Matemática Contemporânea, Vol. 22, 141–162
http://doi.org/10.21711/231766362002/rmc229

KINETIC EQUATION FOR CHARGED PARTICLES IN AXISYMMETRIC TWO-DIMENSIONAL MAGNETIZED PLASMAS

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Abstract

Vlasov equation is solved for charged particles in two-dimensional axisymmetric plasma models using: i) convenient curvilinear (cylindrical, spherical, toroidal) coordinates, ii) Fourier expansions of the perturbed distribution functions over the gyrophase angle in velocity space, iii) smallness of the magnetization parameters, iv) new variables by the conservation integrals to describe the bounce periodic motion of the trapped and untrapped particles along the equilibrium magnetic field line. An approach developed in the paper allows us to evaluate the main contribution of untrapped and trapped particles to the transverse and longitudinal dielectric permittivity elements for radio frequency waves in tokamaks, mirror-traps, and Earth's magnetosphere.

1 Introduction

Since plasma is an ensemble of charged particles (ions and electrons) its behavior can be described by the kinetic equation for the probability distribution functions, $F_\alpha(t, \mathbf{r}, \mathbf{v})$, of α-kind particles in the six-dimensional phase $(\mathbf{r},\mathbf{v})$-volume. In the general case, F_α is a function of 7 variables: t-time, 3 variables in velocity space $\mathbf{v}$, and 3 variables in geometric space $\mathbf{r}$. In plasma theory, the corresponding kinetic equation is known as the Vlasov equation [10] or collisionless Boltzmann equation, where the generalized force acting the particles is defined as the Lorentz force. The Vlasov equation, for the perturbed distribution functions

$$f_\alpha(t, \mathbf{r}, \mathbf{v}) = F_\alpha(t, \mathbf{r}, \mathbf{v}) - F_{0\alpha}(\mathbf{r}, \mathbf{v})$$

*Supported by CNPq of Brazil, grant 300637/01-2

can be written as

$$\frac{\partial f_\alpha}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})f_\alpha + \frac{e_\alpha H_0}{M_\alpha c}[\mathbf{v}\times\mathbf{h}]\frac{\partial f_\alpha}{\partial \mathbf{v}} = -\frac{e_\alpha}{M_\alpha}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{H}}{c}\right)\frac{\partial F_{0\alpha}}{\partial \mathbf{v}}. \quad (1)$$

where $F_{0\alpha}$ is the steady-state (or equilibrium) distribution function of particles with mass M_α and charge e_α; index $\alpha = e, i_1, i_2, ...$ corresponds to electrons (e) and any kind of the possible ions, i.e., $i_1, i_2, ... = H(hydrodgn), D(deuterium)$, $T(tritium)$, ...; $\mathbf{E}$ and $\mathbf{H}$ are the perturbed electric and magnetic fields; H_0 is the modulus of an equilibrium magnetic field $\mathbf{H}_0(\mathbf{r})$; $\mathbf{h}$=$\mathbf{H}_0/H_0$; c is the speed of light; $\boldsymbol{\nabla} \equiv \partial/\partial\mathbf{r}$; $(\mathbf{vh})$ and $[\mathbf{v}\times\mathbf{h}]$ are the scalar and vector products of two vectors, respectively.

After solving Eq. (1), one can calculate the basic moments of plasma distribution function such as the fluctuation of plasma density $n(t,\mathbf{r})$ (as the zeroth moment of f_α)

$$n(t,\mathbf{r}) = \sum_{\alpha}^{e,i_1,i_2,...} \int_{\mathbf{v}} f_\alpha(t,\mathbf{r},\mathbf{v})d\mathbf{v} \quad (2)$$

the perturbed current density components, $\mathbf{j}(t,\mathbf{r})$ (as the first moments of f_α),

$$\mathbf{j}(t,\mathbf{r}) = \sum_{\alpha}^{e,i_1,i_2,...} e_\alpha \int_{\mathbf{v}} \mathbf{v} f_\alpha(t,\mathbf{r},\mathbf{v})d\mathbf{v} \quad (3)$$

the plasma pressure transverse and along the $\mathbf{H}_0$-field lines (as the second moments of f_α), heat conductivity components (as the third moments of f_α), and others.

As is well known, any wave process in magnetized plasmas can be described by solving the Maxwell's equations for ($\mathbf{E}$,$\mathbf{H}$)-components

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \quad (4)$$

where the Gaussian system of units has been used. The set of Eqs. (4) will be complete if we know the connection between $\mathbf{j}$ and ($\mathbf{E}$,$\mathbf{H}$)-fields. Usually this connection is defined by the wave conductivity tensor σ_{ik}: $j_i = \sigma_{ik}E_k$, or by the dielectric tensor ϵ_{ik}:

$$\epsilon_{ik} = \delta_{ik} + 4\pi i\frac{\sigma_{ik}}{\omega} \quad (5)$$

where δ_{ik} are the Kronecker constants; $i, k = 1, 2, 3$ indicate the vector projections. It means that before solving Eqs. (4) we should calculate the ϵ_{ik} (or σ_{ik}) tensor by solving Eq. (1) and using Eq. (3).

The main feature of magnetized plasmas is the fact that their dielectric characteristics have different form for different plasma models. This form depends substantially on the wave frequency ω, the plasma parameters (density N, temperature T) and the geometry of an equilibrium magnetic field $\mathbf{H}_0(\mathbf{r})$. Presently, the linear wave theory is very well developed for the plane waves in both the isotropic (when $\mathbf{H}_0 = 0$) and anisotropic magnetized plasmas in the straight magnetic field, see, e.g., Ref. [16] and the bibliography therein. However, the approximation of plane waves is not suitable for such realistic plasma systems as Earth's magnetosphere, straight mirror traps and tokamaks. All these plasmas can be modeled as two-dimensional (2D) axisymmetric configurations with one minimum of a nonuniform equilibrium magnetic field, where plasma particles should be split in the two populations of the so-called trapped and untrapped (or passing, or circulating) particles. Accordingly, Eq. (1) should be solved separately for each particle group using the specific boundary conditions.

2 Vlasov Equation for Plasma Particles in an Arbitrary Magnetic Field

In this paper, we derive the kinetic equation in the convenient form for 2D magnetospheric, toroidal and mirror-trapped plasma models. Of course, the initial Eq. (1) should be solved separately for each concrete 2D plasma configuration using one set of coordinates or another. However, since the above mentioned models are axisymmetric we can rewrite Eq. (1) in the usual cylindrical coordinates (r, ϕ, z) for plasmas in the arbitrary three-dimensional $\mathbf{H}_0(\mathbf{r})$-field:

$$\frac{\partial f}{\partial t} + v_{\|}(\mathbf{h}\boldsymbol{\nabla})f + \frac{v_\perp}{2}(\boldsymbol{\nabla}\mathbf{h})\hat{V}f - \Big\{\Omega_c + \frac{v_{\|}}{2}\,[2\mathbf{b}(\mathbf{h}\boldsymbol{\nabla})\mathbf{n} + \mathbf{h}(\mathbf{b}\boldsymbol{\nabla})\mathbf{n} - \mathbf{h}(\mathbf{n}\boldsymbol{\nabla})\mathbf{b}$$
$$+2\frac{h_z h_\phi}{r} - \frac{b_z b_\phi}{r} - \frac{n_z n_\phi}{r}\Big]\Big\}\frac{\partial f}{\partial \sigma} + \cos\sigma\left\{v_\perp(\mathbf{n}\boldsymbol{\nabla})f + v_{\|}\left[\mathbf{n}(\mathbf{h}\boldsymbol{\nabla})\mathbf{h} + \frac{h_\phi b_z}{r}\right]\hat{V}f\right.$$
$$\left.+\frac{1}{v_\perp}\left[v_\perp^2\mathbf{n}(\mathbf{n}\boldsymbol{\nabla})\mathbf{b} + v_{\|}^2\mathbf{h}(\mathbf{h}\boldsymbol{\nabla})\mathbf{b} + v_{\|}^2\frac{h_\phi n_z}{r} - v_\perp^2\frac{h_z n_\phi}{r}\right]\frac{\partial f}{\partial \sigma}\right\} + \sin\sigma\,\{v_\perp(\mathbf{b}\boldsymbol{\nabla})f$$
$$+v_{\|}\left[\mathbf{b}(\mathbf{h}\boldsymbol{\nabla})\mathbf{h} - \frac{h_\phi n_z}{r}\right]\hat{V}f - \frac{1}{v_\perp}\left[v_\perp^2\mathbf{b}(\mathbf{b}\boldsymbol{\nabla})\mathbf{n} + v_{\|}^2\mathbf{h}(\mathbf{h}\boldsymbol{\nabla})\mathbf{n} - v_{\|}^2\frac{h_\phi b_z}{r}\right. \qquad (6)$$

