Matemática Contemporânea, Vol. 22, 113–140
http://doi.org/10.21711/231766362002/rmc228

THE STRUCTURE OF COMPOSITE RAREFACTION-SHOCK FOLIATIONS FOR QUARATIC SYSTEMS OF CONSERVATION LAWS

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Abstract

We consider a system of two conservation laws and the associated 3-dimensional manifold which represents fundamental waves, called *wave manifold*.

If the flux functions are polynomial, of degree 2, topological descriptions of shocks and rarefactions are known. Here we present the topological description of composites, both as curve families in each of the two sonic surfaces embedded in the wave manifold and as a pair of curve families in state space. We also prove that composites are stable under C^3-perturbation of the flux functions.

1 Introduction

A system of conservation laws models the evolution in time of a continuum. When the movement is one-dimensional, the system takes the form

$$W_t + F(W)_x = 0, \tag{1}$$

where $W(x,t) \in \Re^n$ represents the state of the continuum and F is a map $F : \Re^n \longrightarrow \Re^n$, whose coordinates are called flux functions. In general the flux functions depend nonlinearly on W, and this causes discontinuities in the solutions of (1). In gas dynamics, for instance, these discontinuities represent shock waves.

In analogy with the experiments in shock-tube, analyzed by Riemann, an initial-value problem for (1) with initial data

$$W(x,t=0) = \begin{cases} W_1 & \text{if } x < 0, \\ W_2 & \text{if } x > 0 \end{cases}$$

is called a Riemann Problem. Furthermore, solutions of the Riemann Problem must be invariant under rescaling, i.e., invariant under the change of variables

$(x,t) \to (ax, at)$, for $a > 0$. These fundamental solutions are built as successions of elementary solutions: shocks, rarefactions, composites.

Shocks are solutions of the form

$$W(x,t) = \begin{cases} W_1 & \text{for } x < st \\ W_2 & \text{for } x > st. \end{cases}$$

Rarefactions are absolutely continuous solutions of the form $\tilde{W}(x/t)$. Composites are solutions of the form

$$W(x,t) = \begin{cases} W_1 & \text{for } x < s_1 \\ \tilde{W}(x/t) & \text{for } s_1 < x < s_2 \\ W_2 & \text{for } s_2 < x. \end{cases}$$

Many properties of these solutions can be derived from the corresponding wave curves defined in the wave manifold [1].

In the first non-trivial case (flux functions are polynomials of degree 2), shock curves were studied in [2] and rarefaction curves in [3]. Here we complete the topological study of elementary solutions by describing composite curves and showing their stability. This will put us in position to construct general solutions for Riemann problems and to show their stability.

More precisely, let us consider the system (1) with two conservation laws, where $W = (u,v)^T$, $W(x,t) \in \Re^2$, and the flux function $F : \Re^2 \longrightarrow \Re^2$ is given by $F = (f,g)^T$, with

$$\begin{cases} f(u,v) = v^2/2 + (b_1+1)u^2/2 + a_1 u + a_2 v \\ \\ g(u,v) = uv - b_2 v^2/2 + a_3 u + a_4 v \end{cases} \tag{2}$$

where, $b_1 \notin \{0, \pm 1, 1 + b_2^2/4\}, b_2^2 + 4/(b_1+1) \neq 0$ and $a_3 - a_2 \neq 0$. This is the normal form of [2] and [3].

In [3] rarefaction curves were studied for this system, and in [2] shock curves were considered. A more general study was done in [1] where, in Section 8, the composite foliation is defined and some general results are proved. At the end of that section, it is remarked that the structure of the composite foliation for quadratic models remained to be determined. This paper addresses that remark.

In order to state our main result, let us recall that the characteristic surface C and the sonic surfaces S and S' are cylinders embedded in a three-dimensional manifold M (the wave manifold [2]). The rarefaction (singular) foliation, described in [3], is given by a differential equation. The composite foliation is defined in [1] as the pullback to S' of the rarefaction foliation in C by shock curves, subject to a condition of speed equality.

Here we obtain an explicit formula for this *shock curves pullback map* which we call ψ and we prove the following theorems.

Theorem 1 *The map $\psi : S' \longrightarrow \psi(S') \subset C$ is a double cover. It is a local diffeomorphism, except at a curve whose image does not contain the singular points of the rarefaction foliations. Furthermore, $\psi(S') = C$ if and only if $b_2^2 + 4/(b_1 + 1) < 0$.*

Theorem 2 *a. If $Q \in C$ is a singular point of the rarefaction line field, then there exists a unique point $Q' \in S'$ such that Q and Q' are singularities of the composite line field in S'. As singularities of this line field, Q and Q' are of the same type (saddle or node) as Q considered as a rarefaction singularity.*

b. The composite (singular) foliation has neither saddle connections nor periodic orbits.

c. If $b_2^2+4/(b_1+1) > 0$ the composite foliation has two other singular points, which are the points in the inflection locus $S' \cap C \cap S$ where the rarefaction curves are tangent to the inflection locus.

Theorem 3 *The composite singular foliation is structurally stable for C^3 perturbations of the flux function provided that $b_2^2 + 4/(b_1 + 1) < 0$.*

We believe that the conclusion of Theorem 3 is true even if this inequality is reversed.

We also describe the projection of the composite foliation in state space ((u, v)-space), which we also call composite foliations, and we prove the following theorem (a more precise formulation of this theorem will be given later.)

Theorem 4 *In state space, the rarefaction foliations are the projections of the composite foliation from S', provided that $b_2^2+4/(b_1+1) < 0$. If this inequality is reversed, then in state space there are regions where exactly one of the following statements is true.*

a. The composite foliation is not defined.

b. Both rarefaction foliations are also composite foliations.

c. Only one rarefaction foliation is also a composite.

In Section 2 we present, without proofs, the main results from [3] and [2] that will be used in this paper. We also present the definitions and notation to be used in what follows and describe the intersection of shock curves with the characteristic surface. In Section 3 we describe the intersection of shock curves with the sonic surface S'. We can then define ψ and prove Theorem 1.

In Section 4 we study the differential equation of the composite foliation in the sonic surface, prove Theorems 2 and 3, and describe the image of ψ as well as the composite foliations in state space, thereby proving Theorem 4.

In this work we had to perform computations involving polynomials of degree 3 and 4, which were done with *Maple* (*Maple* is a trademark of the University of Waterloo).

2 Preliminaries

2.1 The wave manifold and shock curves

Following section 2 of [2], we consider equation (1), with F given by (2). We know that a shock

$$W(x,t) = \begin{cases} W_1 & \text{for } x < st \\ W_2 & \text{for } x > st. \end{cases}$$

is a weak solution if and only if W_1, W_2 and s satisfy the Rankine-Hugoniot condition

$$F(W_1) - F(W_2) = s(W_1 - W_2). \tag{3}$$

If we let $W_1 = (u, v)$ and $W_2 = (u', v')$, and eliminate s from (3), we get

$$[f(u,v) - f(u',v')](v - v') - [g(u,v) - g(u',v')](u - u') = 0. \tag{4}$$

This equation defines a set P that is the union of the 2-plane $u = u'$, $v = v'$ with a three-dimensional manifold M^3, called the *Wave Manifold* for system (1). In M^3 we consider the *shock curves* defined by $du = dv = 0$. Classical shock curves are the projections of these curves onto the (u', v')-plane. We can also consider the *shock′ curves*, defined by $du' = dv' = 0$.

Rarefactions and composite solutions also generate rarefaction curves and composite curves, which are defined in appropriate two-dimensional submanifolds of M^3, and they project onto the (u', v')-plane to produce classical rarefaction and composite curves.

Computations in M^3 are easier using the coordinates from [2]:

$$\begin{array}{ll}
\tilde{u} = b_1 u + b_2 v + a_1 - a_4, & \tilde{v} = v + a_2, \\
\tilde{u}' = b_1 u' + b_2 v' + a_1 - a_4, & \tilde{v}' = v' + a_2, \\
X = u - u', & \tilde{X} = \tilde{u} - \tilde{u}', \\
Y = v - v', & \tilde{Y} = \tilde{v} - \tilde{v}', \\
U = (u + u')/2, & \tilde{U} = (\tilde{u} + \tilde{u}')/2, \\
V = (v + v')/2, & \tilde{V} = (\tilde{v} + \tilde{v}')/2, \\
c = a_3 - a_2. &
\end{array}$$

In terms of $X, Y, \tilde{U}, \tilde{V}$, we see that P is defined by

$$(X^2 - Y^2)\tilde{V} - XY\tilde{U} + cX^2 = 0.$$

Let $Z = Y/X$. In the space $(X, Y, \tilde{U}, \tilde{V}, Z)$, P is given by

$$X^2((1 - Z^2)\tilde{V} - Z\tilde{U} + c) = 0,$$

and

$$Y = ZX.$$

The plane $X = 0$, $Y = 0$ is a solution, reflecting the fact that $u = u'$, $v = v'$ is a solution.By omitting this trivial case, we get the manifold M^3 given by

$$(1 - Z^2)\tilde{V} - Z\tilde{U} + c = 0$$

and

$$Y = ZX.$$

Notice that Z is a direction, so M^3 is actually a submanifold of $\Re^4 \times \Re P^1$, or equivalently, of $\Re^4 \times S^1$. In [2] it was shown that M^3 is a Moebius band crossed with $\Re$. Since there are no special features at $Z = \infty$, in what follows we will use only $\tilde{U}, \tilde{V}, Z, X$ coordinates and consider M^3 as given by $G = 0$ in $(\tilde{U}, \tilde{V}, Z, X)$-space, where $G = (1 - Z^2)\tilde{V} - Z\tilde{U} + c$. Going back to the Rankine-Hugoniot equations (3), we get the following expression for the speed:

$$s = Z\tilde{V} + m + (b_1 + 1)(\tilde{U} - b_2\tilde{V})/b_1, \tag{5}$$

where $m = a_1 + (b_1 + 1)(a_4 + b_2 a_2 - a_1)/b_1$.

