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SOME OPEN PROBLEMS IN INFINITE GROUP THEORY

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Abstract

This paper presents a collection of open problems in infinite group theory. These are in the following main areas: group theoretical questions on discrete subgroups of $PSL(2,\mathbb{C})$, one-relator products and groups of F-type, one-relator groups, discrimination and separation properties and other residual properties, test elements and APE's in general groups and test ranks. Included with the problems is a discussion of the background and known results.

1. Introduction

The emergence of geometric group theory, in particular the studies of hyperbolic groups and automatic groups , and the development of algebraic geometry over groups by G.Baumslag, A. Myasnikov and V. Remeslennikov, has led to a renewed interest in many of the open problems in infinite group theory. As well as new problems arising from these developing theories there have been attempts to look at classical problems, like the Tarski problem, in light of these modern techniques. In this paper we present and explain a collection of open problems in infinite group theory. A large list of such problems has been collected by the New York Group Theory Cooperative and is available at http://zebra.sci.ccny.cuny.edu/web/. Many of these were published in Contemporary Mathematics Vol. 250 by G. Baumslag, A. Myasnikov and V. Shpilrain. The present paper can be considered a continuation of this list although representing the particular interest of the authors. In particular we consider group theoretical questions on discrete subgroups of $PSL(2,\mathbb{C})$, residual properties of one-relator groups, test elements and APE's in general groups, test ranks, groups of F-type and other one-relator products, nilpotent groups and discrimination and separation properties. Included will be a discussion of the background and known results on the problems.

2. Discrete Groups, One-Relator Products and Groups of F-type

The theory of discrete subgroups of $PSL_2(\mathbb{C})$ and the interplay of these subgroups with combinatorial group theory has always been of central interest in infinite group theory. Historically much of the early beginnings of combinatorial group theory can be traced to methods to handle the discrete infinite groups arising from topology and complex analysis. See the book [F-R] for a more complete discussion of this.

Recall that $PSL_2(\mathbb{C})$ is the group of linear fractional transformations

$$z' = \frac{az+b}{cz+d} \text{ with } ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{C}.$$

These can also be considered as projective matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{C}.$$

A subgroup $G \subset PSL_2(\mathbb{C})$ is **discrete** if G contains no sequence of non-trivial elements

$$T_n = \pm \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \; T_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$$

which converges to the identity $I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ elementwise. As examples of discrete groups we have the Modular group $PSL_2(\mathbb{Z})$ and $PSL_2(A)$ where A is any discretely normed subring of $\mathbb{C}$ (see [F]).

If G is a non-elementary discrete subgroup of $PSL_2(\mathbb{R})$ then G is called a **Fuchsian group**. A finitely generated Fuchsian group G has a finite presentation, called the **Poincare presentation**, of the form

$$G =< e_1, .., e_p, h_1, .., h_t, a_1, b_1, ..., a_g, b_g; e_i^{m_i} = 1, i = 1, .., p, R = 1 >$$

where $R = e_1..e_ph_1..h_t[a_1, b_1]...[a_g, b_g]$ and $p \geq 0, t \geq 0, g \geq 0, p + t + g > 0$, and $m_i \geq 2$ for $i = 1, ..., p$. The **Euler Characteristic** of G is given by $\chi(G) = -\mu(G)$ where

$$\mu(G) = 2g - 2 + t + \sum_{i=1}^{p}(1 - 1/m_i).$$

An **F-group** G is any group with a Poincare presentation. An F- group G with $\mu(G) > 0$ has a faithful representation as a Fuchsian group and $2\pi\mu(G)$ represents the hyperbolic area of a fundamental polygon for G. The book by Fine and Rosenberger [F-R] contains a comprehensive description of the ties between discrete groups and combinatorial group theory.

Question DG1. *Given a finitely generated discrete subgroup of* $PSL_2(\mathbb{C})$ *when is it finitely presented. In particular when is a 2-generator discrete subgroup of* $PSL_2(\mathbb{C})$ *finitely presented? More generally when is a 2-generator non-elementary subgroup of* $PSL_2(\mathbb{C})$ *or* $PSL_2(\mathbb{R})$ *finitely presented?*

Note that this is answered via Poincare presentations for Fuchsian groups. It is also answered if it is known that the group is **geometrically finite**, that is, it has a fundamental domain of finite hyperbolic area. The Poincare polyhedron theorem (see [F-R]) can be used then to determine a finite presentation for G. A computer program called **Poincare**, developed by R.Riley [Ri], uses the Poincare polyhedron theorem to determine a finite presentation for a geometrically finite discrete subgroup of $PSL_2(\mathbb{C})$ given by a finite generating set (see [F] for a description of the Riley program).

Question DG2. *Given a finitely generated discrete subgroup of* $PSL_2(\mathbb{C})$ *when is it hyperbolic as a group? In particular when is a 2-generator discrete subgroup hyperbolic?*

We note that a Fuchsian group is hyperbolic as a group. It is known that hyperbolic groups do not contain subgroups of the form $\mathbb{Z} \times \mathbb{Z}$. Question DG2 can be modified to ask whether a finitely generated subgroup of $PSL_2(\mathbb{C})$ whose maximal parabolic subgroups are cyclic must be hyperbolic as a group.