$$+v_{\perp}^2\frac{h_z b_\phi}{r}\bigg]\frac{\partial f}{\partial\sigma}\bigg\} + \frac{\cos 2\sigma}{2}\bigg\{v_\perp\bigg[\mathbf{n}(\mathbf{n}\boldsymbol{\nabla})\mathbf{h} - \mathbf{b}(\mathbf{b}\boldsymbol{\nabla})\mathbf{h} + \frac{n_\phi b_z}{r} + \frac{b_\phi n_z}{r}\bigg]\hat{V}f$$
$$+v_{||}\bigg[\mathbf{h}(\mathbf{b}\boldsymbol{\nabla})\mathbf{n} + \mathbf{h}(\mathbf{n}\boldsymbol{\nabla})\mathbf{b} - \frac{b_\phi b_z}{r} + \frac{n_\phi n_z}{r}\bigg]\frac{\partial f}{\partial\sigma}\bigg\} + \frac{\sin 2\sigma}{2}\bigg\{v_\perp\bigg[\frac{b_\phi b_z}{r} - \frac{n_\phi n_z}{r}$$
$$+\mathbf{n}(\mathbf{b}\boldsymbol{\nabla})\mathbf{h} + \mathbf{b}(\mathbf{n}\boldsymbol{\nabla})\mathbf{h}\big]\hat{V}f - v_{||}\bigg[\mathbf{h}(\mathbf{n}\boldsymbol{\nabla})\mathbf{n} - \mathbf{h}(\mathbf{b}\boldsymbol{\nabla})\mathbf{b} - \frac{n_\phi b_z}{r} - \frac{b_\phi n_z}{r}\bigg]\frac{\partial f}{\partial\sigma}\bigg\}$$
$$= -\frac{e}{M}\bigg\{E_{||}\frac{\partial F_0}{\partial v_{||}} + \cos\sigma\bigg[E_n\frac{\partial F_0}{\partial v_\perp} + \frac{H_b}{c}\hat{V}F_0 + \frac{1}{v_\perp}\Big(E_b + \frac{v_{||}}{c}H_n\Big)\frac{\partial F_0}{\partial\sigma}\bigg]$$
$$-\frac{H_3}{c}\frac{\partial F_0}{\partial\sigma} + \sin\sigma\bigg[E_b\frac{\partial F_0}{\partial v_\perp} - \frac{H_n}{c}\hat{V}F_0 - \frac{1}{v_\perp}\Big(E_n - \frac{v_{||}}{c}H_b\Big)\frac{\partial F_0}{\partial\sigma}\bigg]\bigg\}.$$

where

$$\hat{V}f = v_\perp\frac{\partial f}{\partial v_{||}} - v_{||}\frac{\partial f}{\partial v_\perp}, \qquad \Omega_c = \frac{eH_0}{Mc},$$

and the index α of the particle species is omitted. In Eq. (6), for the vector values $\mathbf{A}=\{\mathbf{E},\mathbf{H},\mathbf{v},\mathbf{j}\}$, we use the normal A_1, binormal A_2, and parallel A_3 projections relative to $\mathbf{H}_0$: $\mathbf{A} = A_1\mathbf{n}+A_2\mathbf{b}+A_3\mathbf{h}$, so that

$$\begin{aligned} A_1 &\equiv A_n = A_r n_r + A_\phi n_\phi + A_z n_z \\ A_2 &\equiv A_b = A_r b_r + A_\phi b_\phi + A_z b_z \\ A_3 &\equiv A_{||} = A_r h_r + A_\phi h_\phi + A_z h_z \end{aligned} \tag{7}$$

where $\mathbf{n}$, $\mathbf{b}$, $\mathbf{h}$ are the normal, binormal, and parallel unit vectors relative to $\mathbf{H}_0$:

$$\mathbf{h} = \mathbf{H}_0/H_0, \qquad \mathbf{n} = [\mathbf{b}\times\mathbf{h}], \qquad \mathbf{b} = [\mathbf{h}\times\mathbf{n}]. \tag{8}$$

In particular, for magnetospheric and mirror trapped plasmas, the vector $\mathbf{n}$ is directed outside the magnetic shell and $\mathbf{b}$ is directed along the angle ϕ. Moreover, in velocity space we use the polar coordinates $(v_\perp, \sigma)$ instead of (v_1, v_2) by the transformation

$$v_1 = v_\perp\cos\sigma, \qquad v_2 = v_\perp\sin\sigma, \qquad v_3 = v_{||}. \tag{9}$$

For axisymmetric tokamaks with circular (as well as elliptic and/or D-shaped) cross-sections of the toroidal magnetic surfaces, Eq. (6) can be readily simplified under the conditions i) $\mathbf{H}_0$ is independent of ϕ, and ii) the normal component (perpendicular to the magnetic surface) of an equilibrium magnetic field is equal to zero, i.e., $H_{0r} = H_{0n} = 0$, or $(\mathbf{n}\mathbf{H}_0)$=0. The corresponding kinetic equation, its solution and dielectric tensor evaluation for such toroidal plasmas will be presented in Section 4, in the general case of the arbitrary tokamak aspect ratio.

As regard to the axisymmetric 2D magnetosphere (as well the straight mirror traps), Eq. (6) can be simplified substantially when the $H_{0\phi}$ component of the geomagnetic field is equal to zero, $H_{0\phi} = 0$. In this case, **n, b, h** have such cylindrical projections:

$$\mathbf{h} = (h_r, 0, h_z), \qquad \mathbf{n} = (h_z, 0, -h_r), \qquad \mathbf{b} = (0, 1, 0). \tag{10}$$

3 Magnetospheric Plasma

As was mentioned above, to study the wave processes in the Earth's magnetosphere/plasmasphere it is necessary to solve the Maxwell's equations with a "nonlocal" dielectric tensor. For the low-frequency oscillations this tensor can be derived by solving the drift-kinetic equation, e.g., following the methods developed in Refs. [3,5,8,15]. As for the high-frequency waves near the ion (or electron) cyclotron frequencies, the dielectric characteristics should be evaluated by the solution of the Vlasov equation for trapped particles taking into account the 2D inhomogeneities of geomagnetic field and plasma parameters [4, 6].

If axis z of cylindrical coordinates is directed along the magnetic axis, the Earth's magnetosphere can be considered as an axisymmetric mirror trap which is symmetric relatively to z and homogeneous in the angle ϕ, see Fig. 1.

As a result, the Vlasov equation for the perturbed distribution function: $f = f(t, r, \phi, z, v_\perp, \sigma, v_{||})$, in any axisymmetric mirror trap with 2D equilibrium magnetic field $\mathbf{H}_0(r, z) = (H_{0r}, 0, H_{0z})$ has the following form

$$\begin{aligned}
&\frac{\partial f}{\partial t} + v_{||} h_r \frac{\partial f}{\partial r} + v_{||} h_z \frac{\partial f}{\partial z} - \Omega_c \frac{\partial f}{\partial \sigma} + \frac{v_\perp}{2} \left(\frac{1}{r} \frac{\partial}{\partial r} r h_r + \frac{\partial h_z}{\partial z} \right) \hat{V} f + \\
&\quad + \cos\sigma \left\{ v_\perp h_z \frac{\partial f}{\partial r} - v_\perp h_r \frac{\partial f}{\partial z} - v_{||} \left(\frac{\partial h_z}{\partial r} - \frac{\partial h_r}{\partial z} \right) \hat{V} f \right\} + \\
&\quad + \sin\sigma \left\{ \frac{v_\perp}{r} \frac{\partial f}{\partial \phi} - \frac{1}{v_\perp} \frac{\partial f}{\partial \sigma} \left[v_{||}^2 \left(\frac{\partial h_z}{\partial r} - \frac{\partial h_r}{\partial z} \right) + v_\perp^2 \frac{h_z}{r} \right] \right\} + \\
&+ \frac{v_\perp}{2} \cos 2\sigma \left(r \frac{\partial}{\partial r} \frac{h_r}{r} + \frac{\partial h_z}{\partial z} \right) \hat{V} f + \frac{v_{||}}{2} \sin 2\sigma \left(r \frac{\partial}{\partial r} \frac{h_r}{r} + \frac{\partial h_z}{\partial z} \right) \frac{\partial f}{\partial \sigma} =
\end{aligned} \tag{11}$$