The shock curves in M^3 are given by $du = dv = 0$, i.e.,

$$\begin{aligned} dX + 2dU &= 0, \\ dY + 2dV &= 0, \end{aligned}$$

or equivalently

$$dK = 0, \quad dL = 0,$$

where

$$K = b_1 X + 2\tilde{U} - 2b_2\tilde{V}, \qquad L = 2\tilde{V} + ZX.$$

The shock$'$ curves are obtained changing X to $-X$ in the expressions for K and L.

Given a pair of real numbers k, l, consider the shock curve defined by $K = k$ and $L = l$. This curve is the set of points $(\tilde{U}, \tilde{V}, Z, X)$ satisfying the system

$$\begin{cases} G = (1 - Z^2)\tilde{V} - Z\tilde{U} + c = 0 \\ K = b_1 X + 2\tilde{U} - 2b_2\tilde{V} = k \\ L = ZX + 2\tilde{V} = l. \end{cases} \tag{6}$$

This is a linear system in $\tilde{U}$, $\tilde{V}$, and X with determinant $2Zp(Z)$, where $p(Z) = Z^2 + b_2 Z + b_1 - 1$. So, if $Zp(Z) \neq 0$, shock curves are parametrized by Z. Solving the above system when $Zp(Z) \neq 0$, one gets

$$\begin{cases} \tilde{U} = (kZ^3 - b_1 l Z^2 + (2cb_2 - k)Z + b_1(l + 2c))/[2Zp(Z)] \\ \tilde{V} = (-kZ + b_1 l + 2c)/[2p(Z)] \\ X = (lZ^2 + (k + b_2 l)Z - (l + 2c))/[Zp(Z)]. \end{cases} \tag{7}$$

Since we assume that the discriminant $b_2^2 - 4(b_1 - 1)$ of p is nozero, p has distinct roots; if these roots are real, let us denote them by $z_1 < z_2$. Also, let $z_0 = 0$. Shock curves are singular along one or three straight lines B_i, $i = 0, 1, 2$, defined by:

$$Z = z_i, \ \tilde{V} = \frac{z_i\tilde{U} - c}{1 - {z_i}^2}, \ X = -2[({z_i}^2 + 1)\tilde{U} - 2cz_i]\frac{b_1 + b_2 z_i}{b_1(1 - {z_i}^2)^2}.$$

In the same fashion, shock$'$ curves are singular along the straight lines B_i', obtained from B_i by changing X into $-X$. Let $B = \cup_{i=0}^{2} B_i$, and $B' = \cup_{i=0}^{2} B_i'$. The local structure of shock curves near B_i is the same as that of the curves $x^2 - y^2 = c$, $z = c'$ near the z-axis in $\Re^3$.

From (7) it is clear that the number of connected components in a generic shock curve is given by the number of real roots of $Zp(Z)$. It is easy to see that if $l \neq -2c$ the shock curve does not intersect the plane $\Pi_0 : Z = 0$ (and in particular the straight line B_0). Also, if $b_2^2 - (4b_1 - 1) > 0$, it is clear that $Z - z_i$ factors from the numerator and from the denominator of the expression of $\tilde{V}$ in (7) if and only if $-kz_i + b_1 l + 2c = 0$. With some more work we can see that it factors also from the numerators and denominators of the expressions of $\tilde{U}$ and of $\tilde{X}$. So, if $-kz_i + b_1 l + 2c \neq 0$, then the shock curve does not intersect the planes $\Pi_i : Z = z_i$, $i = 1, 2$.

Let us now describe shock curves that intersect the singular set B. For simplicity we will consider only B_0. If $l = -2c$, the shock curve has two parts: a straight line given by

$$Z = 0, \ \tilde{V} = -c, \ \tilde{U} = (k - 2cb_2 - b_1 X)/2, \tag{8}$$

denoted by shl_0, and a curve given by

$$\begin{cases} \tilde{U} = (kZ^2 + 2cb_1 Z + (2cb_2 - k))/[2p(Z)] \\ \tilde{V} = (-kZ - 2c(b_1 - 1))/[2p(Z)] \\ X = (-2cZ + (k - 2cb_2))/p(Z), \end{cases} \tag{9}$$

denoted by shc_0. Notice that the intersection $shl_0 \cap shc_0$ of these parts is a point belonging to B_0. Also, the lines shl_0 fill the plane Π_0 as k varies. If $k = 2cb_2$ the intersection point of the shock curve and B_0 lies in the plane $X = 0$; this point is denoted by Q_0.

If $b_2^2 - (4b_1 - 1) > 0$, the same situation occurs for shock curves through points in B_1 or B_2. In each case the shock curve decomposes into a straight line shl_i and a curve shc_i, with $shl_i \cap shc_i$ being a point belonging to B_i. If $k = 2c(2z_i + b_2)/(1 + z_i^2)$ and $l = -2c/(1 + z_i^2)$, then the point $shl_i \cap shc_i$ also lies in the plane $X = 0$. We will denote this point by Q_i, for $i = 1, 2$. If $p(Z)$ has complex roots, all shock curves are connected (remember that Z is a direction, and we must glue $Z = +\infty$ with $Z = -\infty$ to get M^3). It is then clear that shc_0 is a closed curve. If $p(Z)$ has real roots, shock curves are open and have three connected components in general, except for the ones that intersect B, which have two connected components.

We define the *sonic surface* S in M^3 as the set of points where the speed s is extremal along a shock curve, i.e., where the differential forms dG, dK, dL and ds are linearly dependent. In the same fashion, we define the *sonic′ surface* S' by requiring dG, dK', dL' and ds to be linearly dependent. In [2] it is shown that S is given by the equation

$$A_U \tilde{U} + A_V \tilde{V} + A_X X = 0,$$

where

$$\begin{aligned} A_U &= 2(Z^2 + b_1 + 1) \\ A_V &= 2Z(Z^2 - b_2 Z + b_1 + 3) \\ A_X &= -Z^2 - b_2(b_1 + 1)Z + b_1 + 1. \end{aligned}$$

Similarly, the equation for S' is

$$A_U \tilde{U} + A_V \tilde{V} - A_X X = 0. \tag{10}$$

Both S and S' are cylinders embedded in M^3. Let us consider their intersection. There are two cases:

1. $b_2^2 + 4/(b_1 + 1) < 0$. The intersection is formed only by the inflection locus, to be defined in the next section.

2. $b_2^2 + 4/(b_1 + 1) > 0$. The intersection is formed by the inflection locus and by two straight lines constituting what is called double sonic locus.

2.2 Rarefactions

The characteristic surface C in M^3 is defined by $X = 0$. It is also a cylinder. In the characteristic surface we have the rarefaction line field, given by the differential equation $ZdU - dV = 0$ or, equivalently, $ZdK - b_1 dL = 0$. This line field is singular at the points $Q_i = B_i \cap C$, which are the same as $B_i' \cap C$. So, depending on whether $p(z)$ has real or complex roots, the line field has one or three singular points. Let us briefly recount how these results are obtained.

Recall that a rarefaction is an absolutely continuous solution of the form $\tilde{W}(x/t)$. Putting this expression in equation (1), we get $DF(\tilde{W})\tilde{W}' = s\tilde{W}'$, where $\tilde{W}'$ indicates differentiation of $\tilde{W}$ with respect to $s = x/t$. This is the differential equation of the eigenspaces of DF. So, whenever DF has distinct real eigenvalues, there will be a pair of independent line fields. The integral curves of these line fields produce two curve families constituting the *rarefaction foliation.*

Writing $F = (f, g)$ and $W' = (du, dv)$, the equation above becomes:

$$\begin{aligned} f_u du + f_v dv &= s du \\ g_u du + g_v dv &= s dv. \end{aligned}$$

Eliminating s we get

$$f_v (dv/du)^2 + (f_u - g_v) dv/du - g_u = 0.$$

To study this equation, we set $z = dv/du$, and study the differential equation $zdu - dv = 0$ in the surface C defined by $f_v z^2 + (f_u - g_v)z - g_u = 0$. Since f and g are quadratic, this equation is linear in u and v, and this makes it easy to use u and z or v and z as coordinates to study the differential equation in C.

