

HELICOID OF RIEMANN SOLUTIONS: A NEW MECHANISM FOR MULTIPLICITY OF SOLUTIONS

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Abstract

Quasi-Riemann solutions are modifications of Riemann initial-value problems for conservation laws augmented by a small parabolic term and with continuous initial data which is constant except for a finite number of intervals. In this paper we present a class of examples for which Riemann problems do not contain the complete information required to understand the generic topological behavior of Riemann solutions, while quasi-Riemann problems do. Of course, this fact reflects non standard behavior of the Riemann solutions. In particular, the quasi-Riemann solutions cannot be identified with the solutions of a Riemann problem.

This behavior is best illustrated by examining the set of solutions of quasi-Riemann problems as a manifold. We describe part of this manifold for a particular model of 2×2 system of conservation laws of mixed elliptic–hyperbolic type.

We show this part is foliated by surfaces resembling helicoids. Multiple solutions for Riemann problems associated to these surfaces are obtained by lifting a point representing a constant state to each leaf of the helicoid.

1 Introduction

Nonlinear conservation laws have solutions with shocks, or weak solutions. In the absence of restricting criteria, such weak solutions are not uniquely determined by the initial data. Since Courant, Friedrichs [7] and Gel'fand [10], each individual shock wave has been considered admissible if it is the small viscosity limit of a traveling wave solution of an equation with parabolic terms dictated by the physics.

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However, waves of systems for conservation laws come in sequences, rather than individually. In this paper we investigate an example of a Riemann problem,

$$\begin{cases} U_t + F(U)_x = 0, \\ U(x, 0) = \begin{cases} U_l & \text{for } x \leq 0 \\ U_r & \text{for } x > 0 \end{cases} \end{cases} \quad (1)$$

(U_l and U_r are constant states), where such sequences are small viscosity limits of solutions of an associated parabolic Cauchy problem

$$\begin{cases} U_t + F(U)_x = a[B(U)U_x]_x, \\ U(x, 0) = \begin{cases} U_l & \text{for } x \leq -\epsilon, \\ \bar{U}(x/\epsilon) & \text{for } -\epsilon \leq x < \epsilon, \\ U_r & \text{for } x > \epsilon, \end{cases} \end{cases} \quad (2)$$

where U_l and U_r are the constant states, ϵ is a small positive number, and $\bar{U} : [-1, 1] \rightarrow \mathbf{R}^2$ is a C^2 function which is constant except in certain intervals, centered at x_j . Here $-1 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$ (m is a positive integer). Also, a is a small positive number such that $a \ll \inf_{j=1\dots m}(x_{j+1} - x_j)/2$. We assume that $\bar{U}(x)$ is constant except in the intervals $[x_j - a, x_j + a)$ ($j = 1, \dots, m$) and satisfies $\bar{U}(-1) = U_l$ and $\bar{U}(1) = U_r$. We call (2) a *quasi-Riemann problem*. We would expect that when a and ϵ tend to zero at the same rate, we recover a Riemann problem.

Inspired by the theory of systems of conservation laws, we expect solutions of (2) to comprise sequences of approximate rarefaction waves and approximate traveling waves with non-decreasing speeds from left to right. Contrary to sequences of waves in Riemann solution, we allow adjacent waves to have the same speed and we regard such waves as distinct.

Approximate traveling waves should tend to shock waves in the limit as $a \rightarrow 0^+$. More precisely, admissible shocks are limits of traveling wave solutions for conservation laws that are associated to orbits of families of system of ordinary differential equations. Since in this paper we assume that (1) and (2) are two-component systems, in addition to Lax shock waves of families 1 and 2, for which the traveling wave orbits connect saddle points and nodes, there exist shock waves with saddle-saddle connecting orbits. Although there exist heteroclinic saddle-saddle connections and homoclinic saddle-saddle connections, in this paper we consider only the first type of connection, named *transitional*

waves, for which the left state U_- is a saddle point, the right state $U_+ \neq U_-$ is also a saddle point, and they are connected by an orbit. The existence of traveling waves associated to heteroclinic orbits allows adjacent shock waves to have the same speed. (The classical Lax characteristic criterion precludes this possibility.) Thus there are new wave patterns: sequences of transitional waves with the same speed separated by constant states.

