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FROBENIUS GROUPS AND THE ISOMORPHISM PROBLEM

Thierry Petit Lobão

Abstract

We investigate the *isomorphism problem* in Frobenius groups. We prove that the internal structure of such groups is determined in detail by their integral group rings. Besides, we obtain positive solutions to said problem under not very restrictive additional conditions.

Resumo

Investigamos o *problema do isomorfismo* com respeito aos grupos de Frobenius. Provamos que a estrutura interna destes grupos é determinada pormenorizadamente por seus anéis de grupo integrais; ademais, obtivemos soluções positivas do citado problema sob condições adicionais não muito restritivas.

1. Introduction

A central question in theory of group rings is the, so called, *Isomorphism Problem.* This question is already present in G. Higman's doctoral dissertation (1940) and it was made public by R. M. Thrall in the Algebra Meeting at Michigan (1947), who states it in the following terms: *Given a group G and a field $\mathbf{K}$, determine all groups H such that $\mathbf{K}G \simeq \mathbf{K}H$.* The development of the research in the area led to realize that advantageous results may be obtained with *integral group rings of finite groups*, that is to say, those group rings with finite basis whose ring of coeficients were $\mathbb{Z}$. So a conjecture arised, which is nowadays known as the isomorphism problem:

(Iso) If G is a finite group and $\mathbb{Z}$ is the ring of the integers, then:

$$\mathbb{Z}G \simeq \mathbb{Z}G_1 \Longrightarrow G \simeq G_1$$

Solutions to (Iso) were obtained for several classes of groups. The works of G. Higman and S. D. Berman, for instance, showed that abelian groups are determined by their integral group rings. A. Whitcomb also obtained (Iso) for metabelian groups. In a joint work, K. Roggenkamp and L. L. Scott proved (Iso) for nilpotent groups. According to R. Sandling, the class of circle groups is a solution to (Iso). The same is true for Hamiltonian and for 2-Hamiltonian groups, dihedral and symmetric groups, some linear groups (including special linear groups), simple and semisimple groups, and many others. Moreover, it was proved that the isomorphism $\mathbb{Z}G \simeq \mathbb{Z}G_1$ implies a strong similarity between G and G_1. So, for example, their orders are the same, their centres are isomorphic. There is a correspondence between their conjugacy classes which imposes an isomorphism of their lattices of normal subgroups, it also implies that the groups have the same character table. Furthermore, many properties, like commutativity, nilpotency and solubility, among others, are shared by G and G_1 (see [9]).

Several other conjectures associated to the isomorphism problem were raised, among these, the conjectures of A. A. Bovdi and H. Zassenhaus. As regards these conjectures, some interesting solutions were also obtained to many classes of groups.

Frobenius groups constitute a class of fundamental importance in theory of finite groups and several basic problems in this realm stem from this class of groups. Recently, M. Dokuchaev, S. O. Juriaans and C. Polcino Milies undertook a profound investigation on the integral group rings of Frobenius groups (see [1], [2], [5], [6]). They obtained solutions to Bovdi's and Zassenhaus' conjectures or their versions. Concerning these groups, we also obtain a complete solution of the normalizer conjecture, *i. e.*, we prove this conjecture is valid for all Frobenius groups. The *normalizer conjecture* is a very important surmise about the normalizer of G in the group of units of $\mathbb{Z}G$. It was realized recently that there is a curious link between this conjecture and the isomorphism problem.

In this work, we present an outline of our research on the isomorphism problem relative to Frobenius groups (complete proofs will be published elsewhere). We prove that the internal structure of a Frobenius group is strongly determined by its integral group rings. We also obtain necessary and sufficient conditions for the isomorphism problem.

2. The group basis and the Frobenius kernels

A Frobenius group is a finite transitive permutation group which is not regular, *i. e.*, the subgroup fixing a letter is non trivial, but only the identity element fixes more than one letter. The structure of these groups is very peculiar: if G is a Frobenius group, then the set

$$K = \{k \in G : k = 1 \text{ } or \text{ } fix(k) = \emptyset\},$$

consisting of the identity and the elements of G not in any point stabilizer, called the Frobenius kernel, is a Hall subgroup normal in G. Thus $G = K \rtimes H$, to some complement H. It is also verified that the action of H over K is fixed point free. The properties of Frobenius groups may be consulted in Huppert [4] and Passman [7].

Our next results assert that a group basis of $\mathbb{Z}G$, G a Frobenius group, is very peculiar. The following lemma, due to Juriaans and Polcino Milies (see [6]), will be essential.

Lemma 1. *Let G be a Frobnenius group with kernel K and a complement H. If u is a normalized unit of finite order in $\mathbb{Z}G$, then the order of u divides either the order of K or that of H.*

Our first result follows.