In C we obtain a (non-orientable) line field that we can integrate, obtaining a curve family. We can then project this foliation onto the (u, v)-plane to get two curve families, which are the classical rarefaction curves.

In [2] and [1] it is shown that the surface C can be identified with the $X = 0$ section of M^3, the $zdu - dv = 0$ equation becoming $ZdU - dV = 0$. Indeed, replacing f and g by their expressions as functions of u and v, the equation of C becomes:

$$(v + a_2)z^2 + (b_1 u + b_2 v + a_1 - a_4)z - (v + a_3) = 0,$$

or

$$\tilde{u}z + (z^2 - 1)\tilde{v} - c = 0,$$

and the differential equation becomes $zd\tilde{u} - (b_1 + b_2 z)d\tilde{v} = 0$. It is shown in [3] that C is a cylinder embedded in $\Re^2 \times \Re P^1$, when we identify $z = -\infty$ with $z = +\infty$.

Points $(\tilde{u}, \tilde{v}, z)$ such that $E = \tilde{u}^2 + 4\tilde{v}(\tilde{v} + c) < 0$ are never in C. The ellipse $E = 0$ separates the $(\tilde{u}, \tilde{v})$-plane into the hyperbolic region ($E > 0$, where there are two rarefaction curve families, transversal to each other) and the elliptic region ($E < 0$, where there are no rarefaction curves). With respect to C, these regions correspond to the vertical lines $(\tilde{u} = \tilde{u}_0, \tilde{v} = \tilde{v}_0, z)$ either intersecting C transversally in two points or else not intersecting C. Points on the ellipse correspond to points in C where the vertical lines are tangent to C. In $E = 0$, the two foliations become tangent, but they remain transversal to the ellipse, except at the projection of the singularities of the rarefaction foliation. Notice that in M^3 these vertical lines are exactly the shock curves.

Let $\pi : C \longrightarrow \Re^2$ be defined by $\pi(\tilde{u}, \tilde{v}, z) = (\tilde{u}, \tilde{v})$. The inverse image by π of the ellipse $E = 0$ is called the *fold curve*, because it is the singular set of this projection, and all points on $E = 0$ are fold points. The fold is a simple closed curve that does not bound a disk in C. The foliation in each side of the fold projects in one of the rarefaction foliations in the (u, v)-plane.

In C the rarefaction foliation is non-orientable, and as we mentioned, has singularities at the intersection points of B and C. The singular points are in the fold curve, and they are the only points where the foliation is not transversal to this curve.

The structure of the rarefaction curve family in C is described in [3]. It depends on the number and type of singularities of the line field. There are three cases, according to the number of saddle points.

D1: $b_2^2 - 4(b_1 - 1) < 0$ or, equivalently, $1 + b_2^2/4 < b_1$. There is only one singular point, which is a saddle.

D2: $0 < b_1 < 1 + b_2^2/4$. There are three singularities, two of which are saddles, and one is a node.

D3: $b_1 < 0$. There are three singular points, all saddles.

The points in C where the speed function s is critical along the rarefactions form a curve called *inflection locus*. As we mentioned in the previous subsection, it is part of the intersection $S \cap S'$. The inflection locus is transversal to the rarefactions, except at the intersection points with the double sonic locus. This divides case $D3$ in two:

D3.1 $b_2^2 + 4/(b_1 + 1) < 0$. In this case the double sonic locus is empty, and there are no tangency points.

D3.2 $b_2^2 + 4/(b_1 + 1) > 0$. In this case, the double sonic locus is not empty, and there are tangency points.

In cases $D1$ and $D2$, since $b_1 > 0$, we have $b_2^2 + 4/(b_1 + 1) > 0$. So $D3.1$ is the only case with empty double sonic locus.

2.3 Shock curves and the characteristic surface C

This subsection is the only part of this preliminaries that is contained in neither [3] nor [2].

A shock curve intersects C if and only if there exists a solution of equation (6) with $X = 0$. If $Zp(Z) \neq 0$, we can use equations (7). Thus a necessary condition is that $(k+b_2l)^2+4l(l+2c) \geq 0$. As $\tilde{U} = (k+b_2l)/2$ and $\tilde{V} = l/2$ the curve $(k+b_2l)^2+4l(l+2c) = 0$ is the ellipse $E = 0$ we met in the previous section. Recall that it is the boundary of the elliptic region in the $(\tilde{u}, \tilde{v})$-plane (or (k,l)-plane). We have seen that for (k,l) in the elliptic region, the shock curve given by $K = k$, $L = l$ does not intersect C; for (k,l) outside of the elliptic region the shock curve cuts C at two distinct points, and for (k,l) on the boundary of the elliptic region, the shock curve has a double intersection point with C. In terms of transversality it is easy to see that the shock curves are transversal to C everywhere except along the fold curve. If $Zp(Z) = 0$ we have equations (8) and (9). It is easy to see that both shl_i and shc_i intersect C transversally, $i = 0,1,2$. The intersection points coincide if and only if $shl_i \cap shc_i \in C$. In this case, as noticed previously, $shl_i \cap shc_i = B_i \cap C = Q_i$, is a singular point of the rarefaction foliation.

3 The composite map

As we remarked before, the equation for S' is

$$A_U\tilde{U} + A_V\tilde{V} - A_XX = 0.$$

The composite foliation in S' is defined as follows. Given a point in S', one follows the shock curve through this point until it reaches C. This defines a map $\psi : S' \longrightarrow C$. The composite foliation in S' is defined as the pullback by ψ of the rarefaction foliation in C.

From now on we will use lower case letters for coordinates of points in C, and capital letters for coordinates of points in M^3 on the same shock curve. Given a point $(\tilde{u}, \tilde{v}, z, 0) \in C$ we let $k = 2\tilde{u} - 2b_2\tilde{v}$, and $l = 2\tilde{v}$. The coordinates k and l are essentially the same as the original coordinates u and v; indeed, $k = 2b_1u + 2(a_1 - a_4 - b_2a_2)$ and $l = 2(v + a_2)$.

3.1 Shock curves and the sonic surface S'

Here we will see that a shock curve intersects S' at 4, 2, or 0 points, depending on k and l. Substituting $\tilde{U}$, $\tilde{V}$ and X in the equation of S' by their expressions in terms of Z, we get from (7) the following polynomial:

$$\begin{aligned} T(Z) =&(2c + b_2k + l)Z^4 - 2(k - b_2l)Z^3 \\ &+ [(b_1 + 1)4c + (b_1 + 1)b_2k + (b_1 + 1)(b_2^2 + 2)l - 4l]Z^2 \\ &- 2(b_1 + 1)(k + b_2l)Z + (b_1 + 1)^2(2c + l). \end{aligned}$$

Real roots of $T(Z)$ correspond to intersection points of the shock curve and S'. Tangency points are obtained by solving the system

$$\begin{cases} T(Z) = 0 \\ \dot{T}(Z) = 0, \end{cases}$$

where we use a dot to indicate differentiation with respect to Z.

Remark 1 *The polynomial $T(Z)$ has $Z = 0$ as a root if $l = -2c$. It is easy to see that in this case $T(Z)/Z$ is the same polynomial we would get using the equations of shc_0 instead of (7). The same is true for the roots of $p(Z)$.*

If we write $T(Z) = \alpha(Z)k+\beta(Z)l+\gamma(Z)$, we can regard the previous system as a linear system in k and l, viz.,

$$\begin{cases} \alpha k + \beta l = -\gamma \\ \dot{\alpha} k + \dot{\beta} l = -\dot{\gamma}. \end{cases}$$

If $\alpha\dot{\beta} - \beta\dot{\alpha} \neq 0$, this system can be solved, to obtain

$$\begin{cases} k = -2(Z^2 + b_1 + 1)(b_2Z^2 - 4Z - b_2(b_1 + 1))c/D, \\ \\ l = -2(Z^2 + b_1 + 1)^2c/D, \end{cases} \tag{11}$$

where $D = (b_2^2 + 1)Z^4 - 4b_2Z^3 + 2(b_1 + 3)Z^2 + (b_1 + 1)^2$.
A straightforward computation shows that k and l defined by (11) satisfy $E = 0$. In particular, this also shows that the denominator D of k and l has no real roots, since if such a root existed, then either it would be a root of $Z^2 + b_1 + 1$ or l would become unbounded, and it is easy to see that the denominator and $Z^2 + b_1 + 1$ have no common root.

More tangency points are obtained if $\alpha\dot{\beta} - \dot{\alpha}\beta$ and $\alpha\dot{\gamma} - \dot{\alpha}\gamma$ have common real roots. If z_a is a common real root of $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\alpha\dot{\gamma} - \gamma\dot{\alpha}$, then we have a line r_a in the (k, l)-plane, defined by $\alpha(z_a)k + \beta(z_a)l + \gamma(z_a) = 0$, as part of the solution of the linear system. It is easy to see that $\alpha\dot{\beta} - \beta\dot{\alpha}$ and $\alpha\dot{\gamma} - \gamma\dot{\alpha}$ have $Z^2 + b_2(b_1 + 1)Z - (b_1 + 1)$ as their greatest common divisor. Recall that this expression is $-A_X$, the coefficient of X in equation (10). This polynomial has real roots if and only if $b_2^2 - 4/(b_1 + 1) > 0$; in this case let the roots be denoted by z_a and z_b. A simple computation shows that r_a and r_b are both tangent to the ellipse $E = 0$. Since S and S' are defined by equations which are linear in all variables except Z, it is clear that $Z = z_a$ and $Z = z_b$ define two lines which are part of the intersection of S and S'. These lines, as we mentioned in section 1, constitute the double sonic locus.