Question DG3. *Given a subgroup of $PSL_2(\mathbb{C})$ which provides a faithful representation of a Fuchsian group (in particular a surface group), determine conditions when the image group must be discrete.*

Particular examples of discrete representations of triangle groups have been constructed (see the discussion after question ORP9).

Recall that the **Bianchi Groups** are $\Gamma_d = PSL_2(O_d)$ where O_d is the ring of integers in the quadratic imaginary number field $Q(\sqrt{-d}), d$ a square-free positive integer. These are all discrete. (see [F]). For $d \neq 3$ it was proved by Fine and Frohmann [F-Fr 1,2] that these are all non-trivial free products with amalgamation (see [F]). The precise algebraic structure has been worked out for the cases where O_d has a Euclidean algorithm with respect to the norm in O_d. Evidence suggests that properties differ by the class number of O_d. Work on class number 1 has been done by K.Kingston [Ki].

Question DG4. *Classify the algebraic properties of the Bianchi groups by class number. In particular the classificiation of the amalgam structure, the structure of normal subgroups and the classification of torsion elements.*

Evidence in the Euclidean cases shows that they fall into three classes $\{\Gamma_1\}, \{\Gamma_3\}, \{\Gamma_2, \Gamma_7, \Gamma_{11}\}$. Results by R.G. Swan [Sw] show some differences specifically tied to class number. Wilson and Zaleskii [W-Z] using the theory of profinite groups have proved that the Bianchi Groups $\Gamma_1, \Gamma_2, \Gamma_7, \Gamma_{11}$ are conjugacy separable.

Question DG5. *Are the remaining non-Euclidean Bianchi groups conjugacy separable?*

We note that it was proved by Fine and Rosenberger [F-R 3] that the Fuchsian groups are conjugacy separable.

In 1985 Fine and Rosenberger initiated a project to generalize, in an algebraic context, discrete groups, particularly F-groups, by looking at one-relator products of cyclics (see [F-R]). Concurrently Jim Howie and others (see [F-R], [H 4] and [D-H]) began to extend one-relator group theory by looking at general one-relator products of cyclics. Many people have worked on this project and related areas including Reg Allenby, Gilbert Baumslag, Kati Bencsath,Andrew Duncan, Martin Edjvet, Tony Gaglione, Andrea Hempel, Jim Howie, G.Kim, Frank Levin, Colin Maclachlan, Alexei Myasnikov, Frank Roehl, Peter Shalen, Dennis Spellman, Michael Stille, Francis Tang, Rick Thomas and E.B. Vinberg.

If $\{G_\alpha\}$ is a class of groups, then a **one-relator product of the $\mathbf{G}_\alpha$** is a group G of the form $G = (\star G_\alpha)/N(R)$ where $\star G_\alpha$ is the free product of the G_α and $N(R)$ is the normal closure in this free product of the single element R. The G_α are called the **factors** and R is the **relator**. In this context a one-relator group can be viewed as a one-relator product of free groups. If each factor is cyclic then its a **one-relator product of cyclics**.

The first question of interest concerning such one-relator products concerns the Freiheitssatz. We say that a **Freiheitssatz** holds for a one-relator product G if each factor injects into G via the identity map. In [F-R 6] there is a complete discussion and description of various forms of the Freiheitssatz. In general the Freiheitssatz does not hold and therefore some restrictions must be imposed. There are two approaches. The first is to impose conditions on the factors while the second is to impose conditions on the relator. Recall that a group H is **locally indicable** if every finitely generated subgroup has an infinite cyclic quotient. B. Baumslag [B.B. 1], Brodskii[Br 1,2], J.Howie [H 5] and H.Short[Sho] all independently proved that if the factors are locally indicable then the Freiheitssatz holds. The standard open conjecture now is:

Question ORP1. *Does the Freiheitssatz hold for one-relator products with torsion-free factors?*

The second approach is to impose restrictions on the relator. The most common relator condition is that R is a proper power of suitably high order, that is $R = S^m$ with $m \geq 2$. If $m \geq 7$ then the relator satisfies the small cancellation condition $C'(1/6)$ (see [F-R] and [L-S]) and a Freiheitssatz can be deduced from small cancellation theory. Gonzalez-Acuna and Short [Go-S] proved the case $m = 6$ and Howie [H-3,4] using pictures over groups proved the cases $m = 4, m = 5$ and with Brodskii [Br-H] and Duncan [D-H] some parts of $m = 3$. In general the case $m = 2$ remains open. However if the factors admit faithful representations into $PSL_2(\mathbb{C})$ then the Freiheitssatz holds for all $m \geq 2$ (see [B-M-S] and [F-H-R]).

Question ORP2. *Does the Freiheitssatz hold for one-relator products with proper power relator R^m with $m \geq 2$?*

One-relator products are also tied to the question of solving equations over groups. The following three questions are well known in this regard and mentioned in [B-M-S] but are included as part of the present discussion.

Question ORP3. (Kervaire-Laudenbach Conjecture)
If $G = A \star < t > /N(R)$ is trivial, then A is trivial.