Such sequences cause multiplicity of solutions of Riemann problems for inviscid conservation laws [1, 2]. The occurrence of multiple solutions causes ill-posedness in the following problem: given only the states U_l and U_r , find the time-asymptotic solution of the quasi-Riemann problem. Of course, a solution for the parabolic system is uniquely determined at any finite time by its initial data; even when the solution is not determined solely by U_l and U_r , it is determined by specifying extra information, namely how \bar{U} interpolates between U_l and U_r . On the other hand, in the limit as $a \rightarrow 0^+$ and $\epsilon \rightarrow 0^+$, different choices of \bar{U} can lead to convergence to different solutions of the inviscid Riemann problem.

Although Riemann solutions are obtained composing rarefactions and shocks, in this paper we consider only sequences of a 1-Lax shocks (denoted by S_1), transitional (denoted by T) shocks and a 2-Lax shocks (denoted by S_1) to construct part of the solution manifold of a quasi-Riemann problem. This manifold helps in understanding the behavior of multiple Riemann solutions. This part of the manifold arises from 3-cycles and looks like a foliation of helicoids. This helicoidal foliation is interesting since it represents non standard behavior in the context of conservation laws.

To describe this part of the manifold, after setting up the notation in Section 2, we introduce in Section 3 an example satisfying certain conditions. These conditions are shown to give rise to the helicoid. Classes of solutions playing a role in this paper are discussed in Section 4, and mechanisms responsible for the boundaries between classes are presented in Section 5.

We finish this paper describing the helicoid obtained for a certain set of U_l .

We believe that the information contained in this paper is important for the understanding of existence of oscillatory waves in Riemann problems. It should be interesting to connect it with the structural stability of solutions in the context of [16].

2 Preliminaries

In this paper, we consider problems with compact elliptic region, *i.e.*, we assume that the eigenvalues $\lambda_i(U)$ ($i = 1, 2$) of $F'(U)$ are real and distinct except in a compact region.

Systems of conservation laws with compact elliptic regions illustrate some features of Stone's model [4, 8], which describes the permeability of immiscible three-phase flow in porous media commonly used in Petroleum Reservoir Engineering (the three phases are oil, water and gas with compressibility neglected). The existence of an elliptic region in Stone's model is discussed in [5, 14, 15, 13, 17].

A traveling wave $U(\xi)$ is a solution of the system of ordinary differential equations

$$\dot{U} = B(U)^{-1} \{-s[U - U_-] + F(U) - F(U_-)\} \quad (3)$$

satisfying $\lim_{\xi \rightarrow \pm\infty} U(\xi) = U_{\pm}$. Here $\xi = x - st - x_m$, where s is the propagation speed of the traveling wave. We remark that (3) is a system of ordinary differential equations depending on parameters U_- and s .

Equilibria U_- and U_+ of (3) must satisfy the *Rankine-Hugoniot equation*

$$H(U_-, s, U_+) = F(U_+) - F(U_-) - s(U_+ - U_-) = 0; \quad (4)$$

for each U_- , the set of U_+ satisfying (4) forms the *Rankine-Hugoniot curve* associated to U_- .

Strictly speaking, the equilibria U_- and U_+ are “attained” only in the limits $\lim_{\xi \rightarrow \pm\infty} U(\xi) = U_{\pm}$. However, as the rate of approach is exponentially fast, approximate solutions of the parabolic system can be obtained by juxtaposing traveling waves, as long as the propagation speeds are in nondecreasing order. In this situation, the equilibria serve as constant states appearing in the wave sequence. If the speeds are strictly increasing, we may expect such solutions to be time asymptotically stable. Otherwise, they should be stable only for exponentially long time.

