$$

where $E(t)\equiv\|v_t(t)\|^2+\Big(a_1, v_y^2\Big)(t)+2\int_0^1 F\big(v(y,t)\big)dy$. Taking into account Assumptions 1, we reduce (3.3) to the inequality

$$\frac{d}{dt}E(t)+C_0(v_t^2(0,t)+|v_t(0,t)|^{\rho+2})\le C\Big\{\|f_1(t)\|^2+E(t)\Big\},$$

where C_0, C are positive constants. By the Lemma of Gronwall, we get

$$E^N(t)+C_0\int_0^t\{v_\tau^{N^2}(0,\tau)+|v_\tau^N(0,\tau)|^{\rho+2}\}d\tau\le C\|f_1\|^2_{L^2(Q)}, \qquad (3.4)$$

and the constants C_0, C do not depend on $N, t\in(0,T)$. Here and in the sequel, by C we denote any positive constant which does not depend on $t\in(0,T), N$.

Since $E^N(t)\le C$, then $\|v^N(t)\|_{V(0,1)}\le C$, that is,

$$\sup_{t\in(0,T)}\max_{y\in[0,1]}|v^N(y,t)|\le C. \qquad (3.5)$$

3.2 Estimate 2

Taking $t = 0$ in (3.1), we get

$$\Big(v_{tt}^N, w_j\Big)(0) = -\Big(a_1 v_y^N, w_{jy}\Big)(0) - \Big(a_{1y} v_y^N, w_j\Big)(0) - \Big(\Phi(v^N), w_j\Big)(0) +$$

$$+2\Big(a_2 v_{yt}^N, w_j\Big)(0) - \Big(b_1 v_y^N, w_j\Big)(0) + \Big(f_1, w_j\Big)(0) -$$

$$-a_1(0,0)\gamma(0) h(v_t^N(0,0)) w_j(o). \tag{3.6}$$

Since $v^N(y,0) = v_t^N(y,0) = 0$, then

$$-\Big(a_1 v_y^N, w_{jy}\Big)(0) = \Big([a_1 v_{yy}^N + a_{1y} v_y^N], w_j\Big)(0) = 0, \; h(v_t^N(0,0)) = 0.$$

Substituting $v_{tt}^N(0) = w_j$ in (3.6), we obtain

$$\|v_{tt}^N(0)\| \leq \|f_1(0)\|. \tag{3.7}$$

Differentiating (3.1), substituting w_j by $2v_{tt}^N$ and omitting the index N, we come to the equality

$$\frac{d}{dt}\Big\{\|v_{tt}(t)\|^2 + \Big(a_1, v_{ty}^2\Big)(t) + 2\Big(a_{1t} v_y, v_{ty}\Big)(t)\Big\} + 2\Big([a_{1yt} + b_{1t}] v_y, v_{tt}\Big)(t) -$$

$$-3\Big(a_{1t}, v_{yt}^2\Big)(t) - 2\Big(a_{1tt} v_y, v_{yt}\Big)(t) + 2\Big([a_{1y} - 2a_{2t} + b_1] v_{yt}, v_{tt}\Big)(t) + 2\Big(a_{2y}, v_{tt}^2\Big)(t)$$

$$+2\Big(\Phi'(v) v_t, v_{tt}\Big)(t) + 2a_2(t) v_{tt}^2(0,t) + 2\frac{d}{dt}\Big\{a_1(0,t)\gamma(t)\Big\} h(v_t(0,t)) v_{tt}(0,t) +$$

$$+2a_1(0,t)\gamma(t) h'(v_t(0,t)) v_{tt}^2(0,t) = 2\Big(f_{1t}, v_{tt}\Big)(0,t). \tag{3.8}$$

We estimate the boundary terms as follows

$$I_1 = 2a_1(0,t)\gamma(t) h'(v_t(0,t)) v_{tt}^2(0,t) \geq A_0 v_{tt}^2(0,t)\Big(1 + |v_t(0,t)|^\rho\Big), \tag{3.9}$$

where a positive constant A_0 is defined by Assumptions 1.2 - 1.4.

