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π-HENSELIANITY: A NEW APPROACH

Antonio José Engler*

Abstract

For a formally real field F, the class of valuation rings which has a unique extension to the pythagorean closure of F is shown to be the same as the class of R-compatible valuation rings, for a suitable subgroup R of the multiplicative group of F.

Resumo

Mostramos nesta nota que a classe dos anéis de valorização de um corpo formalmente real F que têm extensão única ao fecho pitagórico de F coincide com a classe dos anéis de valorização de F que são R-compatíveis, para um conveniente subgrupo R do grupo multiplicativo de F.

1. Introduction

This note is dedicated to study the π-henselian valuation rings introduced in [9]. In order to be precise let F be a formally real field and A a valuation ring of F with maximal ideal m_A. In ([9], Proposition 2.1) it was stated that A is π-henselian if and only if $(1 + m_A) \cap \Sigma\dot{F}^2 \subset \dot{F}^2$, where $\Sigma\dot{F}^2$ and $\dot{F}^2$ are the multiplicative groups of sums of squares and squares, respectively. In this note we are particularly interested in a related notion: the so called "T-compatible" valuation rings, where T is a subgroup of the multiplicative group $\dot{F} = F \setminus \{0\}$ ([1], [7], [13], [17], etc). They are the valuation rings A with $1 + m_A \subset T$. For instance, the residue field of a valuation ring A is formally real if and only if there is an ordering of F such that A is compatible with the positive cone the ordering ([15], Theorem 2.3, p. 17). For a valuation ring A with residue field of

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characteristic different from 2, A is 2-henselian if and only if A is $\dot{F}^2$-compatible ([15] Lemma 3.14, p. 28).

In the next section, using the methods developed in [9], we will show that a valuation ring A is π-henselian if and only if A is R-compatible, for a suitable subgroup R of $\dot{F}$ (see Definition 2.1). From this we may refine our knowledge of the set of all π-henselian valuation rings developed in [11]. In section 3 we discuss the hypothesis assumed in the propositions.

The results stated here will be applied in a subsequent paper concerning the structure of the Galois group of the quadratic closure of a formally real field.

Conventions: For any field F, $\dot{F}$, $\dot{F}^2$, and $\Sigma\dot{F}^2$ will denote the multiplicative groups of nonzero elements, squares, and sums of squares, respectively.

For $a, b \in \dot{F}$, $D\langle a, b\rangle$ denotes the set of non-zero values of the quadratic form $\langle a, b\rangle$.

For every valuation ring A we denote by A^*, m_A, $k_A = A/m_A$, φ_A, Γ_A, and v_A the group of units of A, the maximal ideal, the residue field, the canonical homomorphism, the value group and a valuation corresponding to A, respectively.

Any unexplained property concerning quadratic forms and/or formally real fields can be found in [15], while any concerning valuation theory can be found in [8].

2. π-Henselianity revisited

Let us recall that an element $t \in \dot{F}$ is called **rigid** if $t \notin \dot{F}^2$ and if $D\langle 1, t\rangle = \dot{F}^2 \cup t\dot{F}^2$. It was proved in [11] that $B_\pi(F) = \{t \in \Sigma\dot{F}^2 \mid t \text{ is not rigid}\}$ is a subgroup of the multiplicative group $\Sigma\dot{F}^2$. We say that a is birigid if both a and $-a$ are rigid. It is well known, for a formally real field F, that a rigid element $s \in \Sigma\dot{F}^2$ is not birigid (see [4], Proposition 1). Due to the extremely simple nature of $D\langle 1, s\rangle$, for a rigid element $s \in \Sigma\dot{F}^2$, it seems to be natural to ask about the nature of $D\langle 1, -s\rangle$. R. Bos, [6], described this value set. In order to present his description we have to introduce a new ingredient.

Definition 2.1 *Set* $R = \bigcap D\langle 1, -s\rangle$, *where* s *ranges over* $\Sigma\dot{F}^2$.

Let us introduce some more notation. Following [1] we say that an element $a \in \dot{F}$ is R-**rigid** if $a \notin R$ and if $\{x + ay \neq 0 \mid x, y \in R \cup \{0\}\} = R \cup aR$ and a is called R-**birigid** if a and $-a$ are R-rigid. We shall denote the set of non-zero elements which are not R-birigid by $B(R)$. Our first lemma describes $D\langle 1, -s\rangle$, for a rigid element $s \in \Sigma\dot{F}^2$ and connects rigid elements and R-birigid elements.

Lemma 2.2 [Bos] *Let* F *be a field such that* $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$ *and assume that* $B_\pi(F) \neq \Sigma\dot{F}^2$.

(1) For $x \in \dot{F}$, $x \in R$ *if and only if* $\Sigma\dot{F}^2 \subset D\langle 1, -x\rangle$.

(2) For every $s \in \Sigma\dot{F}^2 \setminus B_\pi(F)$, $D\langle 1, -s\rangle = R \dot{\cup} -sR$.