We recall that admissible Lax shock waves of family 1 (S_1) are associated to an orbit leading from a repeller node (or spiral) U_- to a saddle point U_+ , whereas admissible Lax shocks of family 2 (S_2) are associated to an orbit from

a saddle U_- to an attracting node (or spiral) U_+ . The U_+ states of each of these shocks is associated to parts of the Rankine–Hugoniot curve: the 1-Lax shock curves or 1-shock curves and the 2-Lax shock curves or 2-shock curves from U_- .

Transitional shocks are associated to saddle-saddle connections. Let γ be the orbit connecting the two saddles, we represent the α -limit saddle of γ by $\alpha(\gamma)$ and the ω -limit saddle of γ by $\omega(\gamma)$. As in case of 1-shock and 2-shocks, the U_+ states of the transitional shocks are associated to parts of the Rankine–Hugoniot curve: the transitional shock curves.

We remark that a concatenated sequence of three transitional shocks with the same speed gives rise to \mathcal{B} -cycles.

We adopt the same notation $L \xrightarrow{S} R$ to indicate that the state L is connected to the state R by a shock S (S may be a 1-shock (S_1), a 2-shock (S_2), or a transitional shock (T)). To indicate a sequence of waves a , b and c , we use the notation $a : b : c$. We represent the shock speed from a state α to a state β by $s(\alpha, \beta)$.

3 The Model

In this paper we consider an example which defines the flux function $F(U)$ (where $U = (u, v)^T$) introduced in [11] with viscosity matrix equal to the identity. For $\bar{U}(x)$, ϵ and a introduced in the Section 1, we consider the family of quasi-Riemann problems

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} u_t + (\frac{1}{2}(v^2 - u^2) + \rho v)_x = a u_{xx}, \\ v_t + (vu - \rho u)_x = a v_{xx}, \end{array} \right. \\ U(x, 0) = \begin{cases} U_l & \text{for } x \leq -\epsilon, \\ \bar{U}(x/\epsilon) & \text{for } -\epsilon \leq x \leq \epsilon, \\ U_r & \text{for } x \geq \epsilon. \end{cases} \end{array} \right. \quad (5)$$

The Riemann problem associated to (5), first studied in [11],

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} u_t + (\frac{1}{2}(v^2 - u^2) + \rho v)_x = 0, \\ v_t + (vu - \rho u)_x = 0, \end{array} \right. \\ U(x, 0) = \begin{cases} U_l & \text{for } x \leq 0, \\ U_r & \text{for } x \geq 0. \end{cases} \end{array} \right. \quad (6)$$

has elliptic behavior inside the circle $E = \{(u, v) : u^2 + v^2 < \rho^2\}$ and hyperbolic behavior outside E . Thus (6) has elliptic-hyperbolic type, *i.e.*, the

matrix $dF(U)$ has eigenvalues $\lambda_k(U) = (-1)^k \sqrt{u^2 + v^2 - \rho^2}$ ($k = 1, 2$) which are complex conjugate for U inside E , and real and distinct for U outside E .

For this example, a traveling wave is a solution of the system of differential equations

$$\begin{aligned} \dot{u} &= -s(u - u_-) + \frac{1}{2}(v^2 - u^2) + \rho v - \frac{1}{2}(v_-^2 - u_-^2) + \rho v_- \\ \dot{v} &= -s(v - v_-) + vu - \rho u - v_- u_- - \rho u_- \end{aligned} \quad (7)$$

depending on the three parameters u_- , v_- and s .

For some values of these parameters, the systems (7) have three invariant lines which are the secondary bifurcations [12] and form an equilateral triangle [11]. For $s = 0$ and appropriate values of u_- and v_- ((u_-, v_-) lying on the corners of the equilateral triangle or on the center of the elliptic region), the systems (7) have also a 3-cycle. These parameters are important later for (C6).

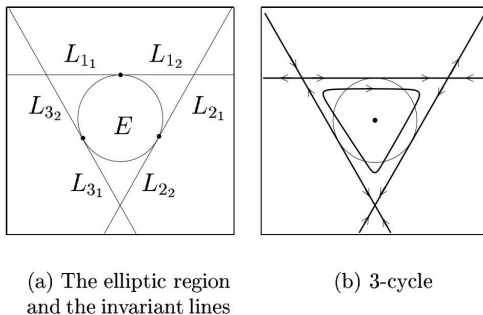


Figure 1: Features of (6) and (7).