From the classical Freiheitssatz this is clearly true if A is a free group. A.Klyachko [Kl] proved that the Kervaire conjecture is true whenever A is a torsion-free group. The Kervaire conjecture is related to the following two quesions.

Question ORP4. *Any single equation over a torsion-free group A is solvable.*

Question ORP5. *Any single power equation, that is an equation of the form $(W(t))^k = g$, is solvable over an arbitrary group A.*

A Fuchsian group via its Poincare presentation is a one-relator product of cyclics. The Fine-Rosenberger project mentioned above had as its primary goals to see which linearity properties of Fuchsian groups are shared by all one-relator products of cyclics. In particular

(1) Which properties of Fuchsian groups are shared by all one-relator products of cyclics?

(2) If a property of a Fuchsian group does not hold in all one-relator products of cyclics, then is there a subclass - specifically a special form of the relator - in which it does hold?

A great deal of interest has centered on the **Tits alternative** that is when the group either contains a non-abelian free subgroup or is virtually solvable. For a one-relator product of cyclics

$$G =< a_1, ..., a_n; a_1^{e_1} = ...a_n^{e_n} = 1, R^m(a_1, .., a_n) = 1 >, m \geq 2$$

a series of results (see [F-R]) showed that the Tits alternative holds whenever $n \geq 3$. Thus interest centered on the case $n = 2$. These groups, which have the form

$$G =< a, b; a^p = b^q = R^m(a, b) = 1 >,$$

are called the **generalized triangle groups**. The most general result is

Theorem (see [F-R-R]). *Let G be a generalized triangle group with presentation*

$$G =< a, b; a^p = b^q = R^m(a, b) = 1 >$$

where $p \leq q$, $p \geq 2$ or $p = 0$, $q \geq 2$ or $q = 0$, $R(a, b)$ is a cyclically reduced word in the free product on a and b involving both a and b and $m \geq 2$. Then G satisfies the Tits alternative except possibly when $p \geq 2, q \geq 2, m = 2, (1/p)+(1/q) \geq 1/2$ and the relator $R(a, b)$ has syllable length greater than 8 in the free product on a, b.

We note also that the finite generalized triangle groups have been completely classified by Howie,Metaftsis and Thomas [H-M-T] and Levai, Rosenberger and Souvignier [L-R-S].

Question ORP6. *Complete the Tits alternative for the generalized triangle groups*

Question ORP7. *Classify the generalized triangle groups which are SQ-universal.*

Question ORP8. *When is a generalized triangle group linear ? hyperbolic? arithmetic?*

Question ORP9. *When does a generalized triangle group have a faithful representation in $PSL_2(\mathbb{C})$?*

There has been a great deal of work done on these questions (see [F-R]). Helling, Kim and Mennicke [H-K-M] have shown that if $m \geq 4$ the group $G =< a, b; a^m = b^2 = ((a^{-1}b)^2(ab)^3)^2 = 1 >$ has a faithful, discrete image in $PSL_2(\mathbb{C})$. Further Helling, Mennicke and Vinberg [H-K-V] show that the groups $G =< a, b; a^k = b^l = (aba^{-1}bab^{-1})^m = 1 >$ with $k, l, m \geq 2$ and with $k \leq l$ have a faithful, discrete representation if at most one of k, l, m is 2 and if $(k, l, m) \neq (2, 3, 3)$. Moreover this group G has a faithful discrete representation of finite volume if $2 \leq k \leq l$ and $(1/k) + (1/l) + (1/m) \geq 1$. In connection with these results it can be shown that the groups $G =< a, b; a^3 = b^3 = (aba^{-1}bab^{-1})^2 = 1 >$ and $G =< a, b; a^3 = b^4 = (aba^{-1}bab^{-1})^2 = 1 >$ are arithmetic. In a similar manner Hagelberg [Ha] and Hagelberg, Maclachlan and Rosenberger [Ha-Mc-R] showed that the groups $G =< a, b; a^k = b^t = [a, b]^m = 1 >$ with $k, t, m \geq 2$, and $k \leq t$ have faithful, discrete representations of finite volume precisely when $(k, t, m) = (3, 3, 3), (3, 4, 2)$ or $(4, 4, 2)$ and that the groups $G =< a, b; a^k = b^l = (a^{-1}bab^{-1}ab^{-1}a^{-1}b)^m = 1 >$ with $k, t, m \geq 2$ and $k \leq t$ have faithful discrete representations of finite volume if $(1/k) + (1/k) + (1/m) \geq 1$ and $(1/t) + (1/t) + (1/m) \geq 1$ except for $(k, t, m) = (2, 2, m)$ and (2,3,2). In addition Hagelberg, Maclachlan and Rosenberger proved that for $(1/k) + (1/k) + (1/m) \geq 1$ and $(1/t) + (1/t) + (1/m) \geq 1$ the groups $G =< a, b; a^k = b^t = [a, b]^m = 1 >$ with $(k, t, m) = (3, 3, 3), (3, 4, 2)$ or (4,4,2) are arithmetic. If $(k, t, m) = (3, 3, 3)$ the group G is a subgroup of index four in the Bianchi group $PSL_2(\mathbf{O_3})$ while if $(k, t, m) = (3, 4, 2)$ or $(4, 4, 2)$, G is commensurable with the Picard group $PSL_2(\mathbf{O_1})$. On the other hand if $G =< a, b; a^k = b^t = [a, b]^m = 1 >$ with $k, t, m \geq 2$, and $k \leq t$, G has a faithful, discrete representation into $PSL_2(\mathbb{C})$ if and only if $(k, t, m) \neq (2, 3, 2), (2, 4, 2), (3, 3, 2), (2, 3, 3)$.