Let us study the intersection of the shock curve by Q_i with S'. Taking $k = 2cb_2$ and $l = -2c$, the polynomial T becomes $2cZ^2(b_2 - Z)^2$. The double root $Z = 0$ reflects the fact that shl_0 and shc_0 intersect, respectivelly, C and S' transversally at Q_0. The double root $Z = 2/b_2$ defines a point Q_0' where shc_0 is tangent to S'. Using equations (7) or (9) we obtain $Q_0' = (\tilde{U}(Q_0'), \tilde{V}(Q_0'), \tilde{X}(Q_0'))$, where

$$\tilde{U}(Q_0') = 2cb_2(b_1 + 2)/(4 + (b_1 + 1)b_2^2),$$

$$\tilde{V}(Q_0') = -cb_2^2(b_1 + 1)/(4 + (b_1 + 1)b_2^2),$$

$$\tilde{X}(Q_0') = -4cb_2/(4 + (b_1 + 1)b_2^2).$$

If $b_2^2 - (4b_1 - 1) > 0$, taking $k = 2c(2z_i + b_2)/(1 + z_i^2)$ and $l = -2c/(1 + z_i^2)$ the polynomial T becomes $[2c(b_2 z_i + b_2^2 + b_1^1 - 1)/(1 + z_i^2)](Z - z_i)^2(Z - z_i')^2$, where $z_i' = [(b_1 + 1)/(1 - b_1)]z_i$. The double root z_i' defines a point Q_i' where shc_i is tangent to S'. As before using equations (7) we compute the coordinates of the point Q_i'. Let us introduce the notation $\Delta = b_2^2 + 4/(b_1 + 1)$. First, we have the following result.

Proposition 1 *Suppose* $\Delta < 0$ (*case* $D3.1$).

a. *If* $E < 0$, *then the* (k,l)*-shock curve does not intersect* S'.

b. *If* $E = 0$, *then the* (k,l)*-shock curve intersects* S' *tangentially at two distinct points.*

c. *If* $E > 0$, *then the* (k,l)*-shock curve intersects* S' *transversely at four distinct points.*

Proof: Computing $\partial k/\partial Z$ and $\partial l/\partial Z$ from (11), we see that they become zero simultaneously if and only if $\Delta > 0$, so we have a parametrization for the whole ellipse. Actually, as Z varies from $-\infty$ to $+\infty$, the ellipse is covered twice. To see that, for (k,l) outside of the elliptic region, $T(Z)$ has four distinct real roots, and for (k,l) inside the elliptic region, $T(Z)$ has four complex roots, it is enough to evaluate $T(Z)$ at one point in each region. At the center of the ellipse, where $l = -c$ and $k = -b_2 l$, $T(Z)$ becomes $c[(b_2+1)Z^4 - 4b_2 Z^3 + 2(b_1+1)Z^2 + (b_1+1)^2]$, which has no real roots. On the other hand, taking $l = 0$ and $k \to \infty$, we get $T(Z) \to b_2 Z^4 - 2Z^3 + b_2(b_1 + 1)Z^2 - 2(b_1 + 1)Z$, which has four real roots. □

Let us summarize the information on how shock curves intersect C and S'.

- If (k, l) lies outside of the elliptic region, the shock curve intersects C at two distinct points and intersects S' at four distinct points.
- If (k, l) lies on the ellipse, the shock curve is tangent to C at one point, and tangent to S' at two distinct points, except in the case of singularities of the rarefactions. If (k, l) is such that the corresponding point $Q_i \in C$ is a singularity for the rarefactions, then $Q_i \in C \cap S'$ and the shock curve through Q_i is formed by a straight line shl_i and a curve shc_i, both transversal to C and to S' in Q_i. Also, $S' \cap shl_i = Q_i$, $S' \cap shc_i = \{Q_i, Q_i'\}$ for some point Q_i', and shc_i is tangent to S' at Q_i'.
- If (k, l) lies inside the elliptic region, the shock curve does not intersect either C or S'.

In order to treat the case $\Delta > 0$, let us introduce some definitions and notations. Let q_0 be the intersection point of r_a and r_b: $q_0 = (0, -2c\frac{b_1+1}{b_1})$. Let us call fa the arc of ellipse bounded by the tangency points that is farther from q_0; similarly the closer is called cl. Let tri be the region bounded by cl, r_a and r_b, and TRI the region bounded by r_a, r_b and fa. Finally let $R1$, $R2$, $R3$, $R4$ be the four regions which form $\Re^2 - TRI$. Thus $R1$ is the region opposed by the vertex to TRI, the others following trigonometrically as shown in Figure 1.

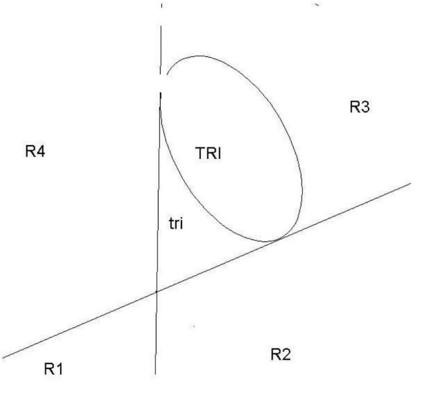

Figure 1: *Regions of the (k, l)-plane for the case $\Delta > 0$*

We have already seen that regarding (11) as a parametrized curve, the point $(k(Z), l(Z))$ lies on the ellipse $E = 0$. A straightforward computation shows that (11) is a parametrization of either fa or cl. It is easy to see that z_a and z_b are the only values of Z where $\partial k/\partial Z$ and $\partial l/\partial Z$ are both zero in (11), and that they are simple roots of $\partial k/\partial Z$ and $\partial l/\partial Z$, indicating that the point

$(k(Z), l(Z))$ actually retraces its path in the opposite direction. Replacing k and l in (11) by K and L and adding the X component from the equation of S', we get a parametrization of a curve in S' and see that $\partial X/\partial Z$ is nonzero at z_a and at z_b, indicating that in S' we have a regular closed curve that is covered twice as Z goes from $-\infty$ to $+\infty$.

Definition 1 *The curve defined in the previous paragraph is called the* sonic fold.

One can ask which arc of ellipse (cl or fa) in the (k, l)-plane is the projection of the tangency curve from S'. A straightforward computation shows that it is cl in cases $D1$ and $D2$ and fa in case $D3.2$. The idea is to compare the l coordinates of q_0 and of $(0, 0)$, noticing that $(0, 0)$ is obtained by (11) if and only if $b_1 + 1 < 0$. (Without loss of generality we can assume $c > 0$.) We then have

Proposition 2 *: Suppose $\Delta > 0$ and $b_1 < 0$, (case $D3.2$).*

a. If $E < 0$, then the (k, l)-shock curve does not intersect S'

b. If (k, l) lies in tri, then the (k, l)-shock curve does not intersect S'.

c. If (k, l) lies in $R1$ or $R3$, then the (k, l)-shock curve intersects S' in four distinct points.

d. If (k, l) lies in $R2$ or $R4$, then the (k, l)-shock curve intersects S' in two distinct points

Proposition 3 *Suppose $\Delta > 0$ and $b_1 > 0$ (case $D1$ and $D2$).*

a. If $E < 0$, then the (k, l)-shock curve does not intersect S'.

b. If (k, l) lies in $R1$ or $R3$, then the (k, l)-shock curve does not intersect S'.

c. If (k, l) lies in tri, then the (k, l)-shock curve intersects S' in four distinct points.

d. If (k, l) lies in $R2$ or $R4$, then the (k, l)-shock curve intersects S' in two distinct points.

Proof of Proposition 2. As in the proof of Proposition 1, the same argument shows that $T(Z)$ has no real root for (k, l) inside the ellipse. Since, in case

$D3.2$, fa is the projection of the sonic fold, if (k,l) is in tri, then $T(Z)$ has no real root. Checking the number of roots of $P(Z)$ at the points

$$(+\infty, 0), (-\infty, 0), (0, +\infty), (0, -\infty),$$

we get that $T(Z)$ has 4 roots at $(0,+\infty)$ and $(0,-\infty)$ (because $b_1 < 0$), and $T(Z)$ has 2 roots at $(-\infty, 0)$, $(+\infty, 0)$ if $b_1 + 1 > 0$ and 4 roots otherwise.

The points on the ellipse with horizontal tangent are $(0,0)$ and $(2cb_2, -2c)$. The lines r_a and r_b intersect at $q_0 = (0, -2c(b_1+1)/b_1)$. It is clear that the ellipse is contained in the half-plane $l < 0$. There are two possible cases, as shown in Figures 2 and 3, according to the relative positions of q_0 and the points of horizontal tangency.