Figure 1(a) illustrates the elliptic region and the three invariant lines, while Figure 1(b) shows the 3-cycles, which are sequences of three transitional shocks with the same speed ($s = 0$), occurring in this example.

The parts of Rankine–Hugoniot curve associated to 1-shock curves and 2-shock curves for (6) from different states are illustrated in Figure 2.

Figure 2(a) shows the 1-shock curves from U_a , U_b and from U_c while Figure 2(b) shows the 2-shock curves from states U_1 , U_2 , U_3 , U_4 , U_5 , U_6 and U_7 . These shock curves will be helpful to understand the behavior of the solutions

in next sections. More details about rarefaction and shock curves for (6) can be found in [11].

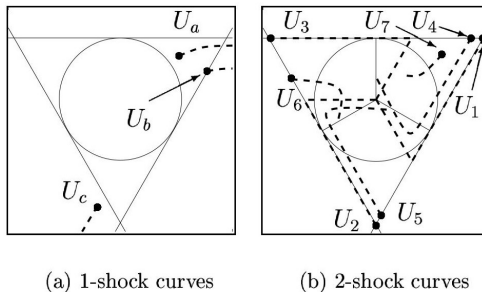


Figure 2: Wave curves of (6) and (7).

Other shock curves from different states are easily obtained, because the Riemann problem (6) satisfies the following symmetry property (see [11]):

Proposition 1 *If $U = U(x, t)$ solves (6) with initial value $U_0 = U(x, 0)$, then also $O_{\pm}U$ will solve (6) with $O_{\pm}U_0 = U(x, 0)$, where O_{\pm} denotes a rotation by $\pm\frac{2}{3}\pi$,*

The Riemann problem (6) and the systems (7) satisfy certain conditions which are associated to non standard behavior of Riemann solutions. We describe below part of the conditions which are relevant in this paper. The complete list of conditions will appear in [3].

- (C1) The Riemann problem, in the state space, has a compact elliptic region E with C^1 closed boundary, *i.e.*, the matrix $dF(U)$ has complex conjugate eigenvalues $\lambda_i(U)$ ($i = 1, 2$) if U lies inside E ; outside E $dF(U)$ has real and distinct eigenvalues.
- (C2) The Riemann solutions contain only 1-Lax shocks, 2-Lax shocks and transitional shocks.

- (C3) Each member of the family of systems of ordinary differential equations (7) has at most four singularities; when the system has four singularities, they must be three saddles and one non-saddle.
- (C4) There exist two types of orbits connecting two saddles of the family of ordinary differential equations (7): straight lines and curves. We adopt the notation T_s for transitional shocks associated with straight line orbits and T_c for transitional shocks associated with curved orbits.
- (C5) The phase portraits of certain members of the family of systems of ordinary differential equations (7) has three invariant straight lines L_i ($i = 1, 2, 3$), forming a triangle Δ , such that whenever two saddles lie on L_i and are connected by orbits, then at least one orbit γ lies on L_i . Each line is tangent to the border of the elliptic region at exactly one point and there is no other intersection. Each line L_i is composed by two parts: the α -limit saddle of each γ lies in the first part L_{i_α} and the ω -limit saddle of each γ lies in the second part L_{i_ω} . The tangent point of L_i and E is the boundary between L_{i_α} and L_{i_ω} . Figure 1(a) illustrates this condition.
- (C6) For $s = 0$ and appropriate values of the parameter U_- , the system of differential equations (7) has 3-cycles. The 3-cycle is formed by straight line orbits and its corners coincide with the corners of Δ (so Δ “coincides” with the 3-cycle). Moreover, shocks connecting corners of a 3-cycle have zero speed. (Actually, one can consider s constant instead $s = 0$, by adding sU to F in (2); we adopt $s = 0$ for simplicity.)
- (C7) From appropriate states, the 1-Lax shock curves intersect transversely the part L_{i_α} of the invariant line L_i and the 2-Lax shock curves intersect transversely the part L_{i_ω} of the invariant line L_i (see Figures 2).
- (C8) Let $S_1^{-1}(L_{i_\alpha})$ be the set of all U_l such that the 1-shock curve from U_l intersects L_{i_α} and let $S_2(L_{i_\omega})$ be the set of all U_r such that the backward 2-wave curve passing through U_r ($S_2^{-1}(U_r)$) intersects L_{i_ω} . These two sets are not empty.
- (C9) In a sequence of more than two transitional shocks, all such shocks have the same speed s , except possibly for the first and the last shocks. Again,