Of interest is to find general necessary conditions for a generalized triangle group G to have a faithful, discrete representation into $PSL_2(\mathbb{C})$ of finite volume. In [Ha-Mc-Ro] there is the following partial result.

Theorem. *Let* $G =< a, b; a^p = b^q = R^m(a, b) = 1 >$ *with* $p = 0$ *or* $p \geq 2, q = 0$ *or* $q \geq 2$ *and* $m \geq 2$ *and* $R(a, b)$ *a cyclically reduced word, not a proper power, in the free product on* a, b *which involves both* a *and* b*. Suppose one of the following holds:*

(1) $m \geq 4$

(2) $m = 3$ *and the word* $R(a, b)$ *does not involve a letter (with respect to the free product on* a *and* b*) of order* 2*.*

Suppose further that G *has a faithful, discrete representation into* $PSL_2(\mathbb{C})$ *of finite volume. Then* $p \geq 2, q \geq 2$ *and* $(1/p) + (1/q) + (1/m) \geq 1$*.*

Question ORP10. *Can the above theorem be generalized? In particular, what is the situation in the omitted cases, that is when* $m = 2$ *or* $m = 3$ *and* $R(a, b)$ *does involve a letter of order 2?*

In the special case of finitely generated one-relator groups G it can be proved that if G has a faithful, discrete representation into $PSL_2(\mathbb{C})$ of finite volume, then $n = 2$ and $m = 1$, that is, G is a torsion-free, two-generator, one-relator group [F-L-R 1].

Question ORP11. *Can a generalized triangle group have a Fuchsian group of finite index?*

Related to the generalized triangle groups are the **generalized tetrahedron groups**. These are groups with presentations of the form

$$< a_1, a_2, a_3; a_1^{e_1} = a_2^{e_2} = a_3^{e_3} = R_1^m(a_1, a_2) = R_2^p(a_1, a_3) = R_3^q(a_2, a_3) = 1 >$$

where $e_i = 0$ or $e_i \geq 2$ for $i = 1, 2, 3; 2 \leq m, p, q$; $R_1(a_1, a_2)$ is a cyclically reduced word in the free product on a_1, a_2 which involves both a_1 and

a_2, $R_2(a_1, a_3)$ is a cyclically reduced word in the free product on a_1, a_3 which involves both a_1 and a_3 and $R_3(a_2, a_3)$ is a cyclically reduced word in the free product on a_2, a_3 which involves both a_2 and a_3 . Further each R_i, $i = 1, 2, 3$ is not a proper power in the free product on the generators it involves. A study of these groups, which generalize the ordinary tetrahedron groups studied by Coxeter, was done in [F-L-R-R] and [V].

Question ORP12. *Classify the finite generalized tetrahedron groups*

Partial results have been obtained by Edjvet, Howie, Rosenberger, Stille and Thomas. The methods used involve essential representations and analyses of trace polynomials. In particular they have shown that if one of m, p, q (in the above notation) is greater than 4 then the given generalized tetrahedron group is finite only if it is an ordinary tetrahedron group.

The closest generalization to Fuchsian groups among one-relator products of cyclics are **groups of F-type**. These are groups with presentations

$$G = < a_1, ..., a_n; a_1^{e_1} = ... = a_n^{e_n} = 1, U(a_1, ..., a_p)V(a_{p+1}, .., a_n) = 1 >$$

where $n \geq 2, e_i = 0$ or $e_i \geq 2$, $1 \leq p \leq n-1$, $U(a_1, .., a_p)$ is a cyclically reduced word in the free product on $a_1, ..., a_p$ which is of infinite order and $V(a_{p+1}, ..., a_n)$ is a cyclically reduced word in the free product on $a_{p+1}, ..., a_n$ which is of infinite order. With p understood we write U for $U(a_1, .., a_p)$ and V for $V(a_{p+1}, ..., a_n)$.

A complete study of such groups was undertaken in [F-R 1], where a large list of open questions on these was presented. Many of these, such as the fact that they are residually finite and conjugacy separable were subsequently answered. However many are still open.

Question ORP13. *Suppose that either U or V is a proper power. Describe addtional conditions, if possible, on the group G so that it has a faithful representation into $PSL_2(\mathbb{C})$. Describe further conditions so that the image is discrete?*

We note that G has a faithful representation into $PSL_2(\mathbb{C})$ if neither U nor

V is a proper power [Ro 4]. Again see the discussion after question ORP9.

Question ORP14. *Describe the solvable subgroups of groups of F-type.*

Question ORP15. *Let G be a group of F-type. Descible additional conditions, if any, on U, V such that one or more of the following holds:*

(1) Subgroups of finite index are again groups of F-type

(2) Torsion-free subgroups of finite index are one-relator groups

(3) Subgroups of infinite index are free products of cyclics

We note that the two-generator subgroups of groups of F-type have been classified (see [F-R]).