(3) There exists $r_0 \in \Sigma\dot{F}^2$ *such that* $-\Sigma\dot{F}^2 \cap R = -r_0\dot{F}^2$. *Furthermore, for any* $r \in \dot{F}$, $r \in -\Sigma\dot{F}^2 \cap R$ *if and only if* $\Sigma\dot{F}^2 = D\langle 1, -r\rangle$.

(4) $R \cap \Sigma\dot{F}^2 = \dot{F}^2$.

(5) $B(R) \cap \Sigma\dot{F}^2 = B_\pi(F)$.

Proof. The proof of the first item follows from the property,

$$y \in D\langle 1, -x\rangle \text{ if and only if } x \in D\langle 1, -y\rangle$$

applied to every $y \in \Sigma\dot{F}^2$.

The statements (2), (3) and (4) are proved in [6] Corollary 1.11, Proposition 1.7 and Lemma 2.1 respectively.

We shall now show (5) by proving that $s \in \Sigma\dot{F}^2$ is rigid if and only if s is R-birigid.

For a rigid element $s \in \Sigma\dot{F}^2$ and for $r \in R$ it follows from ([6], Corollary 1.11) that $1 - sr \in D\langle 1, -sr\rangle \subset D\langle 1, -s\rangle = R \cup -sR$. Therefore, if $x, y \in R$, then $x - sy = x(1 - syx^{-1}) \in x(R \cup -sR) = R \cup -sR$ and so $-s$ is R-rigid. On the other side, Lemma 2.5 of [6] shows that $1 + sr \in R \cup sR$ for every $r \in R$. Thus $R + sR = R \cup sR$ follows as in the other case.

Conversely, let $s \in \Sigma\dot{F}^2$ be a R-birigid element. Then $D\langle 1, s\rangle \subset \Sigma\dot{F}^2$ and $D\langle 1, s\rangle \subset R+sR = R\cup sR$. Hence $D\langle 1, s\rangle \subset (\Sigma\dot{F}^2\cap R)\cup s(\Sigma\dot{F}^2\cap R) = \dot{F}^2\cup s\dot{F}^2$, since $\Sigma\dot{F}^2 \cap R = \dot{F}^2$, by (2). Therefore s is rigid.

□

In the next lemma we record some properties of R-rigid elements which easily follows from the definition and from the last lemma and will be useful in proofs.

Lemma 2.3 *Keep the assumptions of the previous lemma and take $s, t \in \Sigma\dot{F}^2$ such that $st \notin B_\pi(F)$.*

(1) $D\langle s, t\rangle = s\dot{F}^2 \cup t\dot{F}^2$.

(2) $D\langle s, -t\rangle = sR \cup -tR = sR - tR$.

(3) $sR + tR = sR \cup tR$.

Recall that a valuation ring A, with maximal ideal m, is called R-compatible if $1 + m \subset R$ (see [1], Definition 1.7). It follows then that $(1 + m) \cap \Sigma\dot{F}^2 \subset \dot{F}^2$, from Lemma 2.2 (4). Therefore, by ([9], Proposition 2.1), a R-compatible valuation ring A is π-henselian. In the next results we shall state some properties of π-henselian valuation rings.

Lemma 2.4 *Let A be a π-henselian valuation ring of a field F with residue field k such that char $k \neq 2$ and k is not formally real. Denote the maximal ideal of A by m and assume that $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$.*

(1) The element r_0 introduced in Lemma 2.2 (3) can be chosen such that $r_0 \in -(1 + m)$. Moreover $-\Sigma\dot{F}^2 \cap (1 + m)\dot{F}^2 = -\Sigma\dot{F}^2 \cap R$.

For the next items let us fix: $t \in \dot{F}$, $x \in 1 + m$ and $a, b \in \dot{F}^2 \cup \{0\}$.

(2) *If* $z = a + (\pm tx)b \neq 0$ *and* $v_A(a) \neq v_A(\pm txb)$, *then* $z \in (1+m)\dot{F}^2 \cup (\pm t)(1+m)\dot{F}^2$.

(3) *For* $t \in \Sigma\dot{F}^2 \setminus B_\pi(F)$ *such that* $v_A(t) = 0$ *and* $z = a + (\pm tx)b \neq 0$ *we have that* $v_A(z) = min\{v_A(a), v_A(b)\}$.

(4) *For* t *as above and* $z = a + txb \neq 0$ *if* $z \in \Sigma\dot{F}^2$, *then* $z \in \dot{F}^2 \cup t\dot{F}^2$.

(5) *For* t *as above and* $z = a - txb \neq 0$ *if* $z \in \Sigma\dot{F}^2$, *then* $z \in \dot{F}^2 \cup tr_0\dot{F}^2$.

(6) *For* t *as above* $R \cap \pm t(1+m)\dot{F}^2 = \emptyset$.