we adopt $s = 0$ for simplicity.

- (C10) Fix a state in each L_{i_α} ($i = 1, 2, 3$) that is a saddle. By varying s , there is an interval I_i contained in L_{i_ω} such that every saddle in I_i is connected to the saddle in L_{i_α} by a straight line connection.
- (C11) Except in bifurcation cases, the length of the orbit connecting two states U_- and U_+ depends continuously on U_- and U_+ .
- (C12) A straight line or curved connection from a first saddle to a second saddle can disappear through the non-saddle equilibrium collapsing with first saddle, or through the non-saddle equilibrium collapsing with the second saddle. In the context of conservation laws, these collapses occur on the boundary between branches of the Rankine–Hugoniot curve. The first case corresponds to the boundary between a transitional part and a 1-Lax shock part in a Rankine-Hugoniot curve, while the second case corresponds to the boundary between a transitional part and a 1-Lax shock part in a Rankine-Hugoniot curve.

The condition (C10) is not generic and, of course, the values of the parameter s (in (3)) are not the same for different saddle-saddle connections. In example (5), condition (C10) is valid because the viscous matrix $B(U)$ is equal to the identity.

As consequence of condition (C6), the corners of the 3-cycle (and the corners of the triangle formed by the lines L_i) lie on each interval I_i mentioned in (C10).

4 Classes of Solutions

In this section we introduce the concept of *classes of solutions* to group solutions. This concept is useful to describe the structural behavior of Riemann and quasi-Riemann solutions.

A *Class* corresponds to a superset of structurally stable solutions as defined in [16].

In the example described in this paper, two classes differ through the number or type of transitional shocks present.

For simplicity, we present only classes containing shocks and that are relevant to constructing the part of solution manifold described in this paper. (This is the cause of the gap in the notation of the classes. The whole set of classes for this example is presented in [3].)

As consequence of (C2), all classes presented in this paper are composed by solutions containing a 1-shock and a 2-shock or by a 1-shock, transitional shocks, and a 2-shock. The shocks are separated by constant states and the classes may contain degenerate cases where the 1-shock or the 2-shock vanish.

Class	Description of the quasi-Riemann solution classes
V	$S_1 : T_s : T_s : T_s : S_2.$
VI	$S_1 : T_s : T_s : T_s : T_s : S_2.$
XX_n	$S_1 : 3C_n : T_s : S_2.$
XXI_n	$S_1 : 3C_n : T_s : T_s : S_2.$
$XXII_n$	$S_1 : 3C_n : T_s : T_s : T_s : S_2.$
$XXIII_n$	$S_1 : T_s : 3C_n : T_s : S_2.$
$XXIV_n$	$S_1 : T_s : 3C_n : T_s : T_s : S_2.$
XXV_n	$S_1 : T_s : 3C_n : T_s : T_s : T_s : S_2.$

Table 1: Classes of solutions.

We summarize these classes in Table 1, where $3C_n$ denotes n 3-cycle loops ($n = 1, 2, \dots$).

The classes presented in Table 1 are quasi-Riemann solutions but some of them do not survive when $a \rightarrow 0^+$ and $\epsilon \rightarrow 0^+$ in (2). Classes between I and IX are associated to Riemann solutions (*Riemann solution classes*) while the other classes are quasi-Riemann solution classes but not Riemann solution classes.