Taking into account Assumptions 1.2 - 1.4, we get

$$\begin{aligned} I_2 = \; & 2\frac{d}{dt}\Big\{a_1(0,t)\gamma(t)\Big\} h(v_t(0,t)) v_{tt}(0,t) \\ & \leq C\big(1 + |v_t(0,t|^{\rho+1}\big)|v_{tt}(0,t)| \\ & \leq \epsilon\big(1 + |v_t(0,t)|^\rho\big) v_{tt}^2(0,t) + C(\epsilon)\big(1 + |v_t(0,t)|^{\rho+2}\big), \end{aligned} \tag{3.10}$$

where ϵ is an arbitrary positive number. By Assumption 1.1 and (3.5), we obtain

$$I_3 = \left|\Big(\Phi'(v)v_t, v_{tt}\Big)(t)\right| \leq C\|v_t(t)\|\|v_{tt}(t)\|. \tag{3.11}$$

Taking into account (3.9)-(3.11), using the Young inequality with a small positive parameter and choosing an appropriate $\epsilon > 0$, we reduce (3.8) to the inequality

$$\frac{d}{dt}\Big\{\|v_{tt}(t)\|^2 + \Big(a_1, v_{ty}^2\Big)(t) + 2\Big(a_{1t}v_y, v_{yt}\Big)(t)\Big\} + C_0\Big(1 + |v_t(0,t)|^\rho\Big)v_{tt}^2(0,t)$$
$$\leq C\Big\{\|f_{1t}(t)\|^2 + \|v_{tt}(t)\|^2 + \|v_{ty}(t)\|^2 + \|v_t(t)\|^2 +$$
$$+\|v_y(t)\|^2 + \|v_t(0,t)\|^2(1 + \|v_t(0,t)\|^\rho)\Big\}. \tag{3.12}$$

Introducing

$$E_2(t) = \|v_{tt}(t)\|^2 + \Big(a_1, v_{yt}^2\Big)(t),$$

integrating (3.12) and taking into account Assumption 1.2 and estimates (3.4) and (3.7), we obtain

$$E_2(t) \leq C\left\{E_2(0) + \int_0^t \Big\{\|f_1(\tau)\|^2 + \|f_{1\tau}(\tau)\|^2 + E_2(\tau)\Big\}d\tau\right\}.$$

By the Gronwall lemma

$$E_2^N(t) \leq C\Big\{E_2(0) + \|f\|^2_{H^1(0,T;L^2(0,1))}\Big\}, \forall t \in (0,T). \tag{3.13}$$

Returning to (3.12), we have

$$\int_0^T v_{tt}^{N^2}(0,t)\Big\{1 + |v_t^N(0,t)|^\rho\Big\}dt \leq C. \tag{3.14}$$

Combining estimates (3.4), (3.13), (3.14), we obtain

$$\|v^N\|_{L^\infty(0,T;V(0,1))} + \|v_t^N\|_{L^\infty(0,T;V(0,1))} +$$
$$+\||v_t^N(0,t)|^{\frac{\rho}{2}}v_t^N(0,t)\|_{H^1(0,T)} + \|v_{tt}^N\|_{L^\infty(0,T;L^2(0,1))} \leq C$$

uniformly in N. Hence, there exists a function $v(y,t)$ such that

$$\begin{aligned} &v^N \rightharpoonup v \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,T;V(0,1)),\\ &v_t^N \rightharpoonup v_t \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,T;V(0,1)),\\ &v_{tt}^N \rightharpoonup v_{tt} \quad \text{weakly} - \star \quad \text{in} \quad L^\infty(0,T;L^2(0,1)),\\ &v_t^N(0,t) \to v_t(0,t) \quad \text{in} \quad C(0,T). \end{aligned} \tag{3.15}$$

