

# THERMAL RADIATION IN A STEADY NAVIER-STOKES FLOW

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## Abstract

In this work, we prove an existence result for a coupled system of partial differential equations, valid for dimensions two and three. To prove this mathematical result, we use a fixed point argument for multi-valued mappings. The main part of this work is to obtain estimates in the presence of  $L^1$ -data and to prove continuous dependence with respect to given parameters. For a two-dimensional case, it can be recognized as thermoconvective flow for a Navier-Stokes fluid with heat transfer by radiation on a part of the boundary of the domain.

## 1 Introduction

All bodies emit radiant energy continuously. The intensity of the emissions depends on the temperature and on the nature of the surface. Thermal radiation is the process by which heat is transferred from a body by virtue of its temperature, thus it is defined as *radiant energy emitted by a medium by virtue of its temperature* (see [7]).

Usually, thermal radiation appears when a hot surface is in contact with a transparent medium with relatively low heat conductivity or when a body is placed in an enclosure whose walls are at a temperature below that of the body (the temperature of the body will decrease even if the enclosure is evacuated).

Almost every material partly absorbs, partly reflects and partly transmits radiation incident on its surface. An ideal radiator, also called black body, is a body which emits and absorbs at any temperature the maximum possible amount of radiation at any given wavelength (it does not reflect any thermal radiation energy).

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In the literature, it is frequent to find gray bodies whose emissivity is independent of the wavelength of the radiation. That is, at a given temperature the ratio of the monochromatic emissive power of a body to the monochromatic emissive power of a black body at the same wavelength is constant over the entire wavelength spectrum. The monochromatic emissive power denotes the radiation quantity at a given wavelength  $\lambda$ :

$$E_\lambda(\theta) := \epsilon_\lambda \frac{C_1}{\lambda^5 (e^{C_2/\lambda\theta} - 1)};$$

where  $\theta$  denotes the temperature,  $\epsilon_\lambda$  is the percentage of black body radiation emitted by the surface at the wavelength  $\lambda$  and  $C_1$  and  $C_2$  are Planck constants. For a black body, the following relation is well known

$$\int_0^\infty E_\lambda d\lambda = \sigma\theta^4;$$

where  $\sigma$  is the Stefan-Boltzmann constant.

Unfortunately, most surfaces encountered in engineering applications do not behave like black or gray bodies, as such as for instance Navier-Stokes fluids.

In this work, we introduce the thermal radiation for Navier-Stokes flows but it also can be done to a thermal flow of generalized fluids (see [2]). In compliance with the nonlinear character of this class of fluids, a general constitutive law for the heat flux must be considered.

We assume that the medium surrounding the body is nonparticipating, that is, it does neither absorb nor scatter the radiation.

Recently, initial boundary value problems for viscous heat-conducting one-dimensional real gases were studied in [6] and references therein, under various growth constraints. A numerical simulation of a steady state three-dimensional problem for only the conduction/radiation transfer process to rigid and opaque bodies can be found in [9]. A nonlocal boundary value problem results from the self-illuminating radiation on the surface (see [10]).

The outline of this work is as follows. In the next section, we formulate the problem. In Section 3, we give the assumptions and we state the main result. In Section 4, we present the main proof. It consists of the existence of weak solutions for approximated problems,  $L^1$ -theory to the independent term of an equation with partial derivatives, continuous dependness and a fixed point theorem for multivalued mappings to permit obtain the main result. These different techniques give to the physical model an interesting mathematical problem.