5 Boundaries Between Classes of Solutions

Classes are associated to parts of the solution manifold. This association produces the idea of boundary between classes. The boundaries are submanifolds with higher codimension. They occur through certain mechanisms that follow from the results presented in previous sections. For classes containing sequences of transitional shocks, a typical mechanism responsible for appearance of boundaries is the vanishing of the first or the last shock speed, or of both

speeds. Submanifolds associated to these cases have higher codimension. Moreover, submanifolds associated to the latter case have higher codimension than those associated to the other two cases.

Although classes and boundaries are manifolds, we emphasize the role of some boundaries adopting a notation different from that used to classes. Moreover, we use $B(\Lambda, \Gamma)$ to denote the boundary between classes Λ and Γ .

An example of such boundaries occurs between Class V and Class VI. This boundary is a manifold associated to solutions formed by $S_1 : T_s : T_s : T_s : S_2$ such that the last T_s has zero speed. This boundary is a codimension 1 manifold and denoted by $B(V, VI)$.

Another interesting boundary occurs between Class VI and Class XXIII₁. This manifold is associated to sequences $S_1 : T_s : T_s : T_s : T_s : S_2$ such that only the first T_s has non zero speed. Since three zero-speed T_s originate a 3-cycle, this boundary is associated to solutions formed by $S_1 : T_s : 3C_1 : S_2$. This boundary is also a codimension 1 manifold and we denote it $B(VI, XXIII_1)$.

An example of a codimension 2 manifold is associated to solutions formed by $S_1 : 3C_1 : S_2$. In this case, all T_s have zero speed.

Boundary	Description of the quasi-Riemann solution boundary
$B_1(V, VI)$	$S_1 : T_s(\text{with } s = 0) : T_s : T_s : S_2.$
$B_2(V, VI)$	$S_1 : T_s : T_s : T_s(\text{with } s = 0) : S_2.$
$B(VI, XXIII_1)$	$S_1 : T_s : T_s : T_s : T_s(\text{with } s = 0) : S_2.$
$B(XX_n, XXI_n)$	$S_1 : 3C_n : T_s(\text{with } s = 0) : S_2.$
$B(XXI_n, XXII_n)$	$S_1 : 3C_n : T_s : T_s(\text{with } s = 0) : S_2.$
$B(XXIII_n, XXIV_n)$	$S_1 : T_s : 3C_n : T_s(\text{with } s = 0) : S_2.$
$B(XXIV_n, XXV_n)$	$S_1 : T_s : 3C_n : T_s : T_s(\text{with } s = 0) : S_2.$
$B(XXV_n, XXIII_{n+1})$	$S_1 : T_s : 3C_n : T_s : T_s : T_s(\text{with } s = 0) : S_2.$

Table 2: Boundaries between classes of solutions.

Table 2 describes some boundaries between classes. The focus in this description is the boundaries used in this paper.

In this section we also describe how classes and boundaries used in this paper are connected.

We begin the description with a Riemann solution lying in a certain class; by changing U_l and U_r , the solution can change into another class due to a mecha-

nism allowed by conditions (C1)–(C12). These classes differ in the number and type of transitional shocks (3-cycle loops are sequences of transitional shocks). So, appearance (or disappearance) of transitional shocks and change of type of transitional shocks in a sequence of waves are the mechanisms responsible for the existence of boundaries between classes.

The two mechanisms playing a role in this paper are described below. A complete list of the mechanisms in the model considered here will appear in [3].

In the description below, reverse changes are also possible.

- (M1) A sequence of waves formed by a 1-shock followed by a transitional shock *i.e.*, $S_1 : T$ (or $U_l \xrightarrow{S_1} U_{m_1} \xrightarrow{T} U_{m_2}$), changes into a sequence with only a 1-shock S_1 (or $U_l' \xrightarrow{S_1} U_{m_2}'$). The change occurs when the intermediate state U_{m_2} crosses the boundary between the branch of transitional shocks and the branch of 1-Lax shocks in a Rankine–Hugoniot curve from U_{m_1} .

We adopt the notation (M1–) to represent the case when a T_s disappears and the notation (M1+) to represent the case when a T_s appears. This mechanism follows from (C12).