Question ORP16. *Let G be a group of F-type. Suppose $r(G)$ is the algebraic rank of G. Is it true that $n - 2 \leq r(g) \leq n$?*

Question ORP17. *Let G be a group of F-type. Is the automorphism group $Aut(G)$ finitely generated or finitely presented? In particular under what conditions is each automorphism of G induced by an automorphism of the free group of rank $r(G)$?*

This last property, that each automorphism of G is induced by an automorphism of the free group of rank $r(G)$ is true in many cases (see [F-R 1]).

3. Discriminating Groups and Co-Discriminating Groups

A group G is **separated** by a class of groups $\mathcal{S}$ if for each non-trivial $g \in G$ there is a group $H \in \mathcal{S}$ and an homomorphism $\phi_H : G \to H$ such that $\phi_H(g) \neq 1$. If each ϕ_H is an epimorphism then G is **residually** $\mathcal{S}$. The group G is **discriminated** by $\mathcal{S}$ provided to every finite set $X \subset G \setminus \{1\}$ of non-trivial elements of G there is a group $H \in \mathcal{S}$ and an homomorphism $\phi_H : G \to H$ such that $\phi_H(g) \neq 1$ for all $g \in X$. In this case,if each ϕ_H is an epimorphism, G is also called **fully residually** $\mathcal{S}$. Clearly being discriminated by $\mathcal{S}$ implies being separated by $\mathcal{S}$. Both properties play a role in several areas of group theory, in particular the theory of group varieties and the theory of algebraic geometry over groups (see [B-M-R 1,2,3]). Discrimination properties play an important role in the **universal theory** of groups (see [F-G-M-R-S]) and its ties to the solution of the Tarski problem. In particular if H is a subgroup of G then a sufficient condition for G and H to have the same universal theory is that H discriminates G.

A **discriminating group** is a group G where every group separated by G is discriminated by G. Although it is difficult to determine which groups are discriminating they can be characterized in the following manner :

Theorem D1 [B-M-R 1]. *A group G is discriminating if and only if its direct square $G \times G$ is discriminated by G.*

Using this characterization it follows that if $G \times G$ is embeddable in G then G is discriminating.

Baumslag, Myasnikov and Remeslennikov show that all torsion-free abelian groups are discriminating and have further characterized the torsion abelian groups. We refer the reader to [B-M-R 1] for the necessary terminology

Theorem D2 [B-M-R 1]. *Let A be a torsion abelian group and suppose that $\tau_p(A)/\delta(\tau_p(A))$ has no elements of infinite p-height. Then A is discriminating if and only if $\alpha(A,p,k)$ is either zero or infinite for every prime p and every positive integer k and $\beta(A,p)$ is either zero or infinite for every prime p.*

A group G is **codiscriminating** if for any family of groups $\mathcal{S}$, $\mathcal{S}$ separates G if and only if $\mathcal{S}$ discriminates G. That is G is fully residually $\mathcal{S}$ if and only if G is residually $\mathcal{S}$. A **domain** is a group without **zero divisors**, that is G is a domain if given any non-trivial $a, b \in G$ there exists $x \in G$ such that $[a, b^x] \neq 1$. Domains are codiscriminating (see [B-M-R 1]). In particular any CSA group is a domain. Recall that a CSA group is a group where maximal abelian subgroups are malnormal. CSA groups are commutative transitive. Baumslag, Myasnikov and Remeslennikov have proved.

Theorem D3 [B-M-R 1]. *Every one-relator group with greater than 2 generators is a domain and hence is codiscriminating.*

Now the questions.

Question D1. *Characterize the finitely generated discriminating groups.*

This question is purposefully vague since it probably cannot be answered as phrased. The class of finitely generated discriminating groups includes such complicated groups as Thompson's group, the commutator subgroup of the Gupta-Sidki group, Higman's infinite simple group and Grigorchuk's groups G_ω (see [F-G-M-S 1]). What the question really asks is what in general can be said about the structure of finitely generated discriminating groups. This is closely tied to the universal theory of these groups (see [F-G-M-S 2] and the discussion below before Question DG4).

Question D2. *Describe in terms of Ulm invariants the abelian discriminating groups.*

Question D3. *Are there any discriminating finitely generated nilpotent groups? In particular are there any class 2 nilpotent discriminating groups?*

We mention here a result in [F-G-M-S 1] that all the free nilpotent groups $F_m(\mathcal{N}_c)$ with $m, c \geq 2$ are non-discriminating. The proof of this uses the following extension of commutative transitivity: a group G is **commutative transitive of level l** ($l \geq 0$) if it satisfies the following

$$\text{if } xy = yx \text{ and } yz = zy \text{ and there exists } w_1, ..., w_l$$

$$\text{such that } [y, w_1, ..., w_l] \neq 1 \text{ then } xz = zx$$

In this context commutative transitivity is just commutative transitivity of level 0. In [F-G-M-S] it is shown that any group which is commutative transitive of level 1 is nondiscriminating and that the free nilpotent groups of class c are commutative transitive of level $c-1$. In particular also the free solvable groups of rank 2 and hence rank 2 free metabelian groups are nondiscriminating. These results also use the Baumslag-Myasnikov-Remesslennikov criteria concerning $G \times G$ being discriminated by G, which leads to our next questions.