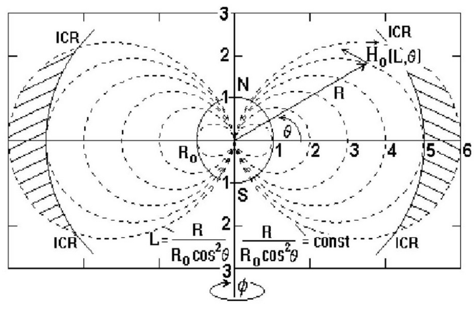

Figure 1: Spherical coordinates (R, θ, ϕ) for an axisymmetric magnetosphere, where ϕ is the azimuthal angle in an equatorial plane; R_0 is the radius of the Earth.

$$
= -\frac{e}{M}\left\{E_{\|}\frac{\partial F_0}{\partial v_{\|}} + \cos\sigma\left[E_n\frac{\partial F_0}{\partial v_\perp} + \frac{H_b}{c}\hat{V}F_0 + \frac{1}{v_\perp}\left(E_b + \frac{v_{\|}}{c}H_n\right)\frac{\partial F_0}{\partial \sigma}\right]\right.
$$

$$
\left. - \frac{H_{\|}}{c}\frac{\partial F_0}{\partial \sigma} + \sin\sigma\left[E_b\frac{\partial F_0}{\partial v_\perp} - \frac{H_n}{c}\hat{V}F_0 - \frac{1}{v_\perp}\left(E_n - \frac{v_{\|}}{c}H_b\right)\frac{\partial F_0}{\partial \sigma}\right]\right\}.
$$

Further, the perturbed distribution function is expanded in a Fourier series over the polar (or gyrophase) angle σ in velocity space:

$$
f(t, r, \phi, z, v_\perp, \sigma, v_{\|}) = \sum_{l}^{\pm\infty} f_l(r, z, v_\perp, v_{\|}) \; \exp(-i\omega t + in\phi + il\sigma) \tag{12}
$$

accounting that the problem is homogeneous in time t and angle ϕ; therefore the perturbed values (including the $\{\mathbf{E},\mathbf{H},\mathbf{j}\}$-components) are proportional to $\sim \exp(-i\omega t + in\phi)$, where n is an integer. Due to this procedure, we reduce the problem to solve the differential equations with respect to four partial derivatives for $f_l(r, z, v_\perp, v_{\|})$-harmonics, whereas the initial Eqs. (1, 6) include seven partial derivatives. Moreover, to evaluate the main contribution of plasma particles to the perturbed current density components it is enough to find the f_l-harmonics with $l = 0, \pm 1$.

Of course, by substituting the Fourier expansion (12) into Eq. (11) we get a set of coupled equations: i.e., the equation for f_l contains the harmonics $f_{l\pm1}$ and $f_{l\pm2}$. However, for magnetized plasmas this coupling can be taken into account by the standard approximation using the small "magnetization" parameter $r_\lambda/l_\perp << 1$, when the Larmor radius $r_\lambda = \sqrt{2TMc}/(eH_0)$ (of particles with the mass M, charge e, and temperature T) is much less than the scale length $l_\perp$ of the inhomogeneity of the plasma-wave parameters in the direction perpendicular to $\mathbf{H}_0$. Thus, to evaluate the $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}$, and ϵ_{33} dielectric tensor components, we should solve the following three equations:

$$-i\omega f_l + v_\| h_r \frac{\partial f_l}{\partial r} + v_\| h_z \frac{\partial f_l}{\partial z} - il\frac{eH_0}{Mc} f_l + \frac{v_\perp}{2}\left(\frac{1}{r}\frac{\partial}{\partial r} r h_r + \frac{\partial h_z}{\partial z}\right)\left(v_\perp \frac{\partial f_l}{\partial v_\|} - v_\| \frac{\partial f_l}{\partial v_\perp}\right) = Q_l \qquad (13)$$

where $l = 0,\ \pm 1$, and

$$Q_0 = \frac{e}{T} E_\| v_\| F_0, \qquad Q_{\pm 1} = \frac{e}{2T}(E_n \mp iE_b) v_\perp F_0. \qquad (14)$$

Note that Eqs. (11-14) can be employed to study a wide class of plasma configurations with an axisymmetric equilibrium magnetic field and given h_r and h_z. For example, these equations were initially used to evaluate the dielectric characteristics in the straight cylindrical magnetic mirror machine [13], and for magnetospheric plasmas with circular magnetic field lines [4-6].

Since the intrinsic geomagnetic field of the Earth (and other planets) has the dipole configuration, let us rewrite Eq. (13) for plasmas in the concrete dipole magnetic field

$$H_0(R,\theta) = B_0 \left(\frac{R_0}{R}\right)^3 \sqrt{1 + 3\sin^2\theta} \qquad (15)$$

where R_0 is the Earth's radius, R is the geocentric distance, θ is the geomagnetic latitude, B_0 is the magnetic field in an equatorial plane on the Earth's surface ($R = R_0,\ \theta = 0$). The new variables (R, θ) are introduced instead of (r, z) as $r = R\cos\theta,\ z = R\sin\theta$.

To solve Eq. (13) we use the standard method of switching to new variables in velocity space associated with the conservation integrals of energy: $v_\perp^2 + v_\|^2 = const$, magnetic moment: $v_\perp^2/2H_0 = const$, and the equation of the $\mathbf{H}_0$-field line: $R/\cos^2\theta = const$. Introducing the variables v, μ, L (instead of $v_\|, v_\perp, R$)

$$v = \sqrt{v_\|^2 + v_\perp^2}, \qquad \mu = \frac{v_\perp^2 H_0(L,0)}{v^2 H_0(L,\theta)}, \qquad L = \frac{R}{R_0\cos^2\theta} \qquad (16)$$

we seek the perturbed distribution function as

$$f_l(r, z, v_\|, v_\perp) = \sum_s^{\pm 1} f_l^s(L, \theta, v, \mu). \qquad (17)$$

As a result, the Vlasov equation for harmonics f_0^s and $f_{\pm 1}^s$ can be rewritten as the first order differential equation with respect to the θ-variable

$$\frac{\sqrt{\cos^6\theta - \mu\sqrt{1+3\sin^2\theta}}}{\cos^4\theta\sqrt{1+3\sin^2\theta}}\frac{\partial f_l^s}{\partial\theta} - is\frac{LR_0}{v}\left(\omega + \frac{l\Omega_{co}\sqrt{1+3\sin^2\theta}}{L^3\ \cos^6\theta}\right) f_l^s = Q_l^s \quad (18)$$

where L, v, μ are as parameters in this equation, $l = 0, \pm 1$, and other definitions are

$$Q_0^s = \frac{eR_0LF_0}{T}\sqrt{1-\mu\frac{\sqrt{1+3\sin^2\theta}}{\cos^6\theta}}E_{||}, \qquad F_0 = \frac{N}{\pi^{1.5}v_T^3}\exp\left(-\frac{v^2}{v_T^2}\right) \quad (19)$$

$$Q_{\pm 1}^s = s\frac{eR_0LF_0\sqrt{\mu\sqrt{1+3\sin^2\theta}}}{2\ T\ \cos^3\theta}(E_n \mp iE_b), \quad v_T^2 = \frac{2T}{M}, \quad \Omega_{co} = \frac{eB_0}{Mc}. \quad (20)$$

By the indexes $s = \pm 1$, we differ the particles with positive and negative values of parallel velocity,

$$v_{||} = sv\sqrt{1-\mu\frac{\sqrt{1+3\sin^2\theta}}{\cos^6\theta}} \quad (21)$$

relative to $\mathbf{H}_0$. Note, in Eqs. (13) and (18) we neglect the drift corrections assuming the drift frequencies are much less than the bounce frequencies. This assumption is valid when

$$\frac{nv_TL^2}{R_0\Omega_{co}} << 1 \qquad \text{and} \qquad \frac{nv_T}{LR_0|\omega - \Omega_{co}/L^3|} << 1 \quad (22)$$

where Ω_{co} is the cyclotron frequency v_T is the thermal velocity of plasma particles, and n is the azimuthal wave number over ϕ (east-west direction).