Case 1. q_0 lies in the $l > 0$ half plane. This occurs if $-1 < b_1 < 0$.

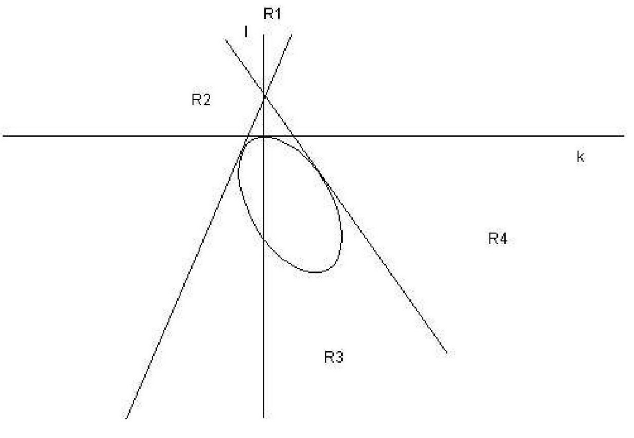

Figure 2: *The (k,l)-plane in case 1 of Proposition 2*

Case 2. q_0 lies in the $l < 0$ half-plane above the point $(2cb_2, -2c)$. This occurs if $b_1 < -1$.

The case with q_0 lying in the $l < 0$ half-plane below $(2cb_2, -2c)$ does not occur because $b_1 < 0$.

In case 1, we can use the points at infinity on the k and l-axis as representatives of $R1$ to $R4$ to conclude that there are 2 roots for $R2$ and $R4$ and 4 roots for $R1$ and $R3$.

In case 2, the 4 points at infinity on the coordinate axes lie in $R1$ and $R3$, and there are 4 roots in these regions. To study regions $R2$ and $R4$, we write the equations of r_a and r_b in the form $k = \epsilon_a l + \xi_a$ and $k = \epsilon_b l + \xi_b$. We then

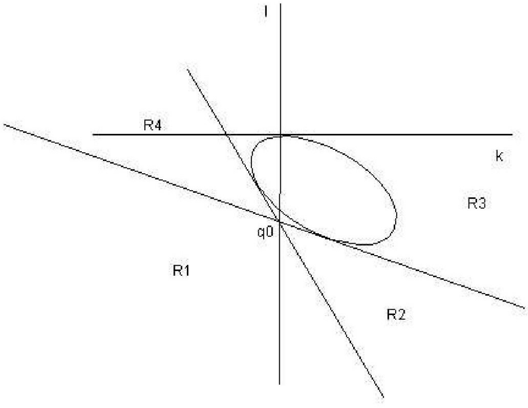

Figure 3: *The (k,l)-plane in case 2 of Proposition 2*

consider the line $k = \epsilon l + \tau$ which passes through q_0 with $\epsilon = (\epsilon_a + \epsilon_b)/2$. Taking $T(Z)$ at the infinite point of this line and writing the condition on b_1 and b_2 for $T(Z)$ to have multiple roots, we get a curve in the (b_1, b_2)-plane that does not intersect the region defined by $\Delta > 0$ and $b_1 + 1 < 0$. Thus it is enough to verify that $T(Z)$ has only two real roots for a pair (b_1, b_2) in this region. □

Proof of Proposition 3. As in the proof of Proposition 2, we can use the points at infinity on the k and l axis as representatives of regions $R1$ to $R4$ to conclude that there are no roots for regions $R1$ and $R3$ and two roots for regions $R2$ and $R4$. It remains to be shown that there are four roots for (k,l) in *tri*. For that it is sufficient to verify that $T(Z)$ has four real roots at the point $k = 0$, $l = -2c$. □

As before, if we put together the information on how a shock curve intersects C and S', we see that, depending on where the point lies in C, the shock curve through it will intersect S' at either 4, 2 or 0 points.

The fold is divided in two arcs. For points in one arc, the shock curve is tangent to C and tangent to S' at two distinct points. For points in the other arc, the shock curve is still tangent to C but does not intersect S'.

There are two curves in C, defined by $z = z_a$ and $z = z_b$, where z_a and z_b are the roots of $Z^2 + b_2(b_1 + 1)Z - (b_1 + 1) = 0$. Along these curves the shock curve intersects C transversally (except at the inverse image of the tangency points), but it intersects S' tangentially at two distinct points.

3.2 Definition of the shock curve pullback map.

We now define the map $\psi : S' \longrightarrow C$ by following shock curves. We still have a problem of choosing between two points in C to define the image of ψ. This difficulty will be solved by the following Lemma. Recall that Π_i is the plane $Z = z_i$, $i = 0, 1, 2$.

Lemma 1 *Let $q \in C$, $q' \in C$ and $Q \in M^3 - \cup_{i=0}^{2}\Pi_i$ be points in the same shock curve such that $Q \neq q, q'$. Let s be the shock speed given by equation* (5). *Then $[s(Q) - s(q)][s(Q) - s(q')] = 0$ if and only if $Q \in S'$. (This is a particular case of a general result from [1], Section 8.)*

Proof: Let $\tilde u$, $\tilde v$ and z be the coordinates of q. Then the coordinates of q' are $\tilde u$, $\tilde v$ and z', where z and z' are roots of $(1 - z^2)\tilde v - z\tilde u + c = 0$. Let $k = 2\tilde u - 2b_2\tilde v$ and $l = 2\tilde v$. Let $\tilde U$, $\tilde V$, Z and X be the coordinates of Q. Recall that $K = 2\tilde U - b_2\tilde V + b_1X$ and $L = 2\tilde U + ZX$.

Using (5), a straightforward computation shows that

$$[s(Q) - s(q)][s(Q) - s(q')] = s(Q)^2 - [(z+z')\tilde v + 2(\frac{b_1+1}{b_1}(\tilde u - b_2\tilde v) + \alpha)]s(Q) +$$

$$zz'\tilde v^2 + (z+z')(\frac{b_1+1}{b_1}(\tilde u - b_2\tilde v) + \alpha) + (\frac{b_1+1}{b_1}(\tilde u - b_2\tilde v) + \alpha)^2.$$

Since $z + z' = -\tilde u/\tilde v$ and $zz' - (\tilde v + c)/\tilde v$, we get an expression for $[s(Q) - s(q)][s(Q) - s(q')]$ in terms of $\tilde u$, $\tilde v$, $\tilde U$, $\tilde V$, Z and X.

Replacing $\tilde u$ and $\tilde v$ by their expressions in terms of k and l, and replacing $\tilde U$, $\tilde V$ and X by their expressions in (7), we get a polynomial in Z of degree 6, which is precisely $T(Z)g(Z)$, where $g(Z) = (1 - Z^2)\tilde v - Z\tilde u + c$. Since $Q \neq q, q'$ we have $g(Z) \neq 0$. It follows that $[s(Q) - s(q)][s(Q) - s(q')] = 0$ if, and only if $Q \in S'$. □

Corollary 1 *Let q and q' be points in C in the same shock curve l_0. Then*

1. *If l_0 intersects S' at four points they can be reordered as Q_i, $1 \leq i \leq 4$ such that $s(Q_1)s(Q_2) = s(q) \neq s(Q_3) = s(Q_4) = s(q')$.*

2. *If l_0 intersects S' at two points Q_1 and Q_2, then $s(Q_1) = s(Q_2) = s(q)$ or $s(Q_1) = s(Q_2) = s(q')$.*

Proof: From the proof of the Lemma 1 we have

$$[s(Q) - s(q)][s(Q) - s(q')] = g(Z)T(Z).$$

Since the equation above is symmetric in q and q' it follows that its left hand side is a product of polynomials of degree 3 in Z. The roots z and z' of $g(Z)$ correspond to $Q = q$ and $Q = q'$, so if $T(Z)$ has four distinct real roots we must have two roots of $s(Q) - s(q)$ and the other two roots of $s(Q) - s(q')$.

If $T(Z)$ has two complex roots, then they must both be roots of one of the polynomials $s(Q) - s(q)$ or $s(Q) - s(q')$; the two real roots will be roots of the other polynomial. □

We can define $\psi : S' \longrightarrow C$ as follows: Given $Q \in S'$, we let $\psi(Q)$ be the point of C which lies in the same shock curve as Q and has the same value of s. Since $\Pi_i \cap S' = B_i'$, we are actually defining ψ in $S' - \cup_{j=0}^{2} B_i'$, but ψ extends by continuity to the whole of S'.