Proof. (1) Since k is not formally real there exist $x_1, \ldots, x_u \in A^*$, $u \geq 1$, such that for $-1 = \varphi_A(x_1)^2 + \cdots + \varphi_A(x_u)^2$. Define $r_0 = x_1^2 + \cdots + x_u^2$. Then $-r_0 \in -\Sigma\dot{F}^2 \cap (1+m)$ and for any other $s \in \Sigma\dot{F}^2$, such that $-s \in -\Sigma\dot{F}^2 \cap (1+m)\dot{F}^2$, we have that $sr^{-1} \in \Sigma\dot{F}^2 \cap (1+m)\dot{F}^2 = \dot{F}^2$. Therefore $-\Sigma\dot{F}^2 \cap (1+m)\dot{F}^2 = -r_0\dot{F}^2$.

By Lemma 2.2 (3) we finish the proof if we can show that $\Sigma\dot{F}^2 = D\langle 1, r_0\rangle$. Take $s \in \Sigma\dot{F}^2$ and write $s = y_1 + \cdots + y_n$, where $y_1, \ldots, y_n \in \dot{F}^2$. If $n = 1$ the statement is obvious. Assume $n \geq 2$. Take $1 \leq i \leq n$ such that $v_A(y_i) \leq v_A(y_j)$, for every $1 \leq j \leq n$. Without loss of generality we may assume $i = 1$. Then $sy_1^{-1} = 1 + y_1^{-1}(y_2 + \cdots + y_n) \in A$ and also $y_1^{-1}(y_2 + \cdots + y_n) \in A$. If $y_1^{-1}(y_2 + \cdots + y_n) \in m$, then $sy_1^{-1} \in (1+m) \cap \Sigma\dot{F}^2$ and so $sy_1^{-1} \in \dot{F}^2$, by ([9], Proposition 2.1). Since this contradicts $n \geq 2$, it follows that $y_1^{-1}(y_2 + \cdots + y_n) \in A^*$. We now consider two cases. If $sy_1^{-1} \in m$, then $\varphi_A(y_1^{-1}(y_2 + \cdots + y_n)) = -1 = \varphi_A(r_0)$. Hence $y_1^{-1}(y_2 + \cdots + y_n)r_0^{-1} \in (1+m) \cap \Sigma\dot{F}^2$. Therefore, arguing as above, $y_1^{-1}(y_2 + \cdots + y_n)r_0^{-1} \in \dot{F}^2$ and $s \in D\langle 1, r_0\rangle$ as desired. If $sy_1^{-1} \in A^*$, for $\alpha = \left((sy_1^{-1} + 1)/2\right)^2$ and $\beta = \left((sy_1^{-1} - 1)/2\right)^2$, $\varphi_A(sy_1^{-1}) = \varphi_A(\alpha) - \varphi_A(\beta) = \varphi_A(\alpha + r_0\beta)$. Once again $sy_1^{-1}(\alpha + r_0\beta)^{-1} \in (1+m) \cap \Sigma\dot{F}^2$ will imply $s \in D\langle 1, r_0\rangle$, proving the statement.

(2) Since $v_A(a) \neq v_A(\pm txb)$, it follows that $z = a(1 + (\pm tx)ba^{-1}) \in (1+m)\dot{F}^2$ if $v_A(a) < v_A(\pm txb)$ and $z \in (\pm t)(1+m)\dot{F}^2$ otherwise.

(3) Going for a contradiction we assume $v_A(z) > \min\{v_A(a), v_A(\pm txb)\}$. By valuation theory, it follows then $v_A(a) = v_A(\pm txb) = v_A(b)$. Thus $\pm\, t =$

$ab^{-1}(1 - za^{-1})x^{-1} \in (1+m)\dot{F}^2$. If $t \in (1+m)\dot{F}^2$, then $t \in \dot{F}^2$, because $(1+m)\dot{F}^2 \cap \Sigma\dot{F}^2 = \dot{F}^2$, by ([9], Proposition 2.1). This contradicts the rigidity of t. In the other case, by (1) above, $t \in r_0\dot{F}^2 \subset B_\pi(F)$, again a contradiction.

(4) Assume $z \in \Sigma\dot{F}^2$. If $v_A(a) \neq v_A(txb)$, by (2) $z \in (1+m)\dot{F}^2 \cup t(1+m)\dot{F}^2$. Hence $z \in \Sigma\dot{F}^2 \cap ((1+m)\dot{F}^2 \cup t(1+m)\dot{F}^2) = \dot{F}^2 \cup t\dot{F}^2$, because A is π-henselian. Consider now the other case. By (3), $v_A(z) = v_A(a) = v_A(txb) = v_A(b)$. Without loss of generality we may assume $v_A(a) = v_A(b) = 0$.

According to ([9] Proposition 2.5 (3)) $\varphi_A(t)$ is $\dot{k}_A^2$-birigid. Hence $\varphi_A(z) = \varphi_A(a) + \varphi_A(t)\varphi_A(b) \in \dot{k}_A^2 \cup \varphi(t)\dot{k}_A^2$. Therefore, there exists $c \in A^*$ such that $\varphi_A(zc^{-2}) = 1$, or $\varphi_A(z(tc^2)^{-1}) = 1$. It follows from this that $zc^{-2} \in 1+m$, or $z(tc^2)^{-1} \in 1+m$. Thus, as A is π-henselian, $z \in \dot{F}^2 \cup t\dot{F}^2$, as required.