This allows us to pass to the limit as $N \longrightarrow \infty$ in (3.1) and rewrite the result in the form

$$\Big(a_1 v_y, \varphi_y\Big)(t) + \Big(a_{1y} v_y, \varphi\Big)(t) + a_1(0,t)\gamma(t)h(v_t(0,t))\varphi(0) =$$

$$\Big(\{f_1 - v_{tt} + a_2 v_{ty} - \Phi(v) - b_1 v_y\}, \varphi\Big)(t) \equiv \Big(G, \varphi\Big)(t), \tag{3.16}$$

where $\varphi(y)$ is an arbitrary function from $V(0,1)$. It means that *a.e.* in $(0,T)$, $v \in L^\infty(0,T;V(0,1))$ is a solution to the following elliptic problem

$$-v_{yy} = \frac{G(y,t)}{a_1(y,t)} \in L^2(0,1);$$

$$v(1) = 0, \quad v_y(0) = \gamma(t)h(v_t(0,t)).$$

It follows from the theory of elliptic problems that

$$v \in L^\infty(0,T;H^2(0,1) \cap V(0,1)),$$

hence, (3.16) can be rewritten as follows

$$v_{tt} - a_1 v_{yy} - 2a_2 v_{yt} + b_1 v_y + \Phi(v) = f_1, \quad a.e. \quad \text{in} \quad Q,$$

$$v_y - \gamma(t)h(v_t)\Big|_{y=0} = 0, \quad v(1,t) = 0,$$

$$v(0,y) = v_t(0,y) = 0.$$

This proves the existence part of Theorem 1.

□

3.3 Uniqueness

If there exist two distinct solutions v_1, v_2 to (2.1)-(2.3), then $z = v_1 - v_2$ is a strong solution to the following problem

$$z_{tt} - a_1 z_{yy} - 2a_2 z_{yt} + b_1 z_y + \Phi(v_1) - \Phi(v_2) = 0, \tag{3.17}$$

$$z_y(0,t) - \gamma(t)\Big\{h(v_{1t}(0,t)) - h(v_{2t}(0,t))\Big\} = 0, \tag{3.18}$$

$$z(0,t) = 0, \quad z(0,y) = z_t(0,y) = 0. \tag{3.19}$$

Multiplying (3.17) by $2z_{tt}$ and integrating over $(0,1)\times(0,t)$, we come, taking into account (3.18), (3.19), to the equality

$$\|z_t(t)\|^2 + \left(a_1, z_y^2\right)(t) + \int_0^t 2a_2(0,\tau)z_\tau^2(0,\tau)d\tau$$

$$+ \int_0^t 2a_1(0,\tau)\gamma(\tau)\Big\{h(v_{1\tau}(0,\tau)) - h(v_{2\tau}(0,\tau))\Big\}\Big(v_{1\tau}(0,\tau) - v_{2\tau}(0,\tau)\Big)d\tau$$

$$= -\int_0^t \left\{2\left(a_2, z_\tau^2\right)(\tau) + \Big([\Phi(v_1) - \Phi(v_2)], z_\tau\Big)(\tau) + \Big(b_1 z_y, z_\tau\Big)(\tau)\right\}d\tau. \quad (3.20)$$

By Assumption 1.1 and (3.5)

$$\left|\Big([\Phi(v_1) - \Phi(v_2)], z_t\Big)(t)\right| \le C\|z(t)\|\|z_t(t)\|$$

and, due to Assumption 1.2, $h(v_t)$ is a monotonic function. This allows us to reduce (3.20) to the inequality

$$E(t) \le C\int_0^t E(\tau)d\tau, \quad E(t) = \|z_t(t)\|^2 + \left(a_1, z_y^2\right)(t).$$

By the Gronwall lemma, $E(t) \equiv 0, \forall t \in (0,T)$, whence, $z_t(y,t) \equiv 0$ in Q which implies that $z(y,t) \equiv 0$. This completes the proof of Theorem 1.