In order to obtain an expression for ψ, we use k, l, z as coordinates in C and K, L, Z, X as coordinates in S'. Then we have $k = K$ and $l = L$. To obtain the expression for z in terms of K, L, Z and X we write $s(K, L, Z, X) = s(k, l, z, 0)$, obtaining $zL = Z(L - ZX) - (b_1 + 1)X$, and from this we get $z = Z - (Z^2 + b_1 + 1)X/L$. (For points with $L = 0$ it is just a matter of using $1/Z$ instead of Z as coordinate.) So we define $\psi(K, L, Z, X)$ by

$$\psi(K, L, Z, X) = \begin{cases} k = K \\ l = L \\ z = Z - (Z^2 + b_1 + 1)X/L. \end{cases} \tag{12}$$

Remark 2 *If $Q \in C$, then the expression for z becomes just $z = Z$, and we get $\psi(Q) = Q$ as we should.*

Let us show that ψ given by (12) is a well defined map from S' to C. Let $(K, L, Z, X) \in S'$. Let $X(Z)$ and $\tilde{V}(Z)$ be the expressions of X and $\tilde{V}$ respectively, given by (7). Replace X and $\tilde{V}$ by $X(Z)$ and $\tilde{V}(Z)$ in the expression of z in (12), and call $z(Z)$ the result. Compute $X(z(Z))$, and verify that its numerator is a multiple of T. Since $(K, L, Z, X) \in S'$ implies that $T(Z) = 0$, this shows that $X(z(Z)) = 0$. So ψ is a well defined map from S' to C, such that for $Q \in S' - B'$, we have $s(Q) = s(\psi(Q))$. In particular this shows that all shock curves originating from S' intersect C, although in the case $\Delta > 0$ there are shock curves which intersect C but not S'.

Let us write the expression of ψ as defined by (12) in coordinates $(\tilde{u}, z)$ for C and $(\tilde{U}, Z)$ for S'. A straightforward computation gives

$$(\tilde{U}, Z) \mapsto (\gamma_1 \tilde{U} + \gamma_0, \frac{\delta_1 \tilde{U} + \delta_2}{\delta_3 \tilde{U} + \delta_4}),$$

with γ_i, $i = 0, 1$ and δ_j, $1 \leq j \leq 4$, given by

$$\gamma_0 = c \frac{b_2 Z^4 + (b_1 - b_2^2) Z^3 + 3 b_2 Z^2 + b_1 (b_1 + 3) Z}{(1 - Z^2)(Z^2 + b_2 (b_1 + 1) Z - (b_1 + 1))},$$

$$\gamma_1 = \frac{(b_2^2-1)Z^4 - 4b_2Z^3 - 2(b1-1)Z^2 - (b_1-1)^2}{(1-Z^2)(Z^2+b_2(b_1+1)Z-(b_1+1))},$$

and

$$\begin{aligned}
\delta_1 &= Z^2 + b_1 + 1,\\
\delta_2 &= -c(b_1+2)Z,\\
\delta_3 &= b_2Z^2 - 2Z,\\
\delta_4 &= c(Z^2 - b_2Z + 1).
\end{aligned}$$

Computing the determinant of its differential, one gets an expression $\eta(\tilde{U}, Z)$, which is a quotient between polynomials in $\tilde{U}$ and Z. Its denominator is just $(Z^2-1)(\delta_3\tilde{U}+\delta_4)^2$ and its numerator is a product $\eta_1\eta_2$, with

$$\begin{aligned}
\eta_1 &= [(b_2+1)Z^2 - 2Z + b_1 + 1]\tilde{U} + c(Z^2 - (b_1+b_2+2)Z + 1),\\
\eta_2 &= [(b_2-1)Z^2 - 2Z - b_1 - 1]\tilde{U} + c(Z^2 - (b_1-b_2-2)Z + 1).
\end{aligned}$$

Since the curves $\eta_1 = 0$ and $\eta_2 = 0$ do not intersect, the curve $\eta = 0$ is a regular curve, with at least two connected components. A straigthforward computation shows that it intersects the fold and the sonic fold in isolated points, none of which is Q_i or Q'_i, for $i = 0, 1, 2$. If $\Delta > 0$, we have to use X and Z as coordinates in order to study a neighborhood of the lines $Z = z_a$, $Z = z_b$, where z_a and z_b are the roots of A_X.

We can now prove Theorem 1.

Theorem 1 *The map $\psi : S' \longrightarrow \psi(S') \subset C$ is a double cover. It is a local diffeomorphism, except at the singular curve of ψ. Furthermore, $\psi(S') = C$ if and only if $\Delta < 0$.*

Proof: Corollary 1 shows that $\psi^{-1}(q)$ consists of two points in S' for every point q in C. Actually we have shown this only for points out of the fold curve, but it is easy to see that it is also true for points in the fold. Using equations (7) and (9), we solve (12) locally in K, L, Z and X and show that the local inverse of ψ is a continuous map. So ψ is a local homeomorphism such that every point in C has two pre-images in S'. By the inverse function theorem, ψ is a local diffeomorphism at all points such that $\eta(\tilde{U}, Z) \neq 0$. The surjectiveness of ψ follows easily from Proposition 1. □

4 The composite line field

4.1 The composite foliation in S'

Let $w = zdk - b_1dl$, so that $w = 0$ is the differential equation of the rarefaction line field in C. The composite line field is then defined by $\psi^\star(w) = 0$, i.e.,

$$[Z - (Z^2 + b_1 + 1)X/L]dK - b_1dL = 0,$$

or

$$(ZL - (Z^2 + b_1 + 1)X)dK - b_1 LdL = 0, \tag{13}$$

since we are examining points with $L \neq 0$.

Let us now prove Theorem 2.

Theorem 2 a. *If $Q \in C$ is a singular point of the rarefaction line field, then there exist $Q' \in S'$ such that Q and Q' are singularities of the composite line field in S'. As singularities of the composite line field, Q and Q' have the same type (saddle or node) as Q considered as a rarefaction singularity.*

b. *The composite (singular) foliation has neither saddle connections nor periodic orbits.*

c. *If $\Delta > 0$ then the composite foliation has two other singular points, which are the points on the inflection locus $S' \cap C \cap S$ where the rarefactions are tangent to the locus.*

Proof: Using $\tilde{U}$ and Z as coordinates in S', equation (13) becomes

$$(\theta_1 \tilde{U} + \theta_0)d\tilde{U} + (\phi_2 \tilde{U}^2 + \phi_1 \tilde{U} + \phi_0)dZ = 0,$$

where θ_i, $i = 0, 1$, and ϕ_j, $j = 0, 1, 2$, are polynomials in Z. In order to find the singularities of this equation we solve the equation $(\theta_1 \tilde{U} + \theta_0) = 0$ in $\tilde{U}$ and substitute in $(\phi_2 \tilde{U}^2 + \phi_1 \tilde{U} + \phi_0) = 0$, getting the equation in Z:

$$8c^2 b_1 (Z-1)^3 (Z+1)^3 (b_2 Z - 2) Z (Z^2 + b_2 Z + b_1 - 1)$$
$$[(b_1 - 1)Z^2 - b_2(b_1+1)Z + (b_1+1)^2][Z^2 + b_2(b_1+1)Z - b_1 - 1] = 0.$$

We have to remember that in order to use $\tilde{U}$ and Z as coordinates we must assume $Z^2 - 1 \neq 0$ as well as $Z^2 + b_2(b_1+1)Z - b_1 - 1 \neq 0$. The first restriction comes from solving $G = 0$ in $\tilde{V}$ and the second from solving the equation for S' in X, since $Z^2 + b_2(b_1+1)Z - b_1 - 1$ is just the X-coefficient in (10).

The equation $Z(Z^2 + b_2 Z + b_1 - 1) = 0$ yields the singularities of the rarefaction line field, as we should expect, since these points belong to the intersection $C \cap S'$. These are the points Q_0, Q_1, Q_2. The equation $(b_2 Z - 2)[(b_1 - 1)Z^2 - b_2(b_1+1)Z + (b_1+1)^2] = 0$ yields three other singular points, Q'_0, Q'_1, Q'_2. In order to see this, first note that z'_1 and z'_2 are the roots of $(b_1 - 1)Z^2 - b_2(b_1+1)Z + (b_1+1)^2$. Next, compute θ_0/θ_1 for $Z = 2/b_2$, $Z = z'_1$ and $Z = z'_2$, verifying that we obtain exactly $\tilde{U}(Q'_0)$, $\tilde{U}(Q'_1)$ and $\tilde{U}(Q'_2)$, respectively.

It is easy to see that Q'_i belongs to the sonic fold but not to the singular set of ψ, for all i; and the same is true for Q_i; so Q'_i, and Q_i (as singularities of the composite foliation) are of the same type as Q_i as a singularity of the rarefaction foliation. So if $b_1 < 0$ there are six saddles; if $0 < b_1 < 1 + b_2^2/4$ there are four saddles and two nodes; and if $1 + b_2^2/4 < b_1$ there are two saddles.

If $\Delta > 0$, we must use X and Z as coordinates to study the neighborhoods of the double sonic locus (the set $S \cap S'$). A straightforward computation shows that there are two other singular points, which are the points in $C \cap S'$ where the inflection locus is tangent to the rarefactions. We have not yet determined the type of these singularities.