The **universal theory** of a group G is the set of all universal sentences true in G (see [F-G-M-S 1,2]). In [F-G-M-S 1] it was proved that if G is discriminating then G has the same universal theory as its direct square $G \times G$. A group then is termed **squarelike** if G has the same universal theory as $G \times G$. In [F-G-M-S 2] the following was proved.

Theorem D4. *(1) The class of squarelike groups properly contains the class of discriminating groups.*

(2) A finitely presented group G is discriminating if and only if it is squarelike.

The only known finitely presented examples of discriminating groups are where $G \times G$ embeds into G. Hence we ask.

Question D4. *If G is a non-abelian finitely presented discriminating group must $G \times G$ embed into G?*

Question D5. *If G is a non-trivial finitely presented group can $G \times G \cong G$?*

We mention here that there are examples of finitely generated groups which are isomorphic to their direct squares (see [Jo]). Work on groups isomorphic to their direct squares has been done by Hirshon [H-M].

Every one-relator group with greater than 2 generators is a domain and hence is codiscriminating. On the other hand the following one-relator groups contain non-trivial abelian normal subgroups and hence cannot be domains.

$$< a, b : a^k = b^t >, k, t \geq 2$$

$$< a, b; a^{-1}ba = b^t >, t \geq 1.$$

The following was asked in [B-M-R 1].

Question D6. *Is it true that the only one-relator groups which are not domains are those which contain abelian normal subgroups?*

Question D7. *Describe all one-relator groups which contain abelian normal subgroups?*

Work on this last question has been done by Murasugi [Mu].

We now list a set of questions on one-relator groups which are related to the discrimination and separation properties.

Question D8. *Describe the fully residually free one-relator groups, that is, the one-relator groups that can be discriminated by free groups.*

This question was mentioned in [B-M-S] and attributed to Baumslag and Spellman. There has been work done on this question by using cohomological methods and a constructive characterization of fully residually free groups. More specifically the class of finitely generated fully residually free groups is properly contained in the class of groups which start with free abelian groups of finite rank and are constructed by repeated iteration of the following four operations:

(1) free products

(2) amalgamated free products with abelian amalgamated subgroups at least one of which is maximal abelian

(3) free extensions of centralizers

(4) separated HNN extensions with abelian associated subgroups at least one of which is maximal abelian. An HNN extension $H =< G, t; t^{-1}At = B >$ is a **separated HNN extension** if $g^{-1}Ag \cap B = \{1\}$ for all $g \in G$.

This construction (see [G-Kh-M] and [Kh-M 1,2]) allows for the following inductive characterization of finitely generated fully residually free groups and allows for inductive type proofs. A fully residually free group G is of **level n** if it can be constructed from an infinite cyclic group by n iterations of the above operations and not $n-1$ such iterations.

Question D9. *Describe the residually free cyclically pinched one-relator groups.*

G.Baumslag [G.B. 3] proved that if F is a finitely generated free group with $V \in F$ with $\overline{F}$ an identical copy of F with $\overline{V}$ the image of V then the amalgamted product $F \star_{V=\overline{V}} \overline{F}$ is residually free. A group of this form is called a **Baumslag double**. If we apply an automorphism ϕ to the first factor and then form $\phi(F) \star_{\phi(v)=\overline{V}} \overline{F}$ this is still residually free and is called a **disguised Baumslag double**. A subgroup of this which is still a one-relator group is also fully residually free.

Question D10. *Is a cyclically pinched one relator group which is residually free and with each factor of equal rank isomorphic to a subgroup embedded in either a Baumslag double or a disguised Baumslag double?*

This question is related to the following. Note that the free product of two residually free groups need not be residually free but that the free product of two fully residually free groups is residually free (see [B.B. 2])

Question D11. *When is an amalgamated product $G_1 \star_{(U=V)} G_2$ of two fully residually free groups residually free?*

These last questions are related to the following ideas. It is known that all non-abelian free groups have the same universal theory (see [F-G-M-R-S]) Any finitely generated non-abelian surface group contains a non-abelian free group and further is residually free. Since non-abelian surface groups are one-relator groups with more than two generators they are domains and hence since residually free, fully residually free. Therefore the non-abelian finitely generated surface groups have the same universal theory as the non-abelian free groups. From the solution to the Tarski problem the non-abelian free groups have the same elementary or first order theory (see [F-G-M-R-S]). A group is said to be **elementarily free** if it has the same elementary theory as the class of non-abelian free groups.

Question D12. *Do there exist finitely generated non-free elementarily free groups? In other words is the elementary theory of the finitely generated non-abelian free groups complete in the sense that if finitely generated G has the same elementary theory as the finitely generated non-abelian free groups then G must also be free?*

In [B-F-G-M-R-R-S] there is a discussion of this question.

Question D13. *Do all the non-abelian surface groups have the same elementary theory?*

Question D14. *Are the non-abelian surface groups elementarily free?*

Of course a positive answer to Question D14 also provides an answer to Question D12. Questions D13 and D14 were also mentioned in [B-M-S].