□

Theorem 2 *Let Assumptions 1 hold;* $v_0 \in H^2(0,1) \cap V(0,1)$, $v_1 \in V(0,1)$ *and* $v_0'(0) - \gamma(t)h(v_1(0)) = 0$. *Then for each* $f_1 \in H^1(0,T; L^2(0,1))$ *there exists a unique strong solution to the problem (2.5) - (2.7).*

Proof. The change of the unknown function

$$z(y,t) = v(y,t) - v_0(y) - tv_1(y)$$

transforms (2.5)-(2.7) into a problem with homogeneous initial data and with a slightly different right hand side, $\Phi(z + v_0(y) + tv_1(y))$ and boundary conditions at $y = 0$, but this is not an obstacle to obtain necessary a priori estimates and to prove Theorem 2. For details see [9,10].

□

Remark 1 *Exploiting the arguments of F. Browder, [1], we can prove that*

$$v \in L^\infty_{loc}(0,\infty; V(0,1) \cap H^2(0,1)),$$

$$v_t \in L^\infty_{loc}(0,\infty; V(0,1)), \quad v_{tt} \in L^\infty_{loc}(0,\infty; L^2(0,1)).$$

As a direct consequence of Theorem 2, we have

Theorem 3 *Let Assumptions 1 hold;* $u_0 \in V(0,1) \cap H^2(0,1)$, $u_1 \in V(0,1)$ *and* $u_{0x} - h\big(u_1(0) + \alpha'(0)u_{0x}(0)\big) = 0$. *Then for each* $f \in H^1(0,T; L^2(\mathcal{D}_t))$ *there exists a unique strong solution to the problem (2.1)-(2.3).*

4 Energy decay

In this chapter, using the method of double perturbation of the energy (the mixed derivative, $-a_2 v_{yt}$, implies double regularization) we prove the exponential decay of the energy for the problem (2.5)-(2.7), and consequently, for the original problem (2.1)-(2.3). Due to Remark 1, we consider (2.5)-(2.7) in $\mathcal{Q} = (0,1) \times (0,\infty)$.

Assumptions 2

2.1 $0 < \alpha_0 \leq \gamma(t) \leq L < \infty \quad \forall t \in R^+$.

2.2 $\alpha \in C^3[0,\infty)$, $\quad sup_{t\geq 0}\{|\alpha'(t)| + |\alpha''(t)| + |\alpha'''(t)|\} \leq L_1$, $\alpha'(t) \geq 0$. *For each* $\epsilon > 0$ *there exists* $T_\epsilon > 0$ *such that* $|\alpha'(t)| + |\alpha''(t)| \leq \epsilon, \forall t \geq T_\epsilon$.

2.3 $a_1(y,t) \geq a_0 > 0 \quad in \quad \overline{\mathcal{Q}} = [0,1] \times [0,\infty)$.

2.4 $h_0 v_t \leq h(v_t) \leq h_1|v_t|$.

2.5 $\Phi \in C^1(R)$, $\quad \Phi(0) = 0$, $\quad F(v) = \int_0^v \Phi(s)ds \geq 0$; *and there exist constants* $\delta_1, \theta \in (0,1)$ *such that* $\theta\Phi(v)v \geq (2+\delta_1)F(v)$.

2.6 $\|f_1(t)\|^2 \geq C_f e^{-\chi t}$,

where $\alpha_0, L, L_1, a_0, h_0, h_1, C_f, \chi$ *are positive constants.*

It is clear that Assumptions 2 imply also Assumptions 1, hence, it follows from Theorem 2 and Remark 1 that for all finite $T > 0$ there exists a unique strong solution to (2.5)-(2.7). It means that we can start from $t = T_\epsilon$ of Assumption 2.2. The value of ϵ will be chosen later. Using Assumptions 2.1 - 2.4, it is easy to verify that

$$2\alpha_0 a_0 h_0 + 2a_2\Big|_{y=0} \geq \delta > 0, \quad \forall t \geq T_\epsilon. \tag{4.1}$$