Theorem 1. *Let G be a Frobenius group; if $\mathbb{Z}G \simeq \mathbb{Z}G_1$ is an isomorphism, then G_1 is also a Frobenius group.*

Proof. Let $G = K \rtimes H$; by group ring techniques, we prove that G_1 admits a decomposition like that: $G_1 = K_1 \rtimes H_1$. Using the previous lemma, we obtain that the action of H_1 over K_1 is fixed point free, thus G_1 is a Frobenius group.

□

In relation to the kernels of the Frobenius groups G and G_1, in the former isomorphism, we also have:

Theorem 2. *Let G be a Frobenius group and $\mathbb{Z}G \simeq \mathbb{Z}G_1$. Then the Frobenius kernels of G and G_1 are isomorphic.*

Proof. In accordance with Theorem 1, $G_1 = K_1 \rtimes H_1$ is a Frobenius group. Applying a proposition due to Giambruno and Sehgal (see [3]), we proved that the Sylow p-subgroups of K and K_1 are isomorphic to all prime p. The theorem follows from the Thompson's result on Frobenius kernels.

□

3. The Frobenius complements

The results in previuos section imply that, if G is a Frobenius group, from the isomorphism $\mathbb{Z}G \simeq \mathbb{Z}G_1$, we also obtain, by group ring theory, an isomorphism of the integral group rings of the complements: $\mathbb{Z}H \simeq \mathbb{Z}H_1$. In this section, we shall prove that a Frobenius complement is a solution to the isomorphism problem.

The structure of a Frobenius complement is strongly related to its Sylow 2-subgroups, which may be cyclic or quaternion groups. Our next results will explore this peculiar structure. Before anything else, we shall need some results about Z-groups. These groups are characterized by the fact that all their Sylow subgroups are cyclic. Zassenhaus proved that such groups are either cyclic or special kind of metacyclic groups (see [11]). Concerning these groups we prove the following essential lemma:

Lemma 2. *Let G be any finite group and $\mathbb{Z}G \simeq \mathbb{Z}G_1$ an isomorphism. If N is a Z-group normal in G, then N is isomorphic to M, the normal subgroup of G_1 associated to N in the correspondence of normal subgroups.*

Now, we claim:

Theorem 3. *If H is a Frobenius complement of odd order, then H satisfies the isomorphism problem.*

Proof. The proof is a straightforward application of the Burnside's result about Frobenius complements (see [8]), which states that, in this case, H is a Z-group. Thus the theorem follows from either the Higman-Berman's or Whitcomb's results (see [9]).

□

Theorem 3 admits an obvious generalization.

Theorem 4. *If H is a Frobenius complement whose Sylow 2-subgroups are cyclic, then H satisfies the isomorphism problem.*

Proof. Analogous to the previuos one, since, in this case, by the same mentioned results, H is a Z-group as well.

□

When the Sylow 2-subgroups of a Frobenius complement are quaternion groups, then the structure of such complement depends on its solubility. First we deal with nonsoluble complements.

Theorem 5. *If H is a nonsoluble Frobenius complement, then H satisfies the isomorphism problem.*

Proof. According to Zassenhaus, in a non soluble Frobenius complement H (see [11]), there exists a normal subgroup H_0 such that

$$H_0 = M \times SL(2,5) \quad \text{and} \quad [H, H_0] = 1 \text{ or } 2,$$

with M a Z-group and $(|M|, 120) = 1$. In the first case, it is not difficult to see that the groups are both determined by their integral group rings, so the result follows. Thus it is sufficient to deal with the case $[H, H_0] = 2$. We may assume $\mathbb{Z}H \simeq \mathbb{Z}H_1$ and $H = M \rtimes \langle SL(2,5), v\rangle$ with $v^2 \in SL(2,5)$. The group $\langle SL(2,5), v\rangle$ is also determined by its integral group ring (see [11]). In accordance with the previous lemma, it follows that H_1 admits an analogous decomposition: $H_1 \simeq M \rtimes' \langle SL(2,5), w\rangle$. Since $\langle M, v\rangle$ is a Z-group, it is determined by its integral group ring. Using group ring techniques, we prove that H and H_1 have the same presentations and, since their orders are equal, the theorem follows.

□

Our next theorem deals with the remaining complements, namely, the soluble Frobenius complements whose Sylow 2-subgroups are quaternion.