If we go back to state space, *i.e*, we consider the image of the composites through the projection $(K, L, Z, X) \to (K, L)$, we obtain the rarefaction curves or parts of them, as we showed in Section 3.2. This implies in particular that there are no saddle connections in S', since there are no saddle connections for rarefactions, and that the tangency points between rarefactions and inflection locus do not occur on saddle separatrices, as a simple examination of the pictures in [3] shows. □

Theorem 3 *If $\Delta < 0$, the composite singular foliation is structurally stable for C^3 perturbations of the flux function.*

Proof: The proof is similar to that in [3], and proceeds in two steps.
Step 1. A C^3 perturbation in the Whitney topology of the flux function produces a C^2 perturbation in the equations of S'. This implies that the perturbed S' is isotopic to the old one and that the new perturbed line field is C^1-close to the old one.
Step 2. As in [3] we use a modified version of Peixoto's theorem [Pe] on structural stability of vector fields on surfaces to obtain the required homeomorphism between the old S' and the new S' which preserves the composite (singular) foliation. □

4.2 Composite foliation in state space

As we have mentioned, the rarefaction foliation is defined in the characteristic surface. Projecting onto state space ((u, v)-plane), we get two foliations transversal everywhere (except along the ellipse), which we also call rarefaction foliation (in state space). In the same way the composite foliation is defined in S' and we also project it onto state space, obtaining the composite foliations in state space. Since all projections are obtained through shock curves and the composite foliation (in S') is obtained through rarefaction foliation (in C) by a pullback by shock curves, it is clear that the composite foliations (in state space) are contained in the rarefaction foliations (in state space).

The question we want to answer is the following: outside of the elliptic region there are two transversal rarefaction foliations. Are they also composite foliations (i.e., projections in state space of the composite foliation in S') ?

The answer is clearly yes if $\Delta < 0$. If $\Delta > 0$ it will depend on the Ri, $i = 1$ to 4, regions (see Fig. 1). More precisely:

Theorem 4 *a. If $b_1 < 0$, then in $R1$ and $R3$ both rarefaction foliations are also composite foliations, whereas in $R2$ only one rarefaction foliation is a composite foliation. In $R4$ only the other rarefaction foliation is a composite foliation. In TRI there is no composite foliations.*

b. If $0 < b_1$, then there is no composite foliation in $R1$ or $R3$. In $R2$ only one rarefaction foliation is a composite foliation. In $R4$ only the other rarefaction foliation is a composite foliation. In tri both rarefaction foliations are composite foliations. In the elliptic region there is no composite foliation.

Notation 1 *In the proof of the Theorem 4 we denote by $proj$ the projection $(k, l, z) \to (k, l)$ from the characteristic surface onto state space along shock curves.*

Proof of Theorem 4: It is clear that a region corresponding to four points of intersection of a shock curve with S' has the two rarefaction foliations as composites and a region where shock curves do not intersect S' has no composite. The only regions to be analyzed are those where the shock curves cut S' at two points, since both points in S' are mapped by ψ to the same point in C. This means that only one rarefaction foliation is a composite. We want to examine regions $R2$ and $R4$ in all cases and check which component of $proj^{-1}(Ri)$, $i = 2$, 4, lies in the image of ψ and which does not. Let us recall a few facts about the characteristic surface C and rarefactions, which can be found in [3]. In (k, l, z)-space the characteristic surface C is defined by an equation which is a second degree polynomial in z and linear in k and l. From this it follows that C is a ruled surface formed by horizontal lines that make a 360 degree turn as z goes from $-\infty$ to $+\infty$. These lines turn around the cylinder with equation $(k + b_2 l)^2 + 4l(l + 2c) = 0$. Each horizontal line l_0 in C projects onto a line l_1 tangent (at a point q) to the ellipse with equation $(k + b_2 l)^2 + 4l(l + 2c) = 0$ in the (k, l)-plane. This is the ellipse $E = 0$ that was defined previously.

The inverse image $proj^{-1}(l_1)$ is the union of l_0 and a curve c_0 intersecting l_0 at a point p_0 that projects on q. Actually $p_0 = proj^{-1}(q)$, since the ellipse $E = 0$ is the set of points which have only one point as inverse image (by the projection). The typical curve c_0 is like the one obtained from the curve $y = \tan x$ when we identify the lines $x = -\pi/2$ and $x = \pi/2$ to transform the rectangle $[-\pi/2, \pi/2] \times [-\infty, +\infty]$ into a cylinder.

In state space let us make pictures of the ellipse $E = 0$, a tangent line t to it and its inverse image in C. We represent C as a rectangle with z as vertical coordinate; the cylinder is obtained by identifying $z = -\infty$ and $z = \infty$. If A is a point in state space we also denote by A the points in $proj^{-1}(A)$. The fold is represented by the central horizontal line, as in Figure 4. In Figure 4-b there are two vertical lines; one of them, denoted by r, is part of $proj^{-1}(t)$ and the other, denoted by a, is an asymptote for $proj^{-1}(t)$.

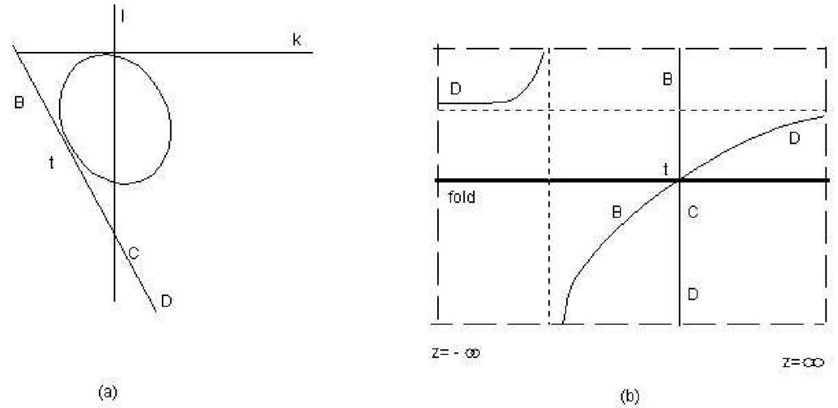

Figure 4: *(a) The ellipse and a line tangent to it. (b) The inverse image of Figure (a) in C.*

If we add a second tangent line, we get Figure 5-a, with two tangent lines to E ,denoted by t_1 and t_2. Consequently, in Figure 5-b we have four vertical lines: r_i part of $proj^{-1}(t_i)$, for $i = 1, 2$, and a_i asymptotes for $proj^{-1}(t_i)$, for $i = 1, 2$. Their relative position is $a_2a_1r_2r_1$, from left to right. Using k and z as coordinates, a simple computation shows that this is the only possible case. In particular, the case $r_2a_1r_1a_2$ cannot happen. Actually we could also have $r_1a_2a_1r_2$, and $r_2r_1a_2a_1$, but these configurations are equivalent to the one shown in Figure 5-b when we identify $z = -\infty$ with $z = +\infty$.

Now let us consider the lines in state space that bound regions Ri, $i = 1$ to 4, for three cases:

Case a. $-1 < b_1 < 0$, i.e., $q_0 = (0, -2cb_1/(b_1 + 1))$ lies in the $l > 0$ region,

Case b. $b_1 < -1$, i.e., q_0 lies in the $l < 0$ region, above the ellipse minimum,

Case c. $b_1 > 0$, i.e., q_0 lies in the $l < 0$ region, below the ellipse minimum.

Using points in the tangent lines and their pre-images by $proj$ as guides, it is easy to determine which regions in C are $proj^{-1}(Ri)$. Figure 6 illustrates case (a); the other cases are obtained in same way. In order to see which component

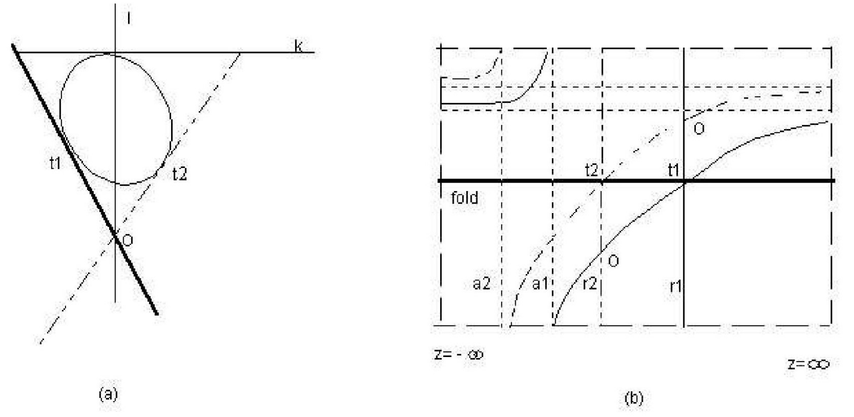

Figure 5: *(a) The ellipse and two lines tangent to it. (b) The inverse image of the Figure (a) in C.*

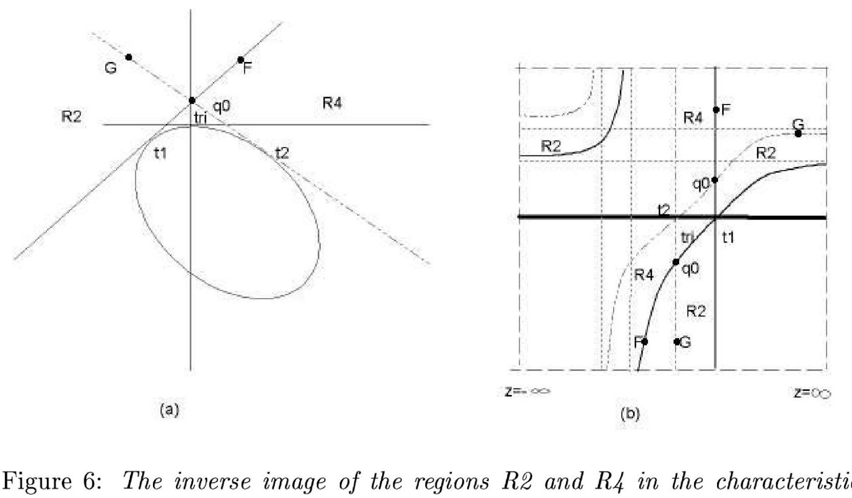

Figure 6: *The inverse image of the regions R2 and R4 in the characteristic surface for case (a).*

of $proj^{-1}(R2)$ or of $proj^{-1}(R4)$ is in $\psi(S')$, we use the lines $l = -2c$ and $l = 0$ and compute s at points in $proj^{-1}$ of these lines.