As earlier,

$$E(t) = \|v_t\|^2 + \Big(a_1, v_y^2\Big)(t) + 2\int_0^1 F\big(v(t,y)\big)dy. \qquad (4.2)$$

Multiplying (2.5) by $2v_t$, we come to the equality

$$\frac{d}{dt}E(t) + 2\Big(a_{2y}, v_t^2\Big)(t) + 2a_1(0,t)\gamma(t)h(v_t(0,t))v_t(0,t) + 2a_2(0,t)v_t^2(0,t)$$

$$-\Big(a_{1t}, v_y^2\Big)(t) + 2\Big([a_{1y} + b_1]v_y, v_t\Big)(t) = 2\Big(f_1, v_t\Big)(t).$$

Taking into account Assumptions 2.3, 2.4 and (4.1), we transform it to the inequality

$$E'(t)+\delta v_t^2(0,t) \le 2\Big(f_1, v_t\Big)(t)+\Big(a_{1t}, v_y^2\Big)(t)-2\Big([a_{1y}+b_1]v_y, v_t\Big)(t)-2\Big(a_{2y}, v_t^2\Big)(t). \qquad (4.3)$$

4.1 Perturbations of $E(t)$

We define for each $\eta \in (0,1]$ and θ from Assumption 2.5

$$\hat{E}(t) = E(t) - 2\eta\Big((y-1)a_2, v_y^2\Big)(t) \qquad (4.4)$$

and

$$E_\eta(t) = \hat{E}(t) + \eta\rho(t), \qquad (4.5)$$

where

$$\rho(t) = \rho_1(t) + \theta\rho_2(t); \quad \rho_1(t) = 2\Big((y-1)v_y, v_t\Big)(t); \quad \rho_2(t) = \Big(v, v_t\Big)(t). \qquad (4.6)$$

Proposition 1 *There is such $T_1(\epsilon)$ that for all $t \ge T_1(\epsilon)$*

$$d(y,t) = a_1(y,t) - 2\eta(y-1)a_2(y,t) \ge \frac{a_0}{2} = d_0 > 0, \quad in \ \overline{Q}. \qquad (4.7)$$

Proof. Easily follows from Assumptions 2.2, 2.3 and from the structure of $a_2(y,t)$.

□

Simple calculations show that by (4.7)

$$\hat{E}(t) \ge 0, \quad \forall t \ge T_1(\epsilon)$$

and for sufficiently small $\eta > 0$ there exist positive constants C_1, C_2, C_3, C_4 such that

$$C_2 E(t) \le \hat{E}(t) \le C_1 E(t), \quad C_4 E_\eta(t) \le \hat{E}(t) \le C_3 E_\eta(t). \tag{4.8}$$

Substituting (4.4) into (4.3) and using the Young's inequality, we get

$$\frac{d}{dt}\hat{E}(t) + \delta v_t^2(0,t) \le -2\eta \frac{d}{dt}\Big((y-1), a_2 v_y^2\Big)(t) + \Big(\{\frac{|a_{1y}+b_1| + |a_{1t}|}{a_1}\} a_1, v_y^2\Big)(t)$$
$$+\Big(\{2|a_{2y}| + |a_{1y}+b_1| + \epsilon\}, v_t^2\Big)(t) + \frac{1}{\epsilon}\|f_1(t)\|^2. \tag{4.9}$$

Taking into account the structures of $a_1(y,t), a_2(y,t), b_1(y,t)$, it easy to see that Assumptions 2.1 - 2.3 imply

$$|a_{1t}(y,t)| + |a_{1y}(y,t)| + |a_{2y}(y,t)| + |b_1(y,t)| \le C_5 \epsilon, \quad \text{for} \quad t \ge T_\epsilon, \tag{4.10}$$

where the constant C_5 does not depend on ϵ, t.