Theorem 6. *If H is a soluble Frobenius complement, then H satisfies the isomorphism problem.*

Proof. We have just mentioned that the Sylow 2-subgroups of a Frobenius complement H are cyclic or quaternion. In the first case, H is a Z-group (see [8]), thus H is determined by its integral group ring. Therefore, it remains only the case in which the said subgroups are quaternion. We assume $\mathbb{Z}H \simeq \mathbb{Z}H_1$. The proof is very long and it was divided into several particular cases, however, the cornerstone was the investigation of the centralizer of F_2 in H, $C_H(F_2)$, where F_2 is the Sylow 2-subgroup of the Fitting subgroup of H. We were able to prove that, in all cases, H is an extension of F_2 by a Z-group, and then we also obtained that H and H_1 have the same presentations. Since H and H_1 have the same order, the theorem follows.

□

4. Necessary and sufficient conditions

In accordance with the previous sections, if $G = K \rtimes H$ is a Frobenius group and $\mathbb{Z}G \simeq \mathbb{Z}G_1$, then $G_1 = K_1 \rtimes H_1$ is also a Frobenius group; moreover $K \simeq K_1$ and $H \simeq H_1$. It is important to observe that we also proved that kernels and complements of Frobenius groups are determined by their integral group rings: in fact, the kernels are nilpotent by Thompson's Theorem; as for the complements, the assertion follows from theorems 5 and 6. However, these facts do not assure that $G \simeq G_1$, because the actions of the complements over the kernels may be very different! Our next results will prove that some necessary conditions to the isomorphism $G \simeq G_1$ are also verified. Futhermore, in some specific cases, we obtained solutions to the isomorphism problem.

It is a known fact that, if $G = K \rtimes H$ is a Frobenius group, then the kernel K is the Fitting subgroup of G. Thus, if K_p is a non trivial Sylow p-subgroup of K, then the subgroup $K_p \rtimes H$ is also a Frobenius group. We shall call these subgroups the *p-components* of G. Suppose that G is a Frobenius group; obviously, a necessary condition to the isomorphism $G \simeq G_1$ is that the p-components of G and G_1 must be isomorphic. We proved that $\mathbb{Z}G \simeq \mathbb{Z}G_1$ may imply isomorphism between the p-components of G and G_1.

Theorem 7. *Let $G = K \rtimes H$ be a Frobenius group and $\mathbb{Z}G \simeq \mathbb{Z}G_1$. Suppose p is a prime that divides the order of K. If K_p is cyclic or $K_p' \neq \Phi(K_p)$, then the p-components of G and G_1 are isomorphic.*

Proof. The isomorphism $\mathbb{Z}G \simeq \mathbb{Z}G_1$ implies $\mathbb{Z}(K_p \rtimes H) \simeq \mathbb{Z}((K_1)_p \rtimes H_1)$. If K_p is cyclic, we prove that $K_p \rtimes H$ is a metabelian group and the result follows. In the remaining case the proof rests on a result by Giambruno and Sehgal. They obtain an isomorphism $\lambda_p : K_p \to (K_1)_p$ that maps the conjugacy classes of G, contained in K_p, into conjugacy classes (see [3]). We are able to extend this isomorphism to the whole p-component, provided K_p has at least two elements of different order, but this fact is assured by condition $K'_p \neq \Phi(K_p)$.

□

We note that the condition $K'_p \neq \Phi(K_p)$ in Theorem 7 is not too restrictive, since we also obtain the same result in some cases of special p-groups.

The last theorem states that a necessary condition to the isomorphism $G \simeq G_1$ is verified. We also obtained sufficient conditions in certain cases.

Theorem 8. *A Frobenius group whose kernel is cyclic is a solution to the isomorphism problem.*

Proof. If the kernel K is a cyclic group, then its automorphism group, $Aut(K)$, is an abelian group. It is not difficult to see that a complement H of a Frobenius group has a homomorphic image in $Aut(K)$, so the Frobenius group is metabelian and the theorem follows.

□

Theorem 9. *Let G be a Frobenius group whose kernel is abelian. If a Sylow 2-subgroup of a complement is cyclic, then G satisfies the isomorphism problem.*

Proof. The proof rests on a result, due to Sudarshan K. Sehgal, Surinder K. Sehgal and H. Zassenhaus (see [10]), which asserts that, if the Frobenius kernel is abelian and a complement satisfies the so called second Zassenhaus conjecture, (ZC-2), then the Frobenius group is determined by its integral group ring. In our specific case, the complement is a Z-group, so the conjecture (ZC-2) is valid and the theorem follows.

□

We note that a result due to Burnside asserts that, if a complement has even order, then the kernel is abelian. Therefore, the last result always holds if a cyclic Sylow 2-subgroup of a complement is nontrivial. In fact, we obtained several other solutions to the isomorphism problem by proving the conjecture (ZC-2) to even order complements.

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Departamento de Ciências Exatas
Universidade Estadual de Feira de Santana
44031-460 Feira de Santana-Ba
E-mail: thierry@uefs.br