(5) As in the case above, $v_A(a) \neq v_A(txb)$ implies either $z \in (1+m)\dot{F}^2$, or $z \in -t(1+m)\dot{F}^2$. In the first case $z \in \dot{F}^2$. In the other case $-t^{-1}z \in -\Sigma\dot{F}^2 \cap (1+m)\dot{F}^2 = -r_0\dot{F}^2$, by (1) above. Therefore $z \in tr_0\dot{F}^2$, ending the proof in this case.

Consider now the case $v_A(z) = v_A(a) = v_A(b)$ (see (3)). Without loss of generality we may assume $v_A(z) = 0$. As $\varphi_A(t)$ is $\dot{k}_A^2$-birigid we have that $\varphi_A(z) \in \dot{k}_A^2 \cup -\varphi_A(t)\dot{k}_A^2 = \dot{k}_A^2 \cup \varphi_A(t)\varphi_A(r_0)\dot{k}_A^2$. Thus, as in the previous item, it follows that $z \in \dot{F}^2 \cup tr_0\dot{F}^2$.

(6) Going for a contradiction, we assume that there is $x \in (1+m)\dot{F}^2$ such that $-tx \in R$. By Lemma 2.2 (1) $\Sigma\dot{F}^2 \subset D\langle 1, tx\rangle$ and by (4) above $\Sigma\dot{F}^2 = \dot{F}^2 \cup t\dot{F}^2$, contradicting one of the assumptions of the lemma. This proves $R \cap -t(1+m)\dot{F}^2 = \emptyset$. The other case is shown in the same way replacing item (4) above by (5).

□

Proposition 2.5 *Let F be a field such that $B_\pi(F) \neq \Sigma\dot{F}^2$ and $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$. Assume that F admits a π-henselian valuation ring A as in the previous lemma. Then $R \subset (1+m)\dot{F}^2$.*

Proof. Consider first the case where there exists $t \in \Sigma\dot{F}^2$ such that $v_A(t) \notin 2\Gamma_A$. By ([9], Proposition 2.5 (2)) $t \notin B_\pi(F)$. We also have $R \subset D\langle 1, -t\rangle$, by the very definition of R. For every $a, b \in \dot{F}^2$ we have that $v_A(a) \neq v_A(tb)$. Hence $R \subset (1+m)\dot{F}^2 \cup -t(1+m)\dot{F}^2$, by (2) of the previous lemma. Heading for a contradiction, we assume that there is $x \in (1+m)\dot{F}^2$ such that $-tx \in R$. It follows from (1) of Lemma 2.2 that $\Sigma\dot{F}^2 \subset D\langle 1, tx\rangle$ and by (2) of the lemma above $\Sigma\dot{F}^2 \subset (1+m)\dot{F}^2 \cup t(1+m)\dot{F}^2$. Since $\Sigma\dot{F}^2 \cap (1+m)\dot{F}^2 = \dot{F}^2$, by ([9], Proposition 2.1), we have that $\Sigma\dot{F}^2 = \Sigma\dot{F}^2 \cap \Big((1+m)\dot{F}^2 \cup t(1+m)\dot{F}^2\Big) = \dot{F}^2 \cup t\dot{F}^2$. We then conclude $(\Sigma\dot{F}^2 : \dot{F}^2) = 2$, contradicting one of the hypothesis.

We now assume $v_A(\Sigma\dot{F}^2) \subset 2\Gamma_A$. Let $t \in \Sigma\dot{F}^2 \setminus B_\pi(F)$. Observe that t is rigid if and only tc^2 is rigid, for every $c \in \dot{F}$. Therefore we may assume $v_A(t) = 0$.

Recall that $R \subset D\langle 1, -t\rangle$. For $r \in R$ let $a, b \in \dot{F}^2$ such that $r = a - tb$. If $v_A(a) \neq v_A(tb)$, it follows from Lemma 2.4 (2) that $r \in (1+m)\dot{F}^2 \cup -t(1+m)\dot{F}^2$. Since we know by (6) of the previous lemma that $R \cap \pm t(1+m)\dot{F}^2 = \emptyset$ it follows that $r \in (1+m)\dot{F}^2$ in this case. Assume now $v_A(a) = v_A(tb)$. By (3) of the last lemma $v_A(r) = v_A(a) = v_A(b)$. Without loss of generality we may assume $v_A(r) = 0$. Then $\varphi_A(r) = \varphi_A(a) - \varphi_A(t)\varphi_A(b)$. Since $\varphi_A(t)$ is $\dot{k}_A^2$-birigid by ([9], Proposition 2.5 (3)) it follows that $\varphi_A(r) \in \dot{k}_A^2 \cup -\varphi_A(t)\dot{k}_A^2$. Thus $r \in \varphi_A^{-1}(\dot{k}_A^2) \subset (1+m)\dot{F}^2$, or $r \in -t(1+m)\dot{F}^2$. As (6) of the last lemma ruled out the second possibility the prove is completed.