Having (4.10), we reduce (4.9) to the form

$$\frac{d}{dt}\hat{E}(t) + \delta v_t^2(0,t) \le -2\eta \frac{d}{dt}((y-1)a_2, v_y^2)(t) +$$
$$+C_6 \epsilon\{(a_1, v_y^2)(t) + \|v_t(t)\|^2\} + \frac{1}{\epsilon}\|f_1(t)\|^2, \tag{4.11}$$

where C_6 is positive constant.

Now we calculate

$$E'_\eta(t) = \hat{E}'(t) + \eta\big(\rho'_1(t) + \theta\rho'_2(t)\big). \tag{4.12}$$

Lemma 1 *For $t \ge T_\epsilon$*

$$\rho'_1(t) \le 2\frac{d}{dt}\Big((y-1)a_2, v_y^2\Big)(t) + 2\Big(F(v), 1\Big)(t) - \Big(a_1, v_y^2\Big)(t) - \|v_t(t)\|^2 + C_7 v_t^2(0,t) +$$
$$+C_8\epsilon\Big(a_1, v_y^2\Big)(t) + \frac{1}{\epsilon}\|f_1(t)\|^2, \tag{4.13}$$

where C_7, C_8 are positive constants.

Proof. Straight calculations give

$$\rho'_1(t) = 2\Big((y-1)f_1, v_y\Big)(t) + a_1(0,t)v_y^2(0,t) - \Big(a_1, v_y^2\Big)(t) - \|v_t(t)\|^2 +$$
$$+v_t^2(0,t) + 2\frac{d}{dt}\Big((y-1)a_2, v_y^2\Big)(t) + 2\Big(F(v), 1\Big)(t) - 2F\big(v(0,t)\big)$$
$$-\Big((y-1)\{a_{1y} + 2(b_1 + a_{2t})\}, v_y^2\Big)(t). \tag{4.14}$$

Using (4.10) and Assumptions 2.2, 2.5, we transform (4.14) into (4.13).

□

Lemma 2 *For* $t \geq T_\epsilon$

$$\theta\rho_2'(t) \leq \theta C_9(\epsilon)v_t^2(0,t) + \theta C_{10}\epsilon\big[(a_1, v_y^2)(t) + \|v_t(t)\|^2\big] + \theta\|v_t(t)\|^2$$
$$-\theta\Big(\Phi(v), v\Big)(t) + C_{11}(\epsilon)\|f_1(t)\|^2. \tag{4.15}$$

Proof. By (4.6) and (2.1)

$$\rho_2'(t) = \Big(v, v_{tt}\Big)(t) + \|v_t(t)\|^2 = \|v_t(t)\|^2 - a_1(0,t)v_y(0,t)v(0,t)$$
$$-2a_2(0,t)v_t(0,t)v(0,t) - \Big(a_1, v_y^2\Big)(t) - \Big(\{a_{1y} + b_1\}v_y, v\Big)(t) - 2\Big(a_2 v_t, v_y\Big)(t)$$
$$-2\Big(a_{2y}v_t, v\Big)(t) - \Big(\Phi(v), v\Big)(t) + \Big(f_1, v\Big)(t). \tag{4.16}$$

Taking into account that

$$v^2(0,t) \leq \|v_y(t)\|^2 \leq \frac{1}{a_0}\Big(a_1, v_y^2\Big)(t), \tag{4.17}$$

boundary condition 2.7, Assumption 2.4 and (4.10), we find

$$\begin{aligned} I_1 &= -a_1(0,t)v_y(0,t)v(0,t) - 2a_2(0,t)v_t(0,t)v(0,t) \\ &\leq 2\epsilon\Big(a_1, v_y^2\Big)(t) + C_{12}(\epsilon)v_t^2(0,t). \end{aligned} \tag{4.18}$$

$$I_2 = \left|\Big(\{a_{1y} + b_1\}v_y, v\Big)(t) + 2\Big(a_2 v_t, v_y\Big)(t) + 2\Big(a_{2y}v_t, v\Big)(t)\right|$$
$$\leq C_{13}\epsilon\{(a_1, v_y^2)(t) + \|v_t(t)\|^2\}. \tag{4.19}$$

Substituting (4.18), (4.19) into (4.16), we come to (4.15).