Depending on μ, the domain of f_l^s is defined by the inequalities:

$$\frac{1}{L^{2.5}\sqrt{4L-3}} \leq \mu \leq 1 \qquad \text{and} \qquad -\theta_t(\mu) \leq \theta \leq \theta_t(\mu) \quad (23)$$

where $\pm\theta_t(\mu)$ are the local mirror (or reflection) points for trapped particles at a given (by L) magnetic field line, which are defined by the zeros of parallel velocity, $v_{||} = 0$:

$$\cos^6\theta_t - \mu\sqrt{1+3\sin^2\theta_t} = 0. \quad (24)$$

The corresponding boundary condition for f_l^s is the continuity of f_l^s at the reflection points, i.e.,

$$f_l^{-1}(\pm\theta_t) = f_l^{+1}(\pm\theta_t). \quad (25)$$

Note, due to the Earth's atmosphere, the untrapped particles will be thermalized by the collisions with atmospheric molecules and atoms, before reaching the Earth's surface. Any particle with $\mu < L^{-2.5}(4L-3)^{-0.5}$ will not survive more than one half of the bounce time and will be precipitated into the atmosphere.

After solving Eq. (18), the two-dimensional normal, $j_n(\theta, L)$, and binormal, $j_b(\theta, L)$, current density components can be expressed as $j_n = 0.5[j_1 + j_{-1}]$ and $j_b = 0.5i[j_1 - j_{-1}]$, where

$$j_l(\theta, L) = \frac{\pi e}{2b^{1.5}(\theta)} \sum_s^{\pm 1} \int_0^\infty v^3 \int_{\mu_0}^{b(\theta)} \frac{\sqrt{\mu} f_l^s(\theta, L, v, \mu)}{\sqrt{1 - \mu/b(\theta)}} \, d\mu \, dv, \qquad l = \pm 1. \qquad (26)$$

For the parallel current density component, we have

$$j_{||}(\theta, L) = \frac{\pi e}{b(\theta)} \sum_s^{\pm 1} s \int_0^\infty v^3 \int_{\mu_0}^{b(\theta)} f_0^s(\theta, L, v, \mu) \, d\mu \, dv \qquad (27)$$

where

$$\mu_0 = \frac{1}{L^{2.5}\sqrt{4L-3}}, \qquad b(\theta) = \frac{\cos^6\theta}{\sqrt{1+3\sin^2\theta}}. \qquad (28)$$

Taking into account that the trapped particles, with a given parameter μ, execute the bounce periodic motion with the bounce period proportional to

$$\tau_b = \tau_b(\mu) = 4 \int_0^{\theta_t} \cos\theta \frac{\sqrt{1+3\sin^2\theta}}{\sqrt{1-\mu/b(\theta)}} d\theta \qquad (29)$$

the solution of Eq. (18), for example, for $f_{\pm 1}^s$, is

$$f_l^s(\theta) = \sum_{p=-\infty}^{+\infty} f_{p,s}^l \exp\left(ip\frac{2\pi}{\tau_b}\tau(\theta) + isl\frac{\omega_{co}R_0}{L^2 v} C(\theta) \right) \qquad (30)$$

where

$$f_{p,s}^l = \frac{-iseR_0 L v \sqrt{\mu} F_0 G_{p,s}^l(\mu)}{2v_T T\pi \left(pv/v_T - sZ_l\right)}, \qquad \tau = \int_0^\theta \cos\eta \sqrt{\frac{1+3\sin^2\eta}{1-\mu/b(\eta)}} d\eta \qquad (31)$$

$$Z_l = \frac{\omega}{\omega_b} + l\frac{2R_0\omega_{co}}{\pi v_T L^2} \int_0^{\theta_t} \frac{\cos\psi}{b(\psi)} \sqrt{\frac{1+3\sin^2\psi}{1-\mu/b(\psi)}} d\psi, \qquad \omega_b = \frac{2\pi v_T}{R_0 L \tau_b} \qquad (32)$$

$$G_{p,s}^l = \int_{-\tau_b/2}^{\tau_b/2} \frac{E_{(l)}(\tau)}{\sqrt{b(\tau)}} \exp\left[-ip\frac{2\pi}{\tau_b}\tau - isl\frac{\omega_{co}R_0}{L^2 v} C(\tau) \right] d\tau \qquad (33)$$

$$C(\theta) = \int_0^\theta \frac{\cos\psi}{b(\psi)} \sqrt{\frac{1+3\sin^2\psi}{1-\mu/b(\psi)}} d\psi - 4\frac{\tau(\theta)}{\tau_b} \int_0^{\theta_t} \frac{\cos\psi}{b(\psi)} \sqrt{\frac{1+3\sin^2\psi}{1-\mu/b(\psi)}} d\psi. \qquad (34)$$

As a result, the contribution of unspecified kind of plasma particles to the transverse current density component is given by

$$\frac{4\pi i}{\omega}b(\theta)j_l(L,\theta)=\frac{\omega_{po}^2R_0L}{4\omega\pi^{1.5}v_T}\sum_s^{\pm 1}s\sum_{p=-\infty}^{+\infty}\int_{\mu_0}^{b(\theta)}\frac{\mu d\mu}{\sqrt{b(\theta)-\mu}}\times$$
$$\times\int_0^\infty\frac{u^4\exp(-u^2)G_{p,s}^l}{pu-sZ_l}\exp\left[ip\frac{2\pi}{\tau_b}\tau(\theta)+isl\frac{\omega_{co}R_0}{L^2uv_T}C(\theta)\right]du \quad (35)$$

where $\omega_{po}^2=4\pi Ne^2/M$ is the square of the plasma frequency for particles with charge e, mass M, and density N (number of particles in cm^3). It should be noted that Eq. (35) is written in the general form and can be applied to define the contribution of trapped particles to the perpendicular current density components in an axisymmetric magnetosphere with an arbitrary configuration of the closed magnetic field lines. The perturbed longitudinal current can be derived by analogy (Ref. [5]).

In order to solve two-dimensional wave equations, we should expand preliminary the perturbed values in a Fourier series over θ. So, for the transverse components of the current density $b(\theta)j_l$ and electric field E_l, we have:

$$b(\theta)j_l(L,\theta)=\sum_m^{\pm\infty}j_l^{(m)}(L)\,e^{\frac{i\pi m\theta}{\theta_0(L)}},\qquad E_l(L,\theta)=\sum_{m'}^{\pm\infty}E_l^{(m')}(L)\,e^{\frac{i\pi m'\theta}{\theta_0(L)}} \quad (36)$$

where the points $\pm\theta_0(L)=\pm\arccos(1/\sqrt{L})$ are the beginning and the end of a given (by L) magnetic field line on the Earth's surface, and $E_l=E_n-ilE_b$ if $l=\pm 1$. This procedure converts the operator, representing the dielectric tensor, into a matrix whose elements are calculated independently on the solutions of Maxwell's equations. As a result,

$$\frac{4\pi i}{\omega}j_l^{(m)}=\sum_{m'}^{\pm\infty}\epsilon_l^{m,m'}E_l^{(m')} \quad (37)$$

and, after the summation over $s=\pm 1$, the contribution of a given kind of plasma particles to the transverse permittivity elements, $\epsilon_l^{m,m'}(L)$, is

$$\epsilon_l^{m,m'}=\frac{\omega_{po}^2R_0L}{2\omega\pi^{1.5}v_T\theta_0}\sum_{p=1}^{\infty}\int_{\mu_0}^1\mu d\mu\int_{-\infty}^{\infty}D_p^{m,l}\hat{D}_p^{m',l}\frac{u^4\exp(-u^2)}{pu-Z_l}du-$$
$$-\frac{\omega_{po}^2R_0L}{2\omega\pi^{1.5}v_T\theta_0}\int_{\mu_0}^1\frac{\mu d\mu}{Z_l}\int_0^\infty D_0^{m,l}\hat{D}_0^{m',l}u^4\exp(-u^2)du \quad (38)$$

where

$$D_p^{m,l} = \int_0^{\theta_t} \cos\left(\pi m\frac{\theta}{\theta_0} - 2\pi p\frac{\tau(\theta)}{\tau_b} - il\frac{\omega_{co}R_0}{L^2uv_T}C(\theta)\right)\frac{d\theta}{\sqrt{b(\theta)-\mu}} +$$
$$+(-1)^p\int_0^{\theta_t} \cos\left(\pi m\frac{\theta}{\theta_0} + 2\pi p\frac{\tau(\theta)}{\tau_b} + il\frac{\omega_{co}R_0}{L^2uv_T}C(\theta)\right)\frac{d\theta}{\sqrt{b(\theta)-\mu}} \quad (39)$$

$$\hat{D}_p^{m,l} = \int_0^{\theta_t} \cos\left(\pi m\frac{\theta}{\theta_0} - 2\pi p\frac{\tau(\theta)}{\tau_b} - il\frac{\omega_{co}R_0}{L^2uv_T}C(\theta)\right)\frac{\cos\theta\sqrt{1+3\sin^2\theta}}{\sqrt{b(\theta)-\mu}}d\theta +$$
$$+(-1)^p\int_0^{\theta_t} \cos\left(\pi m\frac{\theta}{\theta_0} + 2\pi p\frac{\tau(\theta)}{\tau_b} + il\frac{\omega_{co}R_0}{L^2uv_T}C(\theta)\right)\frac{\cos\theta\sqrt{1+3\sin^2\theta}}{\sqrt{b(\theta)-\mu}}d\theta. \quad (40)$$