It is easy to see that $proj^{-1}(l = -2c)$ contains the line $z = 0$, i.e., points of the form $(k, -2c, 0)$. Given $q = (k, -2c, 0)$ in C, let q' be the other point in C such that $proj(q') = proj(q)$. We want to compute $\psi^{-1}(q)$ and $\psi^{-1}(q')$. We recall that the shock curve through q has two components shl_0 and shc_0 given by equations (8) and (9), respectively. Computing $shl_0 \cap S'$, we obtain $\Big(U = (2cb_2 - k)/\big(2(b_1 - 1)\big), V = -c, Z = 0, X = (k - 2cb_2)/\big(2(b_1 - 1)\big)\Big)$. Computing $shc_0 \cap S'$ we obtain three others points. From equation (5) we compute the speed of shock s at these four points. Comparing these values with $s(q)$ and $s(q')$, we see that q is in the image of ψ if and only if k lies outside of some interval.

Drawing the line $l = -2c$ in Figure 5 we get Figures 7, 8 and 9, which correspond, respectively, to cases (a), (b) and (c); this is sufficient in cases (a) and (c) in order to determine which of the components of $proj^{-1}(R2)$ and of $proj^{-1}(R4)$ lie in the image of ψ and which do not, since the intersection of the line $z = 0$ in C with $proj^{-1}(R2)$ or with $proj^{-1}(R4)$ do not lie in the image of ψ, as we have just seen.

For case (b), we use the line $l = 0$ also and perform the same computations, since in this case the line $z = 0$ intersects $proj^{-1}(R2)$ but not $proj^{-1}(R4)$. Again we use $T(Z)$ and the parametric equations of the shock curve for a point with $l = 0$ and compare s at the points in S' and the points in C to conclude that the line $z = \infty$ in C intersects $R4$ outside of the image of ψ, as in Figure 8.

In Figures 10-12 we use dotted lines to represent the composite foliation in state space if the preimage of the region Ri, $i = 2$ or 4, lies below the fold line and full lines otherwise.

Acknowledgements. The authors (and certainly the readers too) would like to thank the referee, whose mathematical and stylistic suggestions greatly improved this paper.

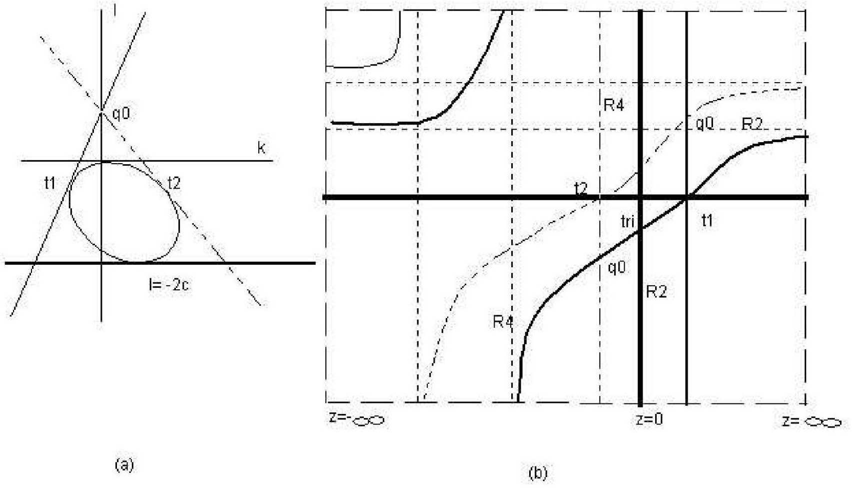

Figure 7: *(b) The components of* $proj^{-1}(R2)$ *and* $proj^{-1}(R4)$ *that intersect the line* $Z = 0$ *do not lie in the image of* ψ*.*

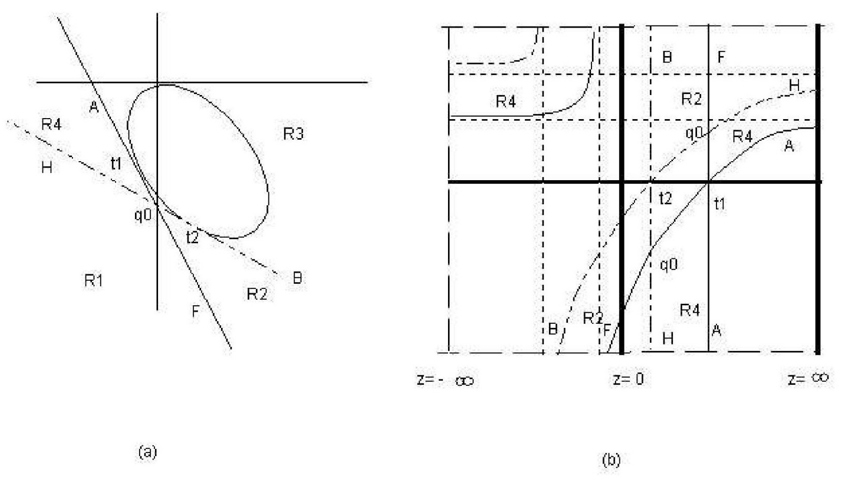

Figure 8: *(b) The component of* $proj^{-1}(R2)$ *that intersects the line* $Z = 0$ *and the component of* $proj^{-1}(R4)$ *that intersects the line* $Z = \infty$ *do not lie in the image of* ψ*.*

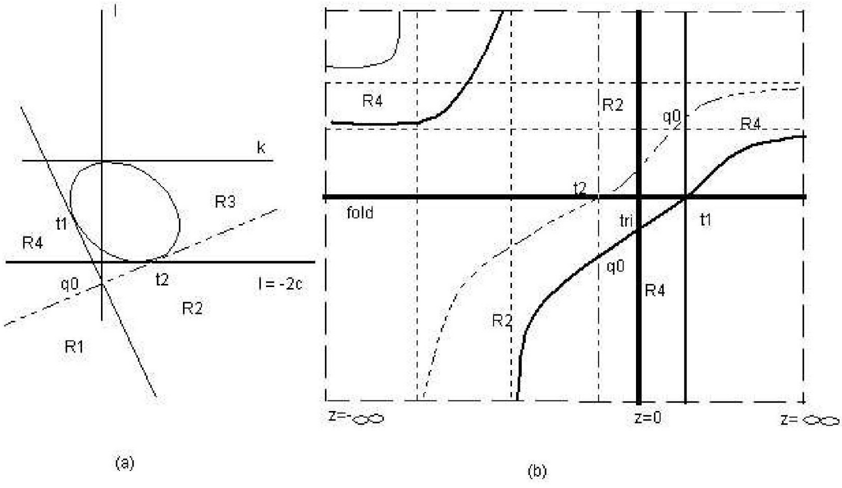

Figure 9: *Components of* $proj^{-1}(R2)$ *and* $proj^{-1}(R4)$ *that do not lie in the image of* ψ *for case (c).*

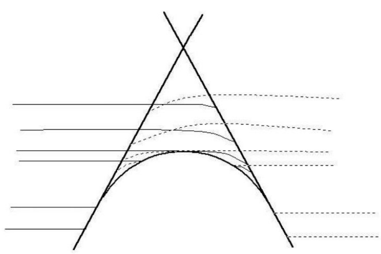

Figure 10: *Composite foliation in state space for case D1.*

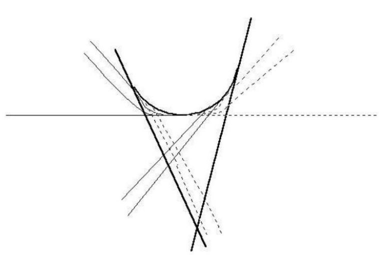

Figure 11: *Composite foliation in state space for case D2.*

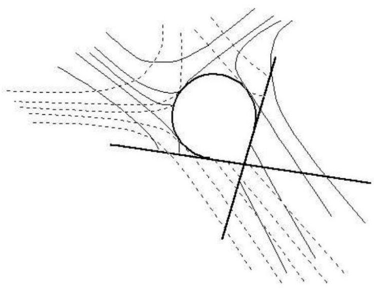

Figure 12: *Composite foliation in state space for case D3.2.*

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