## 2 Statement of the problem

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ ,  $n = 2, 3$ . The flow of Navier-Stokes fluids is described by

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega; \quad (1)$$

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \cdot [\mu(\theta) \nabla \mathbf{u}] + \mathbf{f} - \nabla \pi \quad \text{in } \Omega; \quad (2)$$

$$\mathbf{u} \cdot \nabla e = \mu(\theta) |\nabla \mathbf{u}|^2 - \nabla \cdot \mathbf{q} + g \quad \text{in } \Omega, \quad (3)$$

where  $\mathbf{u} = (u_i)$ ,  $i = 1, \dots, n$ , denotes the velocity,  $\nabla \mathbf{u}$  is the Jacobi matrix,  $\mu$  denotes the viscosity,  $\theta$  the temperature,  $\pi$  the pressure,  $e$  denotes the specific internal energy and  $\mathbf{f} = (f_i)$  and  $g$  are the given external body forces and heat sources, respectively. The constitutive law for the heat flux vector  $\mathbf{q} = (q_i)$  is the Fourier law

$$\mathbf{q} = -k(\theta) \nabla \theta; \quad (4)$$

where  $k$  denotes the heat conductivity. The relation between the energy and the temperature is a linear function  $e = C\theta$ , where we assume  $C \equiv 1$  for the sake of simplicity.

If  $\Gamma_0$  and  $\Gamma$  are two disjoint complementary parts of the sufficiently regular boundary  $\partial\Omega$ , the radiation corresponds to the mixed condition (the temperature is given on a part of the boundary)

$$\theta = 0 \text{ on } \Gamma_0; \quad (5)$$

$$k(\theta) \nabla \theta \cdot \mathbf{n} + \gamma(\theta) = h \text{ on } \Gamma, \quad (6)$$

where  $\mathbf{n} = (n_i)$  is the exterior unit normal to  $\partial\Omega$  and  $\gamma$  is a general function with same characteristics of radiation, which may depend on the wavelength. As particular cases, when  $\Gamma$  denotes a boundary part of a convex black body,  $\gamma$  and  $h$  are  $\gamma(\theta) = \sigma\theta^4$  and  $h = \lambda_\infty$ , respectively, where  $\lambda_\infty$  denotes the intensity of the radiation coming to the surface outside of the system. When  $\Gamma$  denotes a boundary part of a nonconvex gray body,  $h = h(\epsilon, \lambda_\infty, K)$  and  $\gamma$  may be represented by

$$\gamma(\theta) = \sigma(I - K)(I - (1 - \epsilon)K)^{-1} \epsilon \theta^4,$$

where  $\epsilon$  denotes the emissivity of the surface and  $K$  describes the self illumination, whose explicit form of the integral operator is known for example in

[11]. In this case, the intensity of emitted radiation is the sum of the Stefan-Boltzmann heat radiation and the reflect part of the irradiation seen by the surface.

Finally, the particles in the vicinity of the surface are slowed down by virtue of viscous forces. Thus, the fluid particles adjacent to the surface stick to it and have zero velocity relative to the boundary:

$$\mathbf{u} = 0 \text{ on } \partial\Omega. \quad (7)$$

### 3 Main result

We assume that  $\mu : \mathbb{R} \longrightarrow \mathbb{R}_0^+$  is a continuous function such that

$$\exists \underline{\mu}, \bar{\mu} > 0 : \quad \underline{\mu} \leq \mu(\xi) \leq \bar{\mu}, \quad \forall \xi \in \mathbb{R}; \quad (8)$$

$k : \mathbb{R} \longrightarrow \mathbb{R}_0^+$  is a continuous function such that

$$\exists \underline{k}, \bar{k} > 0 : \quad \underline{k} \leq k(\xi) \leq \bar{k}, \quad \forall \xi \in \mathbb{R}; \quad (9)$$

$\gamma : \mathbb{R} \longrightarrow \mathbb{R}$  is a continuous function such that

$$\gamma(\xi)\text{sign}(\xi) \geq 0, \quad \forall \xi \in \mathbb{R}; \quad (10)$$

$$\exists 1 < q < \frac{n-1}{n-2} \quad \exists \bar{\gamma} > 0 : \quad |\gamma(\xi)| \leq \bar{\gamma}|\xi|^q, \quad \forall \xi \in \mathbb{R}. \quad (11)$$

**Remark 1** *We observe that although the assumption (10) is fictitious, in the physical problem when the temperature is a nonnegative function, the assumption has meanful.*