Thus we see that, due to the geomagnetic field inhomogeneity, the full spectrum of an electric field (by $\sum_{m'}^{\pm\infty}$) is present in a given (by m) harmonic of the current density .

To evaluate the longitudinal permittivity elements, we should expand the perturbed longitudinal (parallel to $\mathbf{B}$) components of the current density $j_{||}b(\theta)$ and electric field $E_{||}$ in the following Fourier series over θ:

$$b(\theta)j_{||}(L,\theta) = \sum_m^{\pm\infty} j_{||}{}^{(m)}(L)\, e^{\frac{i\pi m\theta}{\theta_0(L)}}, \qquad E_{||}(L,\theta) = \sum_{m'}^{\pm\infty} E_{||}{}^{(m')}(L)\, e^{\frac{i\pi m'\theta}{\theta_0(L)}}. \quad (41)$$

As a result,

$$\frac{4\pi i}{\omega}j_{||}^{(m)} = \sum_{m'}^{\pm\infty}\epsilon_{||}^{m,m'}E_{||}^{(m')} \quad (42)$$

and the contribution of a given kind of plasma particles to the longitudinal permittivity elements, $\epsilon_{||}^{m,m'}(L)$, is

$$\epsilon_{||}^{m,m'} = \frac{\omega_{po}^2L^2R_0^2}{2\pi^2v_T^2\theta_0}\sum_{p=1}^{\infty}\int_{\mu_0}^1\frac{\tau_b}{p^2}A_p^m\hat{A}_p^{m'}\left[1+\frac{2\omega^2}{p^2\omega_b^2}+i\sqrt{\pi}\frac{2\omega^3}{p^3\omega_b^3}W\left(\frac{\omega}{p\omega_b}\right)\right]d\mu \quad (43)$$

where

$$W(z) = e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right) \quad \text{is the plasma dispersion function,}$$

$$A_p^m = \int_0^{\theta_t}\left[\cos\left(\pi m\frac{\theta}{\theta_0} - 2\pi p\frac{\tau(\theta)}{\tau_b}\right) + (-1)^{p-1}\cos\left(\pi m\frac{\theta}{\theta_0} + 2\pi p\frac{\tau(\theta)}{\tau_b}\right)\right]d\theta \quad (44)$$

$$\hat{A}_p^m = \int_0^{\theta_t}\cos\left(\pi m\frac{\theta}{\theta_0} - 2\pi p\frac{\tau(\theta)}{\tau_b}\right)\cos\theta\sqrt{1+3\sin^2\theta}\,d\theta +$$
$$+(-1)^{p-1}\int_0^{\theta_t}\cos\left(\pi m\frac{\theta}{\theta_0} + 2\pi p\frac{\tau(\theta)}{\tau_b}\right)\cos\theta\sqrt{1+3\sin^2\theta}\,d\theta. \quad (45)$$

As was noted above, the expressions (38) and (43) describe the contribution of any kind of trapped particles to the dielectric tensor elements. The corresponding expressions for plasma electrons and ions can be obtained from Eqs.(38-43) by replacing T (temperature), N (density), M (mass) by the electron T_e, N_e, m_e and ion T_i, N_i, M_i parameters. To obtain the total expressions of transverse and longitudinal dielectric tensor elements, as usual, it is necessary to carry out the summation over all kinds of plasma particles.

The dielectric characteristics of a dipole magnetosphere (as is for magnetospheric plasmas with circular magnetic field lines [4, 6]) depend substantially on the geomagnetic field nonuniformity. If $\omega << \omega_b = 2\pi v_T / R_0 L \tau_b$, the imaginary part of the longitudinal permittivity decreases as $\sim v_T^{-5}$. This decreasing is stronger than $\sim v_T^{-3}$ for plasmas in the straight magnetic field. If $\omega \sim \omega_b$, the numbers of the basic bounce resonances are defined by the condition $p \sim \omega/\omega_b$. In this case, Im $\epsilon_{||}^{m,m}(L)$ has a maximum for waves with longitudinal wave numbers $m \sim p$. It means that the resonant condition for the effective wave-particle interaction in magnetospheric plasmas should be understood as the condition when the wave performs the integer number (p) of oscillations during one bounce period $2\pi/\omega_b$ of the trapped particles.

Information related to the basic cyclotron resonance effects is included in the transverse dielectric tensor components, $\epsilon_l^{m,m'}$, derived taking into account the cyclotron and bounce oscillations of the trapped particles. The effective cyclotron-bounce interaction between the wave and the trapped particles becomes possible in the frequency range $\omega \sim \Omega_{co}/L^3$. In this case, the numbers of the basic bounce resonances are defined by $p \sim |\omega - \omega_{co}/L^3|/\omega_b$. It means that the cyclotron resonant condition should be understood as the condition when the transverse electric field performs the integer number (p) of oscillations during one bounce period of particles. In particular, if $\omega > \Omega_{co,i}/L^3$, there are two symmetric ICR-ICR points (at the considered magnetic field line) where the ion-cyclotron resonance (ICR) is realized exactly (see Fig. 1). The damping rate of such waves will be defined by the level of ion energy (temperature); i.e., the interaction will be effective if the bounce-period of the trapped ions and their transit time between the ICR-ICR points are comparable to each other.

4 Toroidal Plasma

To describe an axisymmetric tokamak we use the quasi-toroidal coordinates ρ, θ, ϕ,

$$r = R_0 + \rho\cos\theta, \qquad z = -\rho\sin\theta \tag{46}$$

where ρ and R_0 are the minor and major radii of the magnetic surface, θ and ϕ are the poloidal and toroidal angles, respectively, as shown in Fig. 2. The poloidal, $H_{0\theta}$, and toroidal, $H_{0\phi}$, projections of an equilibrium magnetic field $\mathbf{H}_0$ are

$$H_{0\theta}(\rho,\theta) = \frac{H_{\theta 0}(\rho)}{1+\epsilon\cos\theta} \qquad H_{0\phi}(\rho,\theta) = \frac{H_{\phi 0}(\rho)}{1+\epsilon\cos\theta} \qquad \epsilon = \frac{\rho}{R} \tag{47}$$

satisfying the conditions $(\boldsymbol{\nabla}\mathbf{H}_0) = 0$ and $[\boldsymbol{\nabla}\times\mathbf{H}_0]_\rho = 0$. For large aspect ratio tokamaks, one can use (see, e.g., Refs. [2,9,11,14]) two small parameters: $\epsilon << 1$ and $H_{0\theta} << H_{0\phi}$. In our paper, for low aspect ratio tokamaks, the Vlasov equation is solved in the general case of arbitrary $\epsilon < 1$ and $H_{0\theta} \sim H_{0\phi}$.