□

From ([10], Corollary 2.6 and Remark 2.7) we know that if a field F admits non-trivial π-henselian valuation rings, then there exists a distinguished π-henselian valuation ring which is comparable to each of the others. This is the motivation for the next lemma.

Lemma 2.6 *Let A and V be π-henselian valuation rings of a field F with non-formally real residue fields, such that $A \subset V$. Then* $(1+m_A)\dot{F}^2 = (1+m_V)\dot{F}^2$.

Proof. Since we know from valuation theory that $A \subset V$ implies $m_V \subset m_A$ all that remains is to show that $(1+m_A)\dot{F}^2 \subset (1+m_V)\dot{F}^2$. Take $x \in 1+m_A$.

Since k_V is not formally real there is $s \in \Sigma\dot{F}^2 \cap V^*$ such that $\varphi_V(x) = \varphi_V(s)$. Thus $xs^{-1} \in 1 + m_V \subset 1 + m_A$. Hence $s \in \Sigma\dot{F}^2 \cap (1 + m_A) \subset \dot{F}^2$. Therefore $x \in (1 + m_V)\dot{F}^2$, proving the other inclusion.

□

Our next theorem completes the proof of the result announced in the introduction.

Theorem 2.7 *Let F be a field, formally real and non-pythagorean, with a proper R-compatible valuation ring V with non-formally real residue field of characteristic $\neq 2$. Then:*

(1) There exists a R-compatible valuation ring $\mathcal{O}$ with non-formally real residue field k (char $k \neq 2$) such that every π-henselian valuation ring of F is comparable to $\mathcal{O}$.

(2) Every π-henselian valuation ring A such that char $k_A \neq 2$ is R-compatible.

If in addition we assume that $B_\pi(F) \neq \Sigma\dot{F}^2$ and $(\Sigma\dot{F}^2 : \dot{F}^2) > 2$, then for every π-henselian valuation ring A such that k_A is not formally real and char $k_A \neq 2$ we have that $R = (1 + m_A)\dot{F}^2$.

Proof. Let us first observe that (2) follows from (1). In fact for a valuation ring A such that $\mathcal{O} \subset A$ we know by valuation theory that $m_A \subset m_{\mathcal{O}}$ and so A has to be R-compatible. If A is π-henselian and $A \subset \mathcal{O}$, valuation theory tell us that k_A is also non-formally real. Thus, by Lemma 2.6, A is also R-compatible ($\dot{F}^2 \subset R$).

(1) As we have already observed, V is π-henselian, since it is R-compatible.

We will now use the description of the set $\mathcal{H}$ of all proper π-henselian valuation rings of F presented in ([10], Corollary 2.6). Recall that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1 = \{A \in \mathcal{H} \mid k_A \text{ is not pythagorean }\}$ and $\mathcal{H}_2 = \mathcal{H} \setminus \mathcal{H}_1$.

If $V \in \mathcal{H}_1$, by Corollary 2.6 of [10], there exists a valuation ring $A_{(1)} \in \mathcal{H}$ such that $A_{(1)} \subset V$, which, as it was observed in ([10] Remark 2.7), is comparable to every element of $\mathcal{H}$. Since $A_{(1)} \subset V$ it follows from valuation theory that the residue field of $A_{(1)}$ fulfills the conditions of item (1). Thus, by Lemma 2.6 we can take $\mathcal{O} = A_{(1)}$

Assume now $V \in \mathcal{H}_2$. Then ([10], Corollary 2.6) implies that there exists $A_{(2)} \in \mathcal{H}_2$ such that $V \subset A_{(2)}$ and $A_{(2)}$ is comparable to every element of $\mathcal{H}$. Since $A_{(2)}$ contains V it follows that $m_{(2)} \subset m_V$. Then $A_{(2)}$ is R-compatible and also $2 \notin m_{(2)}$, showing that the residue field of $A_{(2)}$ has the right characteristic. On the other side, if the residue field of $A_{(2)}$ is formally real, it follows from ([10], Proposition 2.3 (a)) that F is pythagorean, contradicting one of our assumptions. Therefore, choosing $\mathcal{O} = A_{(2)}$ we finish the proof of (1).

Finally, under the additional hypotheses, Proposition 2.5 applies to $\mathcal{O}$. Hence $R \subset (1 + m_{\mathcal{O}})\dot{F}^2$. On the other hand, the R-compatibility implies the other inclusion. Hence $R = (1 + m_{\mathcal{O}})\dot{F}^2$. By (2) above and Lemma 2.6 the statement follows.

□

In the next result we state the existence of a R-compatible valuation ring verifying the hypothesis of the last proposition.