Now, using (4.11), (4.13) and (4.15), we transform (4.12) into the inequality

$$E_\eta' \leq -\{\delta - \eta[C_7 + \theta C_9(\epsilon)]\}v_t^2(0,t) - \eta\Big\{\theta\Big(\Phi(v), v\Big)(t)$$
$$-2\Big(F(v), 1\Big)(t)\Big\} - \eta\{1 - [C_6 + \theta C_{10}]\epsilon\}\Big(a_1, v_y^2\Big)(t)$$
$$-\eta\{1 - [C_6 + \theta C_{10}]\epsilon - \theta\}\|v_t\|^2 + C_{14}(\epsilon)\|f_1(t)\|^2, \tag{4.20}$$

where a positive constant $C_{14}(\epsilon)$ depends only on $\epsilon > 0$.

Putting $\epsilon = \dfrac{1-\theta}{2(C_6 + \theta C_{10})}$ and then $\eta = \dfrac{\delta}{2(C_7 + \theta C_9(\epsilon))}$, we reduce (4.20) to the form

$$E_\eta'(t) \leq -C_{15}E(t) + C_{16}\|f_1(t)\|^2, \tag{4.21}$$

where C_{16} is a positive constant and

$$C_{15} = \min\Big\{\eta\delta_1, \frac{\eta}{2}, \frac{\eta(1-\theta)}{2}\Big\} > 0. \tag{4.22}$$

Taking into account (4.8), we obtain from (4.21)

$$E'_\eta \leq -\frac{C_{15}}{C_1}\hat{E}(t) + C_{16}\|f_1(t)\|^2 \leq -\frac{C_4 C_{15}}{C1}E_\eta(t) + C_{16}\|f_1(t)\|^2.$$

Solving this inequality, we get

$$E_\eta(t) \leq e^{-\lambda t}\Big\{E_\eta(0) + C_{16}e^{-\lambda t}\int_0^t e^{\lambda\tau}\|f_1(\tau)\|^2 d\tau\Big\}, \tag{4.23}$$

where $\lambda = \dfrac{C_4 C_{15}}{C_1} > 0$.

Taking into account (4.8), Assumption 2.6 and decreasing, if necessary, $\lambda > 0$, we obtain

$$E(t) \leq Ce^{-\lambda t}\Big\{1 + E(0)\Big\}, \tag{4.24}$$

with λ defined by (4.22), (4.23) and inequality $0 < \lambda < \chi$. It means that we have proved the following result.

Theorem 4 *Let Assumptions 2 hold. Then there exist positive constant* λ, C *such that*

$$E(t) \leq Ce^{-\lambda t}\Big(1 + E(0)\Big).$$

From (4.24) we can obtain also decay of the energy for the problem (2.1) - (2.3). Defining

$$E_u(t) = \int_{\mathcal{D}_t}\Big(u_t^2 + u_x^2\Big)dx,\ E(\tau) = \int_0^1 \Big\{v_t^2(y,\tau) + a_1(y,\tau)v_y^2(y,\tau)\Big\}dy$$

and using transformation $(x,t) \Leftrightarrow (y,\tau)$, we get

$$E_u(t) = \int_{\mathcal{D}_t}\Big\{u_t^2(x,t) + u_x^2(x,t)\Big\}dx \leq C_V E(\tau) \tag{4.25}$$

and

$$E(0) \leq C_u E_u(0),$$

where $C_V = \max\Big\{1 + \dfrac{L_1^2}{\alpha_0^2}, 1 + \dfrac{2L_1^2}{a_0 L_0^2}\Big\}$, $C_u = \max\Big\{2, k^2 + 3|\alpha'(0)|^2\Big\}$. Substituting (4.24) into (4.25), we obtain

$$E_u(t) \leq Ce^{-\lambda t}\Big\{1 + E_u(0)\Big\}. \tag{4.26}$$

Thus we prove the following theorem.