To solve Vlasov equation we use the standard method of switching to new variables associated with conservation integrals of energy, $v_\perp^2 + v_{||}^2 = const$, and magnetic moment, $v_\perp^2/2H_0 = const$, where the module of an equilibrium magnetic field is

$$H_0 = H_0(\rho,\theta) = \frac{\sqrt{H_{\phi 0}^2(\rho) + H_{\theta 0}^2(\rho)}}{1+\epsilon\cos\theta}. \tag{48}$$

Introducing the new variables v and μ (instead of $v_{||}$ and $v_\perp$)

$$v = \sqrt{v_{||}^2 + v_\perp^2}, \qquad \mu = \frac{v_\perp^2 H_0(\rho,\pi/2)}{v^2 H_0(\rho,\theta)} \tag{49}$$

the perturbed distribution functions, Eq. (12), can be found as

$$f(t,\rho,\theta,\phi,v_{||},v_\perp,\alpha) = \sum_s^{\pm 1}\sum_l^{\pm\infty} f_l^s(\rho,\theta,v,\mu)\exp(-i\omega t + in\phi + il\alpha). \tag{50}$$

In the zeroth order over a magnetization parameter, the equations for f_0^s and $f_{\pm 1}^s$ can be written as

$$\frac{\partial f_l^s}{\partial\theta} + ik_{l,s}(\rho,\theta,v,\mu)f_l^s = Q_{l,s}\,, \qquad s = \pm 1, \quad l = 0, \pm 1 \tag{51}$$

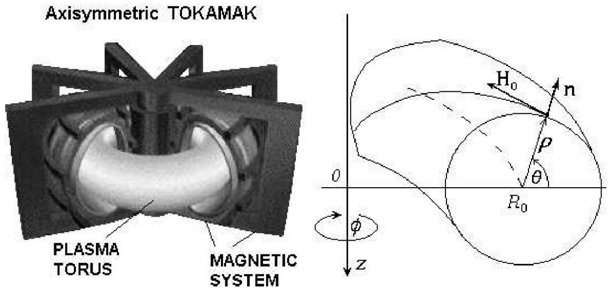

Figure 2: Quasi-toroidal coordinates (ρ, θ, ϕ) for an axisymmetric tokamak plasma

where

$$Q_{0,s} = \frac{e\,\rho}{T h_\theta} F_0 E_{||}, \qquad Q_{\pm 1,s} = \frac{s\,e\,\rho}{2T h_\theta}\sqrt{\frac{\mu}{1+\epsilon\cos\theta-\mu}} F_0 E_{\pm 1} \quad (52)$$

$$F_0 = \frac{N}{\pi^{1.5} v_T^3}\exp\left(-\frac{v^2}{v_T^2}\right), \qquad v_T^2 = \frac{2T}{M}, \qquad E_{\pm 1} = E_n \mp iE_b \quad (53)$$

$$k_{l,s} = \frac{n\,q}{1+\epsilon\cos\theta} - \frac{l}{2}h_\theta\left(\frac{4+\epsilon\cos\theta}{1+\epsilon\cos\theta} - \frac{\rho}{q}\frac{dq}{d\rho}\right) - \frac{s\,\rho}{h_\theta v}\frac{\omega + l\Omega_{co}/(1+\epsilon\cos\theta)}{\sqrt{1-\mu/(1+\epsilon\cos\theta)}} \quad (54)$$

$$h_\phi = \frac{H_{0\phi}}{H_0}, \qquad h_\theta = \frac{H_{0\theta}}{H_0}, \qquad q = \epsilon\frac{H_{0\phi}}{H_{0\theta}}, \qquad \Omega_{co} = \frac{e\sqrt{H_{\phi 0}^2 + H_{\theta 0}^2}}{M\,c}. \quad (55)$$

After solving Eq. (51) the 2D longitudinal, $j_{||}$, and transverse current density components, $j_n = j_1 + j_{-1}$, $j_b = i(j_1 - j_{-1})$, can be expressed as

$$j_{||}(\rho,\theta') = \pi e\frac{1-\epsilon\cos\theta'}{1-\epsilon^2}\sum_s^{\pm 1} s\int_0^\infty v^3\int_0^{\frac{1-\epsilon^2}{1-\epsilon\cos\theta'}} f_0^s(\rho,\theta',v,\mu)d\mu dv \quad (56)$$

$$j_{\pm 1}(\rho,\theta') = \frac{\pi e}{2}\frac{(1-\epsilon\cos\theta')^{1.5}}{1-\epsilon^2}\sum_s^{\pm 1}\int_0^\infty v^3\int_0^{\frac{1-\epsilon^2}{1-\epsilon\cos\theta'}} \frac{f_{\pm 1}^s\sqrt{\mu}d\mu dv}{\sqrt{1-\mu\frac{1-\epsilon\cos\theta'}{1-\epsilon^2}}}. \quad (57)$$

By $s = \pm 1$ we differ the particles with positive and negative parallel velocity relative to H_0:

$$v_{||} = s\sqrt{v^2 - v_\perp^2} = sv\sqrt{1-\frac{\mu}{1+\epsilon\cos\theta}} = sv\sqrt{1-\frac{\mu}{1-\epsilon^2}(1-\epsilon\cos\theta')}. \quad (58)$$

Depending of μ the phase volume of plasma particles should be split in the two populations of the so-called trapped (subscribe symbol t) and untrapped

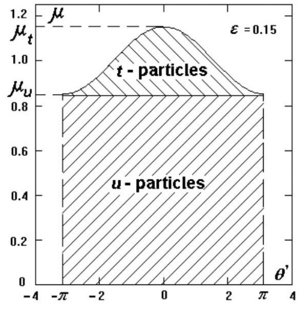

Figure 3: The phase volumes of the trapped and untrapped particles in dependence of the variables μ and θ'.

(subscribe symbol u) particles. In our notation such separation can be done by the μ-variable.

As shown in Fig. 3, the intervals

$$0 < \mu < \mu_u, \qquad -\pi < \theta' < \pi \tag{59}$$

correspond to the untrapped particles, and

$$\mu_u < \mu < \mu_t, \qquad -\theta_t < \theta' < \theta_t \tag{60}$$

correspond to trapped particles, where $\mu_u = 1-\epsilon$, $\mu_t = 1+\epsilon$, and the reflection points of the trapped particles, $\pm\theta_t$, are defined analyzing the condition $v_{||} = 0$,

$$\theta_t = 2 \arcsin \sqrt{\frac{(1-\epsilon)(1+\epsilon-\mu)}{2\epsilon\mu}}. \tag{61}$$

Recently, the contribution of the trapped and untrapped electrons to the longitudinal dielectric permittivity was derived in Ref. [12] by solving Eq. (51) for f_0^s $(l = 0)$. In contrast to them, we solve the Vlasov equation for both the f_0^s and $f_{\pm 1}^s$ harmonics using i) coordinates with the "straight" magnetic field lines, by introducing the new variable θ' instead of θ:

$$\theta'(\theta) = 2 \text{ arctg} \left(\sqrt{\frac{1-\epsilon}{1+\epsilon}} \text{tg} \frac{\theta}{2} \right) \quad \text{or} \quad \theta(\theta') = 2 \text{ arctg} \left(\sqrt{\frac{1+\epsilon}{1-\epsilon}} \text{tg} \frac{\theta'}{2} \right) \tag{62}$$

ii) new time-like variables $\tau_u(\theta')$ and $\tau_t(\theta')$, respectively, instead of θ'

$$\tau_u(\theta',\kappa) = \int_0^{\frac{\theta'}{2}} \frac{d\eta}{(1+\kappa_o \sin^2\eta)\sqrt{1-\kappa\sin^2\eta}} = \Pi\left(\kappa_o, \frac{\theta'}{2}, \kappa\right), \tag{63}$$

$$\tau_t(\theta',\hat{\kappa}) = \Pi\left(\kappa_o\hat{\kappa}, \arcsin\left(\frac{1}{\sqrt{\hat{\kappa}}}\sin\frac{\theta'}{2}\right), \hat{\kappa}\right), \tag{64}$$

to describe the bounce periodic motion of the untrapped and trapped particles along the magnetic field line. Here we have accounted that the bounce-period of u-particles is proportional to

$$\tau_{b,u}(\kappa) = 2\Pi\left(\kappa_o, \pi/2, \kappa\right), \tag{65}$$

whereas the bounce-period of t-particles is proportional to

$$\tau_{b,t}(\hat{\kappa}) = 4\Pi\left(\kappa_o\hat{\kappa}, \pi/2, \hat{\kappa}\right), \tag{66}$$

where $\Pi(\kappa_o,\theta,\kappa)$ and $\Pi(\kappa_o,\pi/2,\kappa)$ are the incomplete and complete elliptic integrals of the third kind. The new variables κ (for untrapped particles) and $\hat{\kappa}$ (for trapped particles) are introduced instead of μ as

$$\kappa(\mu) = \frac{2\epsilon\mu}{(1-\epsilon)(1+\epsilon-\mu)}, \qquad \hat{\kappa}(\mu) = \frac{1}{\kappa(\mu)}, \qquad \kappa_o = \frac{2\epsilon}{1-\epsilon}. \tag{67}$$

The solution of Eq. (51) for untrapped (u) and trapped (t) particles can be found in the form

$$f^s_{l,u}(\rho,\theta',v,\kappa) = \sum_p^{\pm\infty} f^{s,p}_{l,u}(\rho,v,\kappa)\exp\left[i(p+nq_t)\frac{2\pi\tau_u(\theta')}{\tau_{b,u}(\kappa)} - inq_t\theta'\right] \tag{68}$$

$$f^s_{l,t}(\rho,\theta',v,\hat{\kappa}) = \sum_p^{\pm\infty} f^{s,p}_{l,t}(\rho,v,\hat{\kappa})\exp\left[ip\frac{2\pi\tau_t(\theta')}{\tau_{b,t}(\hat{\kappa})} - inq_t\theta'\right] \tag{69}$$

where p is the number of bounce resonances, $s=\pm1$, $l=0,\pm1$, and $q_t = q/\sqrt{1-\epsilon^2}$ is the tokamak safety factor.