Let us define our admissible spaces using Sobolev and Lebesgue spaces. For each  $p > 1$ ,

$$\begin{aligned} V &= \{ \mathbf{v} \in [W_0^{1,2}(\Omega)]^n : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \}, \\ W_p &= \{ \theta \in W^{1,p}(\Omega) : \theta = 0 \text{ on } \Gamma_0 \} \quad \text{and} \quad W = W_2 \end{aligned}$$

are endowed with the standard norms. We also assume that

$$\mathbf{f} \in [L^2(\Omega)]^n; \quad (12)$$

$$g \in L^1(\Omega), h \in L^1(\Gamma). \quad (13)$$

From standard techniques, the problem (1)-(7) may be formulated as

**(P):** Find  $(\mathbf{u}, \theta)$  sufficiently regular satisfying (5),(7) and

$$\int_{\Omega} \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{u} \, dx + \int_{\Omega} \mu(\theta) \nabla \mathbf{u} : \nabla \mathbf{v} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in V; \quad (14)$$

$$\begin{aligned} \int_{\Omega} [k(\theta) \nabla \theta - \theta \mathbf{u}] \cdot \nabla \phi \, dx + \int_{\Gamma} \gamma(\theta) \phi \, ds &= \\ &= \int_{\Omega} (g + \mu(\theta) |\nabla \mathbf{u}|^2) \phi \, dx + \int_{\Gamma} h \phi \, ds, \quad \forall \phi \in W_{p/(p-1)}, \end{aligned} \quad (15)$$

reminding that

$$(\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} = \sum_{i,j=1}^n u_i w_{j,i} v_j = \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{w}, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n.$$

Now, we are able to state the main result of this paper.

**Proposition 1** *Under the assumptions (8)-(13), there exists a solution  $(\mathbf{u}, \theta) \in V \times W_p$  to the problem defined by (14)-(15), for all  $nq/(q+n-1) < p < n/(n-1)$ .*

This existence result uses similar argument of the paper [4], where the thermoconvective flow without thermal radiation of a class of non-Newtonian fluids is studied. The restriction  $p < n/(n-1)$  means that  $p/(p-1) > n$ , consequently  $W_{p/(p-1)} \hookrightarrow L^\infty(\Omega)$  and  $W_{p/(p-1)} \hookrightarrow L^\infty(\Gamma)$ . So the right hand side of equation (15) is well defined. While the restriction  $nq/(q+n-1) < p$  implies that  $W_p \hookrightarrow L^q(\Gamma)$ .

We remark that when  $n = 2$  and  $q = 4$  we have  $8/5 < p < 2$ . Thus there exists a solution to the thermal radiation problem in the two-dimensional space.

## 4 Proof of the main result

We begin to introduce an existence result (proof may be found in the paper [1, pages 218-220]).

**Theorem 1** *Let  $E$  be a locally convex Hausdorff topological vector space and let  $K$  be a nonempty convex compact set in  $E$ . If*

$$\Psi : K \longrightarrow \{R \in \mathcal{P}(K) : R \neq \emptyset, R \text{ closed convex} \}$$

*is an upper semicontinuous mapping then  $\Psi$  has at least one fixed point, i.e.,  $e \in \Psi(e)$  for some  $e \in K$ .*

Let us establish the existence result to the problem **(P)**. For each  $p > 1$ , we define the nonempty convex compact set, when it is considered the weak topology on  $V \times W_p$ , by

$$K := \{\mathbf{w} \in V : \|\mathbf{w}\|_V \leq R_1\} \times \{\xi \in W_p : \|\xi\|_{W_p} \leq R_2\}, \quad (16)$$

where  $R_1, R_2 > 0$  are conveniently chosen. We construct the mapping

$$(\mathbf{w}, \xi) \in K \mapsto \mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \mapsto \theta = \theta(\xi, \mathbf{u})$$

where  $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in V$  is the solution to the problem defined by

$$\int_{\Omega} \mathbf{w} \otimes \mathbf{v} : \nabla \mathbf{u} \, dx + \int_{\Omega} \mu(\xi) \nabla \mathbf{u} : \nabla \mathbf{v} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in V; \quad (17)$$