Theorem 2.8 *Let F be a formally real field such that $(\Sigma\dot{F}^2 : B_\pi(F)) > 2$. Then there exists a R-compatible valuation ring $\mathcal{O}$ of F such that its residue field k is a non-formally real field of characteristic not 2.*

By ([1], Theorem 2.16) the existence of $\mathcal{O}$ will be assured if there exists a subgroup H of $\dot{F}$ such that $B(R) \subset H \neq \dot{F}$. It seems however that the properties of R are not strong enough to yield such a subgroup. In [9] we gave an explicit construction of a π-henselian valuation ring $\mathcal{O}$ (Theorem 2.8). We will show that $\mathcal{O}$ is R-compatible. This will be achieved in a number of steps. In the first one we state that the subset $\mathcal{M}$ constructed in theorems 3.7 and 3.9 of [9] verifies $1 - \mathcal{M} \subset R$ (Notation as in [9]). Let us first recall the description of $\mathcal{M}$. Define

$$O = \left\{x \in \Sigma\dot{F}^2 \setminus B_\pi(F) \mid 1 + x \in \dot{F}^2\right\}, \qquad U = \{x \in B_\pi(F) \mid xO \subseteq O\}$$

and also for every $s \in O$

$$O_s = \left\{ x \in \Sigma \dot{F}^2 \setminus B_\pi(F) D\langle 1, s\rangle \mid 1 + x \in \dot{F}^2 \right\},$$

$$U_s = \{ x \in B_\pi(F) D\langle 1, s\rangle \mid x O_s \subseteq O_s \}.$$

If $sO_s \neq O_s$ for every $s \in O$, take $\mathcal{A} = O \cup U$ and let $\mathcal{A} = O_e \cup U_e$, if there exists $e \in O$ such that $eO_e = O_e$. Define now $\mathcal{M} = \{x \in \mathcal{A} \mid x^{-1} \notin \mathcal{A}\}$.

It may be worth mentioning that Proposition 3.11 of [9] implies, in the above second case, that $\mathcal{A}$ does not depend on the element $e \in O$ verifying $eO_e = O_e$.

In what follows the assumption $(\Sigma \dot{F}^2 : B_\pi(F)) > 2$ is always assumed and we will be using the results of section 3 of [9] for $S = B_\pi(F)$.

Lemma 2.9

(1) If there exists $e \in O$ such that $1 - e \notin R$, then $eO_e = O_e$.

(2) $1 - \mathcal{M} \subset R$.

Proof. (1) Since $e \in O$, it follows that e is rigid and also R-birigid, by Lemma 2.2 (5). Therefore $1 - e \in -eR$, since we are assuming $1 - e \notin R$. Let $r \in R$ such that $1 - e = -er$. Observe that $e^{-1} = 1 - r$.

Take now $t \in O_e$. Then $t \notin B_\pi(F) \cup eB_\pi(F)$ and so $t + e \in t\dot{F}^2 \cup e\dot{F}^2$, by Lemma 2.3 (1). On the other side $t+e = e(1+te^{-1}) = e(1+t-tr) = e(1+t)-etr$. Since $t \in O_e$ we have that $1+t \in \dot{F}^2$ and $et \notin B_\pi(F)$. Thus Lemma 2.3 (3) also implies $t+e \in eR \cup -etR$. So $t+e \in (t\dot{F}^2 \cap eR) \cup (t\dot{F}^2 \cap -etR) \cup (e\dot{F}^2 \cap eR) \cup (e\dot{F}^2 \cap -etR)$. Observe now that $t\dot{F}^2 \cap eR \neq \emptyset$ implies $te \in \Sigma\dot{F}^2 \cap R = \dot{F}^2 \subset B_\pi(F)$, contradicting $t \in O_e$. Next, if $t\dot{F}^2 \cap -etR \neq \emptyset$, then $-e \in -\Sigma\dot{F}^2 \cap R = -r_0\dot{F}^2$, by Lemma 2.2 (3). Since $r_0 \in B_\pi(F)$, this leads to a contradiction. If e is replaced by t in the above argument we also get $e\dot{F}^2 \cap -etR = \emptyset$. Hence $t + e \in e\dot{F}^2$ and then, Lemma 3.4 of [9] implies that $t \in eO_e$. Therefore $O_e \subset eO_e$. As Lemma 3.4 of [9] also gives the other inclusion we get the desired equality.

(2) Let us first assume that $sO_s \neq O_s$, for every $s \in O$. In this case, by ([9], Theorem 3.7), $\mathcal{M} = O \cup \{x \in U \mid x^{-1} \notin U\}$. Item (1) implies then $1 - x \in R$ for every $x \in O$. For $x \in \mathcal{M} \setminus O$, there is $s \in O$ such that $x^{-1}s \notin O$. Since $x \in B_\pi(F)$ and $s \in \Sigma\dot{F}^2 \setminus B_\pi(F)$, we have that $x^{-1}s \in \Sigma\dot{F}^2 \setminus B_\pi(F)$, because $B_\pi(F)$ is a subgroup of $\dot{F}$. Therefore the condition $x^{-1}s \notin O$ is equivalent to $1+x^{-1}s \notin \dot{F}^2$. Hence $1+x^{-1}s \in x^{-1}s\dot{F}^2$ ($x^{-1}s \notin B_\pi(F)$). Thus $1+s^{-1}x \in \dot{F}^2$ and so $s^{-1}x \in O$. It follows from (1) above that $1-s, 1-s^{-1}x \in R$. Hence $1 - x = (1-s) + s(1-s^{-1}x) \in R \cup sR$ ($s \notin B_\pi(F)$).