In the new variables, the domain of $f^s_{l,u}$ is

$$0 \le \kappa \le 1, \qquad -\pi \le \theta' \le \pi \qquad \text{for untrapped particles}$$

and the domain of $f^s_{l,t}$ is

$$0 \le \hat{\kappa} \le 1, \qquad -\theta_t(\hat{\kappa}) \le \theta' \le \theta_t(\hat{\kappa}) \qquad \text{for trapped particles}$$

where $\pm\theta_t(\hat{\kappa}) = \pm 2 \arcsin\sqrt{\hat{\kappa}}$ are the reflection points of the trapped particles with given $\hat{\kappa}$. Fourier-amplitudes $f_{l,u}^{s,p}$ and $f_{l,t}^{s,p}$ can be defined after the corresponding bounce-averaging. To evaluate the dielectric tensor elements we use the Fourier expansions in θ':

$$\frac{\mathbf{j}(\rho,\theta')}{1-\epsilon\cos\theta'} = \sum_{m}^{\pm\infty} \mathbf{j}^{(m)}(\rho) e^{im\theta'}, \quad \text{and} \quad \frac{\mathbf{E}(\rho,\theta')}{1-\epsilon\cos\theta'} = \sum_{m'}^{\pm\infty} \mathbf{E}^{(m')}(\rho) e^{im'\theta'}. \quad (70)$$

As a result

$$\frac{4\pi i}{\omega} j_l^m = \sum_{m'}^{\pm\infty} \left[\epsilon_{l,u}^{m,m'} + \epsilon_{l,t}^{m,m'}\right] E_l^{m'}, \quad \text{and} \quad \frac{4\pi i}{\omega} j_{||}^m = \sum_{m'}^{\pm\infty} \left[\epsilon_{||,u}^{m,m'} + \epsilon_{||,t}^{m,m'}\right] E_{||}^{m'} \quad (71)$$

Here $\epsilon_{l,u}^{m,m'}, \epsilon_{l,t}^{m,m'}$ and $\epsilon_{||,u}^{m,m'}, \epsilon_{||,t}^{m,m'}$ are the contributions of u- and t-particles to the transverse and longitudinal dielectric permittivity elements, respectively:

$$\epsilon_{l,u}^{m,m'} = \frac{0.5\,\omega_L^2\,\rho\,\sqrt{1+\epsilon}}{\pi^{2.5}\omega h_\theta v_T \sqrt{1-\epsilon}} \sum_{p}^{\pm\infty} \int_0^1 \frac{\kappa\, d\kappa}{(\kappa_o+\kappa)^2} \int_{-\infty}^{+\infty} \frac{u^4 \exp\left(-u^2\right) A_{p,l}^m A_{p,l}^{m'} du}{(p+nq_t-l\delta-lq)\,u - U_l(\kappa)} \quad (72)$$

$$\epsilon_{l,t}^{m,m'} = \frac{0.5\,\omega_L^2\,\rho\,\sqrt{1+\epsilon}}{\pi^{2.5}\omega h_\theta v_T \sqrt{1-\epsilon}} \sum_{p}^{\pm\infty} \int_0^1 \frac{d\hat{\kappa}}{(1+\kappa_o\hat{\kappa})^2} \int_{-\infty}^{+\infty} \frac{u^4 \exp\left(-u^2\right)}{pu - V_l(\hat{\kappa})} B_{p,l}^m \hat{B}_{p,l}^{m'} du \quad (73)$$

$$\epsilon_{||,u}^{m,m'} = \frac{2\omega_L^2 \rho^2 \sqrt{\kappa_o}(1+\epsilon)}{\pi^3\, h_\theta^2\, v_T^2\, (1-\epsilon)} \sum_{p}^{\pm\infty} \int_0^1 \frac{\Pi(\kappa_o, \frac{\pi}{2}, \kappa)\, C_p^m C_p^{m'}}{(p+nq_t)^2(\kappa_o+\kappa)^{1.5}} \times$$
$$\times \left[1 + 2u_p^2 + 2i\sqrt{\pi} u_p^3 W(u_p)\right] d\kappa \quad (74)$$

$$\epsilon_{||,t}^{m,m'} = \frac{4\omega_L^2 \rho^2 \sqrt{\kappa_o}(1+\epsilon)}{\pi^3\, h_\theta^2\, v_T^2\, (1-\epsilon)} \sum_{p=1}^{\infty} \int_0^1 \frac{\Pi(\kappa_o\hat{\kappa}, \frac{\pi}{2}, \hat{\kappa}) D_p^m D_p^{m'}}{p^2\,(1+\kappa_o\hat{\kappa})^{1.5}} \times$$
$$\times \left[1 + 2v_p^2 + 2i\sqrt{\pi} v_p^3 W(v_p)\right] d\hat{\kappa} \quad (75)$$

where the following definitions have been used

$$A_{p,l}^{m} = \int_0^{\pi} \cos\left\{ (m+nq_t-l\delta)\,\eta - lg\theta(\eta) - 2\pi\,(p+nq_t-l\delta-lg)\,\frac{\tau_u(\eta,\kappa)}{\tau_{b,u}(\kappa)} \right.$$
$$\left. -\, l\frac{\Omega_c\rho\sqrt{2(\kappa_o+\kappa)}}{h_\theta u v_T\sqrt{\epsilon(1+\epsilon)}}\left[F\left(\frac{\eta}{2},\kappa\right) - 2K(\kappa)\frac{\tau_u(\eta,\kappa)}{\tau_{b,u}(\kappa)}\right]\right\}\sqrt{\frac{1+\kappa_o\sin^2\frac{\eta}{2}}{1-\kappa\sin^2\frac{\eta}{2}}}\,d\eta \quad (76)$$

$$B_{p,l}^{m} = \int_0^{\theta_t}\sqrt{\frac{1+\kappa_o\sin^2\frac{\eta}{2}}{\hat\kappa-\sin^2\frac{\eta}{2}}}\cos\left\{(m+nq_t-l\delta)\eta - lg\theta(\eta) - 2\pi p\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)} - \right.$$
$$\left. -l\frac{\Omega_c\rho\sqrt{2(1+\kappa_o\hat\kappa)}}{h_\theta u v_T\sqrt{\epsilon(1+\epsilon)}}\left[F\left(\arcsin\left(\sqrt{\frac{1}{\hat\kappa}}\sin\frac{\eta}{2}\right),\hat\kappa\right) - 4K(\hat\kappa)\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)}\right]\right\}d\eta$$
$$+(-1)^p\int_0^{\theta_t}\sqrt{\frac{1+\kappa_o\sin^2\frac{\eta}{2}}{\hat\kappa-\sin^2\frac{\eta}{2}}}\cos\left\{(m+nq_t-l\delta)\eta - lg\theta(\eta) + 2\pi p\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)} + \right.$$
$$\left. +l\frac{\Omega_c\rho\sqrt{2(1+\kappa_o\hat\kappa)}}{h_\theta u v_T\sqrt{\epsilon(1+\epsilon)}}\left[F\left(\arcsin\left(\sqrt{\frac{1}{\hat\kappa}}\sin\frac{\eta}{2}\right),\hat\kappa\right) - 4K(\hat\kappa)\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)}\right]\right\}d\eta \quad (77)$$

$$W(z) = e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right) \quad (78)$$