and  $\theta = \theta(\xi, \mathbf{u}) \in W_p$  is a SOLA solution to the problem defined by

$$\begin{aligned} \int_{\Omega} [k(\xi) \nabla \theta - \theta \mathbf{u}] \cdot \nabla \phi \, dx + \int_{\Gamma} \gamma(\theta) \phi \, ds &= \\ &= \int_{\Omega} (g + \mu(\xi) |\nabla \mathbf{u}|^2) \phi \, dx + \int_{\Gamma} h \phi \, ds, \quad \forall \phi \in W_{p/(p-1)}. \end{aligned} \quad (18)$$

The notion of Solution Obtained as Limit of Approximations (SOLA) was introduced by Dall'Aglio in [5] and means the only solution which is found by means of approximations.

*Existence and uniqueness of solution to the problem (17)*

The existence of a unique solution  $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in V$  to the problem defined by (17), satisfying the following estimate, independently of  $\mathbf{w}$  and  $\xi$ ,

$$\|\nabla \mathbf{u}\|_{2,\Omega} \leq \frac{\|\mathbf{f}\|_{2,\Omega}}{\underline{\mu}}, \quad (19)$$

is a classical result of Navier-Stokes theory (see for instance [8]).

*Existence and uniqueness of solution to the problem (18)*

Let us prove the existence of a solution  $\theta \in W_p$  to the problem defined by (18) satisfying

$$\|\nabla \theta\|_{p,\Omega} \leq C(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \bar{\mu} \|\mathbf{u}\|_V^2)^{p^2 n / [2p^2 n - (n-p)(2-p)]}, \quad (20)$$

where  $C = C(\underline{k}, \Omega, p, n)$  is a positive constant, for all  $1 < p < n/(n-1)$ .

First, we introduce the operator  $A : W \rightarrow W'$  such that

$$\langle A(\theta), \phi \rangle := \int_{\Omega} [k(\xi) \nabla \theta - \theta \mathbf{u}] \cdot \nabla \phi \, dx + \int_{\Gamma} \gamma(\theta) \phi \, ds.$$

Since  $\phi \in W \hookrightarrow L^{2(n-1)/(n-2)}(\Gamma)$  and  $n \leq 4$ , the boundedness property holds

$$\|A(\theta)\|_{W'} \leq \bar{k}\|\theta\|_W + \|\theta\|_{2n/(n-2),\Omega}\|\mathbf{u}\|_{2n/(n-2),\Omega} + \|\gamma(\theta)\|_{2(n-1)/n,\Gamma}.$$

As we also have  $\theta \in W \hookrightarrow L^{2(n-1)/(n-2)}(\Gamma)$ , we get  $\|\gamma(\theta)\|_{2(n-1)/n,\Gamma} \leq \bar{\gamma}C\|\theta\|_W^q$ , remarking that  $2q(n-1)/n < 2(n-1)/(n-2)$  means  $q < n/(n-2)$ .

Thus the existence of  $\theta_m \in W$  satisfying

$$\langle A(\theta_m), \phi \rangle = \int_{\Omega} (g + F_m) \phi \, dx + \int_{\Gamma} h \phi \, ds, \quad (21)$$

where  $F_m = \mu(\xi)|\nabla \mathbf{u}|^2/(1 + (1/m)\mu(\xi)|\nabla \mathbf{u}|^2)$ , is obtained (see [8]) using the fact that the operator  $A$  is pseudomonotone, semicontinuous and coercive which follow from (9)-(10) and the anti-symmetry property of the coercive term, and recalling that  $q < 2(n-1)/(n-2)$  and  $n \geq 2$ , the operator

$$\theta \in W \mapsto \theta \in L^q(\Gamma) \mapsto \gamma(\theta) \in \mathbb{R}$$

is strongly continuous.

Next, in order to get an estimate to the solution  $\theta_m$ , independently on  $m$ , we refer the additional difficulty of the  $L^1$ -data. Thus, we apply the argument introduced by Boccardo and Gallouet. This technique uses an estimate in a strip to obtain the final estimate in whole space, in spite of the exponent to the admissible space is less than  $n/(n-1)$ .