On the other side, by ([9], Lemma 3.5 (5)), the condition $x^{-1}s \notin O$ implies that there is $t \in O_s$ such that $x^{-1}t \notin O$. Hence if we replace s by t in the above argument we also get $1 - x \in R \cup tR$. If $sR \cap tR \neq \emptyset$, then $s^{-1}t \in R \cap \Sigma\dot{F}^2$. By Lemma 2.2 (4) $s^{-1}t \in B_\pi(F)$, which contradicts $t \in O_s$. Thus $1 - x \in R$ as desired. Therefore in this case we have that $1 - \mathcal{M} \subset R$.

Let us now consider the case $O_e = eO_e$ for some $e \in O$ and take $x \in \mathcal{M}$. By ([9], Theorem 3.9) we have to consider two possibilities: $x \in O_e$ or $x \in U_e$ but $x^{-1} \notin U_e$. In the first case, if $1 - x \notin R$, item (1) above implies that $xO_x = O_x$. Thus, by ([9], Theorem 3.11), $ex \in B_\pi(F)$, which contradicts $x \in \Sigma\dot{F}^2 \setminus (B_\pi(F) \cup eB_\pi(F))$. Hence $1 - x \in R$, for every $x \in O_e$.

Consider now the case $x^{-1} \notin U_e$. By Lemma 3.10 (20) of [9], there exists $s \in O_e$ such that $x + s \in s\dot{F}^2$. Hence $x + s \notin B_\pi(F) \cup eB_\pi(F)$ and then ([9], Lemma 3.10 (14)) implies that $x + s \in O_e$. So, by the first part of the proof, $1 - (x + s) \in R$. Thus, by Lemma 2.2 (5),

$$1 - x = 1 - (x+s) + s \in R \cup sR. \qquad (\dagger)$$

Take now $x_1 = xe^{-1}$. We claim that $1 + x_1 \in \dot{F}^2$. It follows from the claim together with Lemma 2.2 (2) that

$$1 - x = 1 - x_1 e = (1+e) - e(1+x_1) \in D\langle 1, -e\rangle = R \cup (-e)R. \qquad (\ddagger)$$

Hence ($\dagger$) and ($\ddagger$) imply $1 - x \in (R \cup sR) \cap (R \cup (-e)R)$. Observe next that $sR \cap (-e)R \neq \emptyset$ implies $-es \in -\Sigma\dot{F}^2 \cap R = -r_0\dot{F}^2$ by Lemma 2.2 (3). Thus

$es \in r_0\dot{F}^2 \subset B_\pi(F)$, since $r_0 \in B_\pi(F)$. This contradicts $s \in O_e$. Hence $sR \cap (-e)R = \emptyset$ and then $1 - x \in R$, as desired.

We now prove the claim. Lemma 3.10 (10) of [9] implies that $e, e^{-1} \in U_e$ and also $x_1 \in U_e$. Since $x^{-1} \notin U_e$, also $x_1^{-1} \notin U_e$. Hence $x_1 \in \mathcal{M}$. As ([9], Theorem 3.9) implies that $\mathcal{A} = U_e \cup O_e$ verifies the assumptions of ([9], Lemma 3.1), we have that $1 + x_1 \in \dot{F}^2$, as required.

□

We can therefore conclude that the subset $\mathcal{A}$ constructed in [9], theorems 3.7 and 3.9, fulfills the above condition (2) in addition to the conditions $(i) - (vi)$ of Lemma 3.1 of [9]. It follows from this lemma that $\mathcal{O} = \{a - b \mid a, b \in \mathcal{A}\}$ is a valuation ring with maximal ideal $m = \{a - b \mid a, b \in \mathcal{M}\}$. In the next step we shall prove a technical lemma from which we will be able to show that $\mathcal{O}$ is the valuation ring we are looking for. In order to make things easier we shall use the expression *"the lemma"* to mean Lemma 3.1 of [9].

Lemma 2.10 *Let $\mathcal{A}$ and $\mathcal{M}$ be given as in the comments before Lemma 2.9. Then, for every $a, b \in \mathcal{A}$, $(a-b)^2 \in \mathcal{A}$. Moreover, if $a, b \in \mathcal{M}$, then $(a-b)^2 \in \mathcal{M}$.*

Proof. Let us first claim that $x + y \in \mathcal{A} \setminus \mathcal{M}$, for every $x \in \mathcal{A} \setminus \mathcal{M}$ and $y \in \mathcal{M}$.