$$C_p^m(\kappa) = \int_0^{\pi}\cos\left[(m+nq_t)\,\eta - 2\pi\,(p+nq_t)\,\frac{\tau_u(\eta,\kappa)}{\tau_{b,u}(\kappa)}\right]d\eta \quad (79)$$

$$D_p^m(\hat\kappa) = \int_0^{\theta_t}\cos\left[(m+nq_t)\,\eta - 2\pi p\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)}\right]d\eta +$$
$$+(-1)^{p-1}\int_0^{\theta_t}\cos\left[(m+nq_t)\,\eta + 2\pi p\frac{\tau_t(\eta,\hat\kappa)}{\tau_{b,t}(\hat\kappa)}\right]d\eta \quad (80)$$

$$U_l(\kappa) = \frac{\rho\sqrt{2(\kappa_o+\kappa)}}{\pi h_\theta v_T\sqrt{\epsilon(1+\epsilon)}}\left[\omega(1+\epsilon)\Pi\left(\kappa_o,\frac{\pi}{2},\kappa\right) + l\Omega_c K(\kappa)\right] \quad (81)$$

$$V_l(\hat\kappa) = \frac{2\rho\sqrt{2(1+\kappa_o\hat\kappa)}}{\pi h_\theta v_T\sqrt{\epsilon(1+\epsilon)}}\left[\omega(1+\epsilon)\Pi\left(\kappa_o\hat\kappa,\frac{\pi}{2},\hat\kappa\right) + l\Omega_c K(\hat\kappa)\right] \quad (82)$$

$$u_p(\kappa) = \frac{\omega\rho\sqrt{2(1+\epsilon)(\kappa_o+\kappa)}}{|p+nq_t|\;\pi h_\theta\; v_T\sqrt{\epsilon}}\Pi\left(\kappa_o,\frac{\pi}{2},\kappa\right), \quad F(\eta,\kappa) = \int_0^{\eta}\frac{d\theta}{\sqrt{1-\kappa\sin^2\theta}} \quad (83)$$

$$v_p(\hat\kappa) = \frac{2\omega\rho\sqrt{2(1+\epsilon)(1+\kappa_o\hat\kappa)}}{p\;\pi\;h_\theta\;v_T\;\sqrt{\epsilon}}\Pi\left(\kappa_o\hat\kappa,\frac{\pi}{2},\hat\kappa\right), \qquad K(\kappa) = F\left(\frac{\pi}{2},\kappa\right) \quad (84)$$

$$\omega_L^2 = \frac{4\pi Ne^2}{M}, \quad q_t = \frac{q}{\sqrt{1-\epsilon^2}}, \quad \delta = \frac{1.5\;h_\theta}{\sqrt{1-\epsilon^2}}, \quad g = \frac{h_\theta}{2}\left(1-\frac{\rho}{q}\frac{dq}{d\rho}\right). \quad (85)$$

Note that our $\epsilon_{l,u}^{m,m'}, \epsilon_{l,t}^{m,m'}, \epsilon_{||,u}^{m,m'}, \epsilon_{||,t}^{m,m'}$ describe the contribution of any kind of untrapped and trapped particles to the dielectric tensor elements. The corresponding expressions for plasma electrons and ions can be obtained by replacing T (temperature), N (density), M (mass) , e (charge) by the electron T_e, N_e, m_e, e_e and ion T_i, N_i, M_i, e_i parameters, respectively. To obtain the total expressions of the permittivity elements, as usual, it is necessary to carry out the summation over all species of plasma particles.

The bounce resonance conditions of the effective wave-particle interactions in a tokamak plasma are the same those derived in Ref. [7] and can be rewritten as

$$(p + nq_t - l\delta - lg)u - U_l(\kappa) = 0, \qquad l, p = 0, \pm 1, \pm 2, ... \tag{86}$$

for the untrapped particles, where $u = v/v_T$, and

$$pu - V_l(\hat{\kappa}) = 0, \qquad l, p = 0, \pm 1, \pm 2, ... \tag{87}$$

for the trapped particles.

It should be noted that the phase coefficients $A_{p,l}^m$, $B_{p,l}^m$, $C_{p,l}^m$ and $D_{p,l}^m$ can be calculated introducing the Jacobi elliptic functions. In particular,

$$C_p^m(\kappa) = \int_{-K(\kappa)}^{K(\kappa)} \cos \Big[2(m + nq_t) \mathrm{am}(\kappa, w) - \\ -(p + nq_t) \frac{\pi \, \hat{\Pi}(\kappa_o, w)}{\hat{\Pi}(\kappa_o, K(\kappa))} \Big] \mathrm{dn}(\kappa, w) dw \tag{88}$$

$$D_p^m(\kappa) = \sqrt{\kappa} \int_{-2K(\kappa)}^{2K(\kappa)} \cos \Big[2(m + nq_t) \arcsin \left(\sqrt{\kappa} \mathrm{sn}(\kappa, w) \right) - \\ - p \frac{0.5 \pi \hat{\Pi}(\kappa_o \kappa, w)}{\hat{\Pi}(\kappa_o \kappa, K(\kappa))} \Big] \mathrm{cn}(\kappa, w) dw. \tag{89}$$

The corresponding Jacobi elliptic functions are (see also Ref. [1])

$$\mathrm{sn}(\kappa, w) = \frac{2\,\pi}{\sqrt{\kappa}K(\kappa)} \sum_{l=0}^{\infty} \frac{\hat{q}^{l+1/2}(\kappa)}{1-\hat{q}^{2l+1}(\kappa)} \sin\frac{(2l+1)\pi w}{2K(\kappa)} \tag{90}$$

$$\mathrm{cn}(\kappa, w) = \frac{2\,\pi}{\sqrt{\kappa}K(\kappa)} \sum_{l=0}^{\infty} \frac{\hat{q}^{l+1/2}(\kappa)}{1+\hat{q}^{2l+1}(\kappa)} \cos\frac{(2l+1)\pi w}{2K(\kappa)} \tag{91}$$

$$\mathrm{dn}(\kappa, w) = \frac{\pi}{2K(\kappa)} + \frac{2\,\pi}{K(\kappa)} \sum_{l=1}^{\infty} \frac{\hat{q}^{l}(\kappa)}{1+\hat{q}^{2l}(\kappa)} \cos\frac{l\pi w}{K(\kappa)} \tag{92}$$

$$\mathrm{am}(\kappa, w) = \frac{\pi\;u}{2K(\kappa)} + \sum_{l=1}^{\infty} \frac{2\hat{q}^{l}(\kappa)}{l(1+\hat{q}^{2l}(\kappa))} \sin\frac{l\pi w}{K(\kappa)} \tag{93}$$

where

$$\hat{q}(\kappa) = \exp\left[-\pi\frac{K(1-\kappa)}{K(\kappa)}\right] \tag{94}$$

and

$$\hat{\Pi}(\kappa_o, w) = \int_0^w \frac{d\,u}{1+\kappa_o \mathrm{sn}^2(\kappa, u)} \tag{95}$$

using the new w variables, instead of θ':

$$w(\theta') = \int_0^{\theta'/2} \frac{d\,\eta}{\sqrt{1-\kappa\sin^2\eta}} \qquad \text{for untrapped particles,} \tag{96}$$

and

$$w(\theta') = \int_0^{\arcsin\left(\frac{1}{\sqrt{\hat{\kappa}}}\sin\frac{\theta'}{2}\right)} \frac{d\,\eta}{\sqrt{1-\hat{\kappa}\sin^2\eta}} \qquad \text{for trapped particles.} \tag{97}$$

5 Conclusions

In this paper, the kinetic Vlasov equation is written for plasma particles in the arbitrary three-dimensional equilibrium magnetic field. The transverse and longitudinal dielectric permittivity elements have been derived for radio frequency waves by solving the Vlasov equation for trapped and untrapped particles in two-dimensional axisymmetric toroidal and magnetospheric plasma models.

In particular, our dielectric characteristics can be used for both the large ($\epsilon << 1$) and low ($\epsilon < 1$) aspect ratio tokamaks to analyze the finite-ϵ effects in the frequency range of the Alfvén, Fast Magnetosonic, Lower Hybrid, and Ion/Electron Cyclotron Waves.

The new dielectric tensor elements evaluated in the paper are suitable to develop the 2D numerical codes to solve the Maxwell's equations in tokamak geometry and Earth's magnetosphere taking into account the cyclotron and bounce resonances.

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