Indeed, it begins by considering  $\phi(\theta_m)$  as a test function in (21), where  $\phi$  is the real function defined by

$$\phi(e) = \begin{cases} -1 & \text{if } e < -M-1 \\ e+M & \text{if } -M-1 \leq e \leq -M \\ 0 & \text{if } -M < e < M \\ e-M & \text{if } M \leq e \leq M+1 \\ 1 & \text{if } M+1 < e. \end{cases}$$

Since we get

$$\int_{\Omega} \theta_m \mathbf{u} \cdot \nabla \phi(\theta_m) \, dx = \int_{\Omega} \mathbf{u} \cdot \nabla \phi\left(\frac{\theta_m^2}{2}\right) \, dx = 0 \text{ and } \int_{\Gamma} \gamma(\theta_m) \phi(\theta_m) \, ds \geq 0,$$

it follows

$$\underline{k} \int_{B_M} |\nabla \theta_m|^2 \, dx \leq \|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \bar{\mu} \|\mathbf{u}\|_V^2,$$

in the strip  $B_M = \{x \in \Omega : M \leq |\theta_m(x)| \leq M+1\}$ , for each  $M \in \mathbb{N}$ . We may proceed as in paper [3] to recover the explicited estimate (20). Indeed, for  $p < 2$

$$\begin{aligned} \int_{B_M} |\nabla \theta_m|^p dx &\leq \left( \int_{B_M} |\nabla \theta_m|^2 dx \right)^{p/2} |B_M|^{(2-p)/2} \leq \\ &\leq \left( \frac{\|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \bar{\mu} \|\mathbf{u}\|_V^2}{\underline{k}} \right)^{p/2} \left( \int_{B_M} |\theta_m|^{pn/(n-p)} dx \right)^{(2-p)/2} \frac{1}{M^{(2-p)pn/(2n-2p)}}. \end{aligned}$$

By recourse to the comparison test of series and taking into account that

$$\begin{aligned} \sum_{M \in \mathbb{N}} \left( \int_{B_M} |\theta_m|^{pn/(n-p)} dx \right)^{(2-p)/2} \frac{1}{M^{(2-p)pn/(2n-2p)}} &\leq \\ &\leq \left( \sum_{M \in \mathbb{N}} \int_{B_M} |\theta_m|^{pn/(n-p)} dx \right)^{(2-p)/2} \left( \sum_{M \in \mathbb{N}} \frac{1}{M^{(2-p)n/(n-p)}} \right)^{p/2} \end{aligned}$$

is satisfied for exponents  $2/(2-p)$  and  $2/p$ , it follows

$$\begin{aligned} &\|\nabla \theta_m\|_{p,\Omega}^p \leq \\ &\leq \left( \frac{\|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \bar{\mu} \|\mathbf{u}\|_V^2}{\underline{k}} \right)^{p/2} \|\theta_m\|_{pn/(n-p),\Omega}^{(2-p)(n-p)/(2pn)} \left( \sum_{M \in \mathbb{N}} \frac{1}{M^{n(2-p)/(n-p)}} \right)^{p/2}. \end{aligned}$$

Reminding that  $p < n/(n-1)$  the Dirichlet series converges and applying Sobolev imbedding it yields the estimate (20).

Using the estimate (20) independent of  $m$ , we can pass to the limit when  $m$  tends to infinity and we obtain the required solution. Indeed, there exists  $\theta \in W_p$  such that for a subsequence

$$\theta_m \rightharpoonup \theta \text{ in } W_p.$$

Easily, we pass to the limit in (21), remarking that  $p > qn/(q+n-1)$  implies that

$$\theta_m \rightarrow \theta \text{ in } L^q(\Gamma), q < p(n-1)/(n-p), \text{ and a.e. in } \Gamma.$$