- (M2) A sequence of waves described by a transitional shock followed by a 2-shock, *i.e.*, $T : S_2$ (or $U_{m_1} \xrightarrow{T} U_{m_2} \xrightarrow{S_2} U_r$), changes into a sequence with only a 2-shock S_2 (or $U_{m_2}' \xrightarrow{S_2} U_r'$). The change occurs when the intermediate state U_{m_2} crosses the boundary between the branch of transitional shocks and the branch of 2-Lax shocks in the Rankine–Hugoniot curve from U_{m_1} . This change can occur, for example in a Riemann solution for (6), when U_r crosses the horizontal part of the 2-shock curve from U_2 . This mechanism follows from (C12). As in (M1), we use (M2–) to represent the case in which a T_s disappears and (M2+) to represent the case which a T_s appears.

The boundaries between classes occur through one or more mechanisms. Generically, two or more mechanisms appear when one changes U_l and U_r simultaneously. In this paper we consider boundaries originating from only one mechanism. In Table 3 we introduce the boundaries that play a role in this paper and specify the mechanism responsible for the connection.

Class	Boundary Connected	Mechanisms
V	$B_1(V, VI)$	(M1+)
	$B_2(V, VI)$	(M2+)
VI	$B_1(V, VI)$	(M1-)
	$B_2(V, VI)$	(M2-)
	$B(VI, XXIII_1)$	(M2+)
XX_n	$B(XX_n, XXI_n)$	(M2+)
	$B(XX_n, XXIII_n)$	(M1+)
XXI_n	$B(XX_n, XXI_n)$	(M2-)
	$B(XXI_n, XXII_n)$	(M2+)
$XXII_n$	$B(XXI_n, XXII_n)$	(M2-)
	$B(XXII_n, XXIII_n)$	(M2+)
$XXIII_n$	$B(VI, XXIII_1), (n = 1)$	(M2-)
	$B(XXIII_n, XXIV_n)$	(M2+)
	$B(XXIII_n, XXV_{n-1}), (n > 1)$	(M2-)
$XXIV_n$	$B(XXIII_n, XXIV_n)$	(M2+)
	$B(XXIV_n, XXV_n)$	(M2+)
XXV_n	$B(XXIV_n, XXV_n)$	(M2-)
	$B(XXV_n, XXIII_{n+1})$	(M2+)

Table 3: Connections of classes of solutions and their boundaries.

6 The Helicoid and the Multiplicity of Solutions

In this section we describe parts of the quasi-Riemann solution manifold problem for system (7), which looks like helicoids. These helicoids are based on quasi-Riemann solutions containing 3-cycles. Each leaf corresponds to a U_l . So, to describe one of these helicoids, we fix U_l and let U_r vary.

We begin considering U_l and U_r inside the triangle \triangle formed by the invariant lines L_i ($i = 1, 2, 3$) such that the 1-shock curve $S_1(U_l)$ and the 2-shock curve $S_2^{-1}(U_r)$ which reaches U_r cross at a state U^* (Figures 2 illustrate such case, for U_l equal U_a , U^* equal U_7 lying on $S_1(U_a)$, and U_r any state lying on $S_2(U_7)$ and inside \triangle) and, besides this case, we take U_l and U_r far away from the center of E . We introduce the latter requirement because we have numerical evidence indicating that for U_l or U_r close to the center the solution of the Cauchy problem has asymptotically unstable behavior. This instability is probably

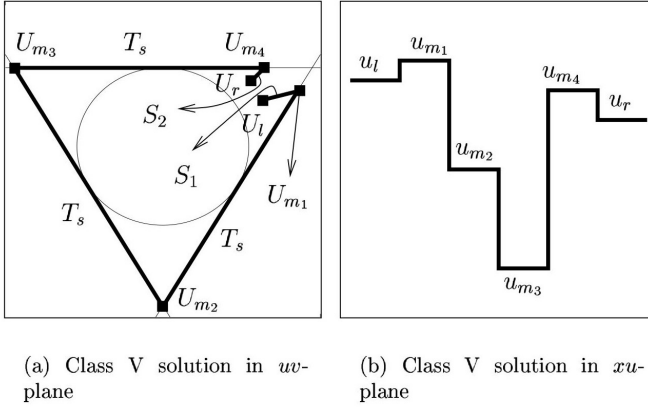


Figure 3: Class V solutions.

related to oscillations studied in [6, 9].