4. One-Relator Groups

We now list some problems on one-relator groups not directly related to discrimination properties.

G.Baumslag and P.Shalen [B-S] have proved that a finitely presented group G with deficiency greater than one admits a proper free product with amalgamation decomposition $G = (A \star B : C)$ where the factors A, B are finitely generated. From a result of Baumslag [B] it is known that in this case the amalgamated subgroup C is also necessarily finitely generated. An example by Baumslag and Shalen shows that in this decomposition, the factors need not be finitely presented. A proper free product with amalgamation decomposition $(A * B : C)$ with finitely generated factors is called a **Baumslag-Shalen decomposition**. In particular any one-relator group with at least three generators admits a Baumslag-Shalen decomposition.

Question OR1. *If $(A \star B : C)$ is a Baumslag-Shalen decomposition for a torsion-free one-relator group G with C a free group must A, B be either one-relator groups or free groups?*

This is known as the **amalgam conjecture** (see [F-P]) and in [F-P] it was proved that this was true up to homology.

Question OR2. *Let G be a torsion-free one-relator group. Must G admit a Baumslag-Shalen decomposition $(A \star B : C)$ with A, B either one-relator groups or free groups and C is free?*

This is known as the **amalgam(*) conjecture**

Question OR3. *In general, or under what specific conditions, must the factors in a Baumslag-Shalen decomposition be finitely presented?*

Question OR4. *In a Baumslag-Shalen decomposition for a one-relator group with torsion must the factors be finitely presented?*

The following are long standing problems on one-relator groups and are also listed in [B-M-S].

Question OR5. *The isomorphism problem for one-relator groups.*

There are partial results on this problem (see [B-M-S]). Rosenberger [Ro 3] showed the isomorphism problem is solvable for cyclically pinched one-relator groups. Further Pietrowski showed that the isomorphism problem is solvable for one-relator groups with non-trivial centre [P], while S.Pride showed the solvability of the isomorphism problem for two-generator one-relator groups [Pr 2]. Fine, Rosenberger and Stille [F-R-S 2] gave the solution for a special class of parafree one-relator groups introduced by Baumslag. Sela [Se] solved the isomorphism problem for torsion-free hyperbolic groups that do not split as as either amalgamated products or HNN groups over either the trivial group or an infinite cyclic group. It is not known which one-relator groups are hyperbolic.

Question OR6. *Is every one-relator group without Baumslag-Solitar subgroups hyperbolic?*

This was discussed in [B-M-Sh]. It is known that one-relator groups with torsion are hyperbolic so the problem is restricted to torsion-free one-relator groups. Torsion-free hyperbolic groups are CSA, and in [G-Kh-M] it was proved that a torsion-free one-relator group is CSA if and only if it does not contain metabelian Baumslag-Solitar groups $BS(1,p)$ and subgroups isomorphic to $F_2 \times \mathbb{Z}$. Here F_n stands for s free group of finite rank n.

Question OR7. *The conjugacy problem for one-relator groups.*

The conjugacy problem for one-relator groups with torsion was solved by B.B. Newman [N] while a solution for cyclically pinched one-relator groups was given by Lipschutz [Li]. A claimed solution for all one-relator groups by Juhasz [J] has never been given a full proof.

Question OR8. *Are all one-relator groups with torsion residually finite?*

This has been called the **Baumslag conjecture**. A background and a survey of partial solutions can be found in [G.B 1]. An extension of this question was asked by F.Tang who also gave partial results to the Baumslag conjecture. Recently D.Wise [W] proved that the conjecture is true in general for sufficiently long relators.

Question OR9. *What separability properties are satisfied by one-relator groups with torsion? In particular, is a one-relator group with torsion conjugacy separable? subgroup separable?*

We note that there are positive answers for all of these for both cyclically pinched and conjugacy pinched one-relator groups (see [F-R-S 1] and [F-R 5]).

Question OR10. *Suppose a torsion-free one-relator group has the property that* *every subgroup of finite index is again a one-relator group and every subgroup of infinite index is a free group. Must the group be a surface group?*

Hyperbolic groups satisfy the **big powers condition**, that is. if G is hyperbolic and $(w_1, ..., w_n)$ is an ordered n-tuple of elements from G where no adajacent elements commute then there exists a power k such that if $k > k_1, k > k_2, , , k > k_n$ then $w_1^{k_1} w_2^{k_2} ... w_n^{k_n} \neq 1$. The next two questions were posed by A. Myasnikov (see [K-M])

Question OR11. *If G is a torsion-free one-relator group with cyclic centralizers and satisfying the big powers condition, must G be hyperbolic?*

Question OR12. *Suppose G is a torsion-free one-relator group with cyclic centralizers. Must G satisfy the big powers condition?*

5. Test Elements

The final set of questions involve test elements and related concepts in general groups. A **test element** in a group G is an element g with the property that if $f(g) = g$ for an endomorphism f of G to G then f must be an automorphism. A test element in a free group is called a **test word**. Nielsen [Ni] gave the first non-trivial example of a test word by showing that in the free group on x, y the commutator $[x, y]$ satisfies this property. Other examples of test words and test elements have been given by Zieschang [Z 1,2],Rosenberger[Ro 5,6,7], Kalia and Rosenberger[K-R], Hill and Pride [H-P] and Durnev[D]. Gupta and Shpilrain[G-S] have studied the question as to whether the commutator $[x, y]$ is a test element in various quotients of the free group on x, y.