Thus, recalling the assumption (11) and applying Lebesgue theorem, we obtain

$$\gamma(\theta_m) \rightarrow \gamma(\theta) \text{ in } L^1(\Gamma) \text{ and a.e. in } \Gamma. \quad (22)$$

Finally, the uniqueness of SOLA solution follows by same arguments as in [5]. Taking two different approximations of  $F_m$  in (21) we have different solutions, namely  $\theta_m$  and  $\tilde{\theta}_m$ . Then, choosing  $\phi(\theta_m - \tilde{\theta}_m)$  in problems (21) of  $\theta_m$  and



$\tilde{\theta}_m$ , we obtain an estimate analogous to (20) which permits to pass to the limit when  $m$  tends to the infinity. Thus the limit solution is unique.

*Existence of a fixed point to the problem (17)-(18)*

For each  $(\mathbf{w}, \xi) \in K$  we define  $\Psi(\mathbf{w}, \xi) = \{\mathbf{u}\} \times \{\theta\}$ . Indeed,  $\Psi$  is well defined. We proved that  $\Psi(\mathbf{u}, \xi)$  is a nonempty subset of  $K$ , if  $R_1$  and  $R_2$  in (16) are such that

$$R_1 \geq \frac{\|\mathbf{f}\|_{2,\Omega}}{\mu} \text{ and } R_2 \geq C(\underline{k}, \Omega, p, n)(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \bar{\mu}R_1^2)^{p^2n/[2p^2n-(n-p)(2-p)]}.$$

Uniqueness of  $\mathbf{u}$  and  $\theta$  guarantee that  $\Psi(\mathbf{w}, \xi)$  is a convex set. The closedness is a particular case of the continuous dependence of the solutions.

The upper semicontinuity of  $\Psi$  for the weak topology is also a consequence of continuous dependence of the solutions with respect to the given parameters, due to the closed graph's theorem to multivalued mappings (see [1, p.413]).

So it remains to prove the continuous dependence of the solutions. We consider sequences  $\{(\mathbf{w}_\eta, \xi_\eta)\} \subset K$  and  $\{(\mathbf{u}_\eta, \theta_\eta)\} \subset \Psi(\mathbf{w}_\eta, \xi_\eta)$  such that

$$\mathbf{w}_\eta \rightharpoonup \mathbf{w} \text{ in } V \text{ and } \xi_\eta \rightharpoonup \xi \text{ in } W_p.$$

By a compactness property, we have

$$\mathbf{w}_\eta \rightarrow \mathbf{w} \text{ in } L^r(\Omega) \text{ and } \xi_\eta \rightarrow \xi \text{ in } L^s(\Omega),$$

where  $r < 2n/(n-2)$  and  $s < pn/(n-p)$ .

Since  $\mathbf{u}_\eta$  are the solutions of (17) $_{\mathbf{w}_\eta, \xi_\eta}$ , for all  $\eta \in \mathbb{N}$ , satisfying the estimate (19), we can extract a subsequence such that

$$\mathbf{u}_\eta \rightharpoonup \mathbf{u} \text{ in } V.$$

Then the convective term  $(\mathbf{w}_\eta \cdot \nabla)\mathbf{u}_\eta \cdot \mathbf{v} \rightarrow (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}$  in  $L^1(\Omega)$ , choosing  $r = n < 2n/(n-2)$  if  $n < 4$ .

Taking into account that

$$\mu(\xi_\eta)\nabla\mathbf{v} \rightarrow \mu(\xi)\nabla\mathbf{v} \text{ in } L^2(\Omega),$$

we pass to the limit the problem (17) when  $\eta \rightarrow +\infty$ . Let us now pass to the limit the problem (18), when  $\eta \rightarrow +\infty$ . A compactness result gives  $\mathbf{u}_\eta \rightarrow \mathbf{u}$  in  $L^r(\Omega)$  and a.e. in  $\Omega$ .