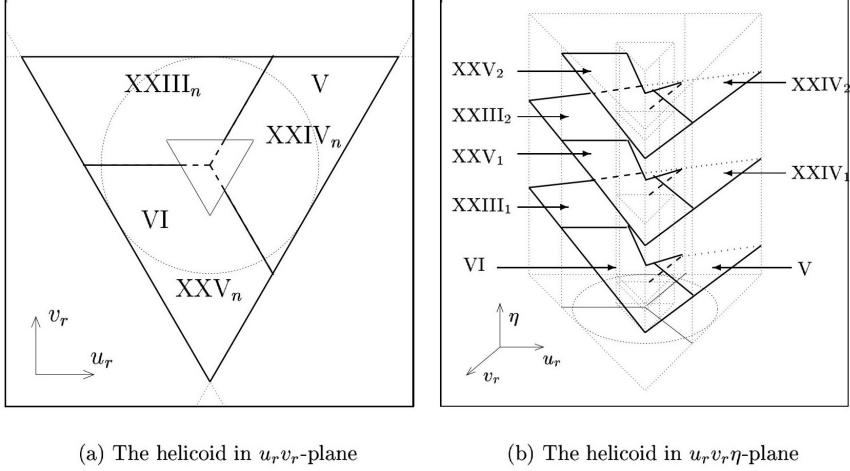
For the initial data mentioned above, because of condition (C7), the shock curves cross the invariant lines so that one can construct a solution lying in Class V, namely $U_l \xrightarrow{S_1} U_{m_1} \xrightarrow{T_s} U_{m_2} \xrightarrow{T_s} U_{m_3} \xrightarrow{T_s} U_{m_4} \xrightarrow{S_2} U_r$. Because of condition (C9), the second T_s has zero speed. Fig. 3(a) shows the solution in uv -plane while Fig. 3(b) shows the solution in xu -plane.

Now we move U_r inside \triangle in the clockwise direction, keeping far away from the center. In this case the intermediate state U_{m_4} moves to the upper right corner of the triangle and the solutions lie in Class V until U_{m_4} reaches the corner. In this case, the solution leaves Class V and enters in $B_2(V, VI)$. Figure 2(b) illustrates it moving U_4 to U_1 or U_5 to U_2 or U_6 to U_3 . After that, moving U_r , the solution enters in Class VI.

Keeping U_r moving, still because of (M2+), the solution leaves Class VI and enters a Class given by $S_1 : T_s : T_s : T_s : T_s : S_2$. By (C6) and (C9), this is a Class XXIII₁ solution. Of course, between Class VI and Class XXIII₁, the solution crosses $B(VI, XXIII_1)$.

Moving U_r the solution changes to Class XXIV₁, then to Class XXV₁, then to Class XXIII₂, and so on crossing the appropriated boundaries.

To represent geometrically the topological behavior of the solution, we in-


 (a) The helicoid in $u_r v_r$ -plane

 (b) The helicoid in $u_r v_r \eta$ -plane

 Figure 4: The helicoid for a fixed U_l .

introduce a parameter η . In state space, it is given by the sum of arc length of the 1-shock curve from U_l to the first intermediate state, the length of the orbits connecting the saddles of the transitional shocks, and the length of the 2-shock curve from the last intermediate state to U_r . The (u_r, v_r, η) -space turns out to be extremely useful to describe parts of the solution manifold.

Figure 4 sketches the helicoid described above, for U_l given previously. In Figure 4(a) we show the projection of the helicoid in $u_r v_r$ -plane while in Figure 4(b) we illustrate it in u_r, v_r, η -space.

Of course, the helicoid are associated with multiple Riemann solutions and we believe they are related to oscillatory waves introduced in [9].

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