Theorem 5 *Let Assumptions 2 hold. If* $u_0 \in V(0,1) \cap H^2(0,1)$, $u_1 \in V(0,1)$ *and* $u_{0x}(0) - h\big(u_1(0) + \alpha'(0)u_{0x}(0)\big) = 0$, *then for each* $f \in H^1(0,\infty; L^2(\mathcal{D}_t))$, *satisfying Assumptions 2.6, there exists a unique strong solution to (2.1)-(2.3) such that the energy decays exponentially in time.*

References

[1] Browder F. E., *On non-linear wave equations*, Math. Zeitschrift, 80, (1962), 249-264.

[2] Cooper J., Medeiros L. A., *The Cauchy problem for nonlinear wave equations in the domain with moving boundary*, Annali dela Scuola Normale Supperiore di Pisa, XXVI, (1972), 829-838.

[3] Cooper J., Bardos C., *A nonlinear wave equation in a time dependent domain*, J. Math. Anal. Appl., 42, 1, (1973), 29-60.

[4] Dragieva N. A., *Solution of the wave equation in a domain with moving boundaries by Galerkin's Method*, Zh. Vychisl. Mat. Fiz, 15, 4, (1975), 946-956.

[5] Ferreira J., Lar'kin N. A., *Global solvability of a mixed problem for a nonlinear hyperbolic-parabolic equation in a noncylindrical domains*, Portugaliae Mathematica, 53, 4, (1996), 381-395.

[6] Ferreira J., Lar'kin N. A., *Decay of solutions of nonlinear hyperbolic-parabolic equation in a noncylindrical domains*, Commun. Appl. Anal., 1, 1, (1997), 75-81.

[7] Inoue A., *Sur* $\Box u + u^3 = f$ *dans un domaine non cylindrique*, J. Math. Anal. Appl., 46, (1974), 777-819.

[8] Komorrik V., Zuazua E., *A direct method for the boundary stabilization of the wave equation*, J. math. pures et appl., 69, (1990), 33-54.

[9] Kozhanov A. I., Lar'kin N. A., *Wave equations with nonlinear dissipation in noncylindrical domains*, Doklady Mathematics, 62, 2, (2000), 159-162.

[10] Lar'kin N. A., Simões M. H., *The non-homogeneous wave equation in non-cylindrical domains*, TEMA-Tendências em Matematica Aplicada e Computacional, 2, (2001), 117-124.

[11] Lar'kin N. A., *Global solvability of a boundary value problem for a class of quasi-linear hyperbolic equations*, Siberian Math. J., 1, (1981), 82-88.

[12] Limaco Ferrel J., Medeiros L. A., *Kirchhoff-Carrier elastic strings in non-cylindrical domains*, Portugaliae Mathematica, n. 4, (1999), 465-500.

[13] Lions J.-L., *Une remarque sur les problemes d'evolution nonlineaires dans les domaines non cylindriques*, Rev. Romaine Pures Appl. Math., 9, (1964), 11-18.

[14] Lions J.-L., *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires*, Paris: Dunod, (1969).

[15] Medeiros L. A., *Non-linear wave equations in domains with variable boundary*, Arch. Rational Mech. Anal., 47, 1, (1972), 47-58.

[16] Nakao M., Narazaki T., *Existence and decay of solutions of some nonlinear wave equations in noncylindrical domains*, Math. Rep., 11, 2, (1978), 117-125.

[17] Sidelnik Y. I., *Existence and uniqueness of a generalized solution of the mixed problem for an equation of plate oscillation type in a noncylindrical domains*, J. of Soviet Math., 63, 1, (1993), 98-101.

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