Let us verify that

$$\mu(\xi_\eta)|\nabla \mathbf{u}_\eta|^2 \rightarrow \mu(\xi)|\nabla \mathbf{u}|^2 \text{ in } L^1(\Omega).$$

First step, if  $\sqrt{\mu(\xi_\eta)}|\nabla \mathbf{u}_\eta|$  weakly converges to  $\sqrt{\mu(\xi)}|\nabla \mathbf{u}|$  in  $L^2(\Omega)$  then from lower semicontinuity of the  $L^2$ -norm it follows

$$\liminf_{\eta \rightarrow +\infty} \int_{\Omega} \mu(\xi_\eta)|\nabla \mathbf{u}_\eta|^2 dx \geq \int_{\Omega} \mu(\xi)|\nabla \mathbf{u}|^2 dx.$$

Indeed, taking an arbitrary  $\mathbf{v} \in [L^2(\Omega)]^n$ ,  $\sqrt{\mu(\xi_\eta)}\mathbf{v}$  is strongly convergent to  $\sqrt{\mu(\xi)}\mathbf{v}$  in  $[L^2(\Omega)]^n$ , and consequently the weak convergence arises

$$\int_{\Omega} \sqrt{\mu(\xi_\eta)}|\nabla \mathbf{u}_\eta| \mathbf{v} dx \rightarrow \int_{\Omega} \sqrt{\mu(\xi)}|\nabla \mathbf{u}| \mathbf{v} dx.$$

Second step, if we choose  $\mathbf{v} = \mathbf{u}_\eta$  as a test function in (17) and we pass to the limit when  $\eta$  tends to the infinity, it yields

$$\limsup_{\eta \rightarrow +\infty} \int_{\Omega} \mu(\xi_\eta)|\nabla \mathbf{u}_\eta|^2 dx \leq \int_{\Omega} \mathbf{f} \cdot \mathbf{u} dx = \int_{\Omega} \mu(\xi)|\nabla \mathbf{u}|^2 dx.$$

Thus, we conclude that

$$\begin{aligned} \int_{\Omega} \mu(\xi)|\nabla \mathbf{u}|^2 dx &\leq \liminf_{\eta \rightarrow +\infty} \int_{\Omega} \mu(\xi_\eta)|\nabla \mathbf{u}_\eta|^2 dx \leq \limsup_{\eta \rightarrow +\infty} \int_{\Omega} \mu(\xi_\eta)|\nabla \mathbf{u}_\eta|^2 dx \leq \\ &\leq \int_{\Omega} \mu(\xi)|\nabla \mathbf{u}|^2 dx. \end{aligned}$$

From the estimate (20) we extract a subsequence (denoted by the same manner) such that

$$\theta_\eta \rightharpoonup \theta \text{ in } W_p.$$

Then, the convective term  $\theta_\eta \mathbf{u}_\eta \cdot \nabla \phi \rightarrow \theta \mathbf{u} \cdot \nabla \phi$  in  $L^1(\Omega)$ , with  $r = n/(n-1)$ .

From compactness results, it arises

$$\theta_\eta \rightarrow \theta \text{ in } L^s(\Omega), \text{ a.e. in } \Omega \text{ and in } L^t(\Gamma), t < p(n-1)/(n-p), \text{ a.e. in } \Gamma.$$

Consequently, by similar argument used to prove (22),

$$\gamma(\theta_\eta) \rightarrow \gamma(\theta) \text{ in } L^1(\Gamma) \text{ and a.e. in } \Gamma.$$

Finally, we pass to the limit the problem (18), when  $\eta \rightarrow +\infty$ , taking into account that

$$k(\xi_\eta)\nabla \phi \rightarrow k(\xi)\nabla \phi \text{ in } L^{p/(p-1)}(\Omega).$$

Thus, we conclude  $(\mathbf{u}, \theta) \in \Psi(\mathbf{w}, \xi)$ , and the fixed point theorem 1 guarantees the existence of  $(\mathbf{u}, \theta) \in K$  such that  $(\mathbf{u}, \theta) \in \Psi(\mathbf{u}, \theta)$ . This is the desired solution.

□

We observe that it is still an open problem to apply thermal effects to the following types of fluids: fluids of the differential type, of grade  $n$ , of rate type and integral type.

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