Recall that a subgroup H of a group G is a **retract** if there exists a homomorphism $f : G \to H$ which is the identity on H. Clearly in a free group F any free factor is a retract. However there do exist retracts in free groups which are not free factors. T.Turner [T] characterized test words as those elements of a free group which do not lie in any proper retract. This is now known as the

retraction theorem. Using this characterization he was able to give several straightforward criteria to determine if a given element of a free group is a test word. Using these criteria, Comerford [C] proved that it is effectively decidable whether elements of free groups are test words. Since free factors are retracts, Turner's result implies that no test word can fall in a proper free factor. Therefore being a test word is a very strong form of non-primitivity. Shpilrain [Shp 1] defined the **rank** of an element w in a free group F as the smallest rank of a free factor containing w. Clearly in a free group of rank n a test word has maximal rank n. Shpilrain [Shp 1] conjectured that the converse was also true but Turner gave an example showing this to be false. However Turner also proved that Shpilrain's conjecture is true if only test words for monomorphisms are considered.

As a direct consequence of the characterization Turner obtains that in a free group of rank 2 any non-trivial element of the commutator subgroup is a test word [T], which shows that there is a fairly extensive collection of test words in a free group of rank two. O'Neill and Turner [O-T] extended the retraction theorem to a large class of torsion-free hyperbolic groups.

Question TE 1. *Find other examples of classes of groups for which the retraction theorem holds*

Question TE 2. *Give further examples of test elements in non-free groups.*

In a free group an **almost primitive element** - (APE) - is an element of a free group F which is not primitive in F but which is primitive in any proper subgroup of F containing it. An element g of F is a **tame almost primitive element** if it almost primitive and whenever $g^\alpha \in H$ for a finitely generated subgroup H with $\alpha \geq 1$ minimal then either g^α is primitive in H or the index of H is just α. Further let $\mathcal{U}$ be a variety defined by a set of laws $\mathcal{V}$ (refer to the book of H.Neumann [Ne] for relevant terminology). For a group G we let $\mathcal{V}(\mathcal{G})$ denote the verbal subgroup of G defined by $\mathcal{V}$. An element $g \in G$ is $\mathcal{U}$**-generic** in G if $g \in \mathcal{V}(\mathcal{G})$ and whenever H is a group, $f : H \to G$ a homomorphism and

$w = f(u)$ for some $u \in \mathcal{V}(\mathcal{H})$ it follows that f is surjective. Equivalently $g \in G$ is $\mathcal{U}$-generic in G if $g \in \mathcal{V}(\mathcal{G}) \subset \mathcal{G}$ but $g \notin \mathcal{V}(\mathcal{K})$ for every proper subgroup K of G [St]. In [F-R-Sp-St 1,2] various connections between these concepts were considered. In addition many additional examples of test words were given. In a recent paper Konieczny, Rosenberger and Wolny [K-R-W] proved that in F_n, the free group on $\{a_1, .., a_n\}$, the word $w = a_1^{\alpha_1} ... a_n^{\alpha_n}$ with $\alpha_i \geq 2$ for $i = 1, ..., n$ is a tame almost primitive element if and only if $\alpha_1 = ... = \alpha_n = 2$.

Question TE 3. *Characterize the almost primitive elements in F_2. Characterize the tame almost primitive elements in F_2.*

In a general finitely generated group G a element g is a **primitive element** if there is a minimal generating system for G containing g. The defintions of almost primitive and tame almost primitive are then the same as in a free group. We note that Brunner, Burns and Oates-Williams give a different definition of almost primitive and tame almost primitive elements (see also [F-R-Sp-St 1,2]). In a free group the definitions for almost primitive elements coincide but not for tame almost primitive elements. Certain examples of tame almost primitive elements in surface groups were given in the paper [K-R-W].

Question TE 4. *Give further examples of almost primitive and tame almost primitive elements in finitely generated groups.*

A **test set** in a group G consists of a set of elements $\{g_i\}$ with the property that if f is an endomorphism of G and $f(g_i) = \alpha(g_i)$ for some automorphism α of G and for all i then f must also be an automorphism. Any set of generators for G is a test set and if G possesses a test element then this is a singleton test set. The **test rank** of a group is the minimal size of a test set. Clearly the test rank of any finitely generated group is finite and bounded above by the rank and below by 1. Further the test rank of any free group of finite rank is 1 since these contain test elements. For a free abelian group of rank n the test rank is precisely n. In [F-R-Sp-St] it was shown that given integers n and k with

$k < n$ there exist a group of rank n and test rank k. Recent work of Rocca and Turner [R-T] have given further examples of groups with rank n and arbitrary test rank k with $1 \leq k \leq n$ and are not of the above form. They also give an explicit method to determine the test rank of a finite abelian p-group.

Question TE 5. *Find a procedure to determine the test rank of a group, or to characterize test sets within given groups.*

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