In fact, $x \in \mathcal{A} \setminus \mathcal{M}$ means that $x^{-1} \in \mathcal{A}$. By AM1 of *the lemma* $x^{-1}y \in \mathcal{M}$. It follows then from (iv) of *the lemma* that $1 + x^{-1}y \in \mathcal{A} \setminus \mathcal{M}$. Hence $x + y = x(1 + x^{-1}y) \in \mathcal{A} \setminus \mathcal{M}$, as desired.

Take now $a, b \in \mathcal{A}$ and assume $(a-b)^2 \notin \mathcal{A}$. From (i) of *the lemma*, it follows that $(a-b)^{-2} \in \mathcal{A}$. In fact $(a-b)^{-2} \in \mathcal{M}$. Thus, by (iv) of *the lemma*, $1 + (a-b)^{-2} \in \mathcal{A} \setminus \mathcal{M}$. Since $a, b \in \mathcal{A}$, by AM1 of *the lemma*, $2ab(a-b)^{-2} \in \mathcal{M}$. Hence the claim above implies that $(a^2 + b^2 + 1)(a-b)^{-2} = 1 + (a-b)^{-2} + 2ab(a-b)^{-2} \in \mathcal{A} \setminus \mathcal{M}$. On the other side, as $a, b \in \mathcal{A}$ and $(a-b)^{-2} \in \mathcal{M}$, it follows from AM1 and (iii) of *the lemma* that $(a^2+b^2+1)(a-b)^{-2} = (a^2 + b^2)(a-b)^{-2} + (a-b)^{-2} \in \mathcal{M}$, a contradiction.

Assume next that $a, b \in \mathcal{M}$. If $(a-b)^2 \in \mathcal{A} \setminus \mathcal{M}$, then $(a-b)^{-2} \in \mathcal{A} \setminus \mathcal{M}$ by the very definition of $\mathcal{M}$. Since $a, b \in \mathcal{M}$, AM1 of *the lemma* implies that $2ab(a-b)^{-2} \in \mathcal{M}$. Thus, (iv) of *the lemma* yields $(a^2+b^2)(a-b)^{-2} = 1+2ab(a-b)^{-2} \in \mathcal{A} \setminus \mathcal{M}$. On the other side $a, b \in \mathcal{M}$ implies $(a^2+b^2)(a-b)^{-2} \in \mathcal{M}$. We can conclude from this contradiction that $(a-b)^2 \in \mathcal{M}$, as desired.

□

We next prove Theorem 2.8.

Proof. We get from theorems 3.7, 3.9 and Lemma 3.1 of [9] that $\mathcal{O} = \{a - b \mid a, b \in \mathcal{A}\}$ is a valuation ring of F with a non-formally real residue field k of characteristic not 2 and maximal ideal $m = \{a-b \mid a, b \in \mathcal{M}\}$. We want to show that $1+m \subset R$. Take $x \in 1+m$. Then $x-1 \in m$ and so there are $a, b \in \mathcal{M}$ such that $x-1 = a-b$. By the lemma above $(x-1)^2 = (a-b)^2 \in \mathcal{M}$. On the other side, as $2 \in \mathcal{O} \setminus m$, $x+1 \in 2+m \subset \mathcal{O} \setminus m$. Hence $(x+1)^2 \in \mathcal{O} \setminus m$. Actually, we can say a little bit more about $(x+1)^2$. By *the lemma* $(x+1)^2 \in \mathcal{A}$. Since $\mathcal{M} \subset m$, $(x+1)^2 \in \mathcal{A} \setminus \mathcal{M}$. Thus $(x+1)^{-2} \in \mathcal{A}$. Hence $(x+1)^{-2}(x-1)^2 \in \mathcal{M}$, by AM1 of *the lemma*. By (2) of Lemma 2.9, $1-(x+1)^{-2}(x-1)^2 \in R$. Since $1-(x+1)^{-2}(x-1)^2 = (x+1)^{-2}((x+1)^2-(x-1)^2) = (x+1)^{-2}4x$, we can conclude $x \in R$, as desired.

□

3. Extreme cases

We shall discuss now the restrictions imposed in the last section. For a formally real field F such that $(\Sigma \dot{F}^2 : \dot{F}^2) = 2$ the considerations of the last section are not true in several forms. For example, any $t \in \Sigma \dot{F}^2 \setminus \dot{F}^2$ is rigid and $\Sigma \dot{F}^2 = \dot{F}^2 \cup t\dot{F}^2$. Therefore $R = D\langle 1, -t\rangle$ and so $-t \in R$. On the other side we also have $t \in D\langle 1, 1\rangle$. Hence a straightforward calculation shows that $t \in D\langle 1, -t\rangle$. Thus $\Sigma \dot{F}^2 \subset R$, contrary to the statement (4) of Lemma 2.2. If we look at the valuation rings of F the picture is also different from what we have seen so far. The field F may or may not admit π-henselian valuation rings.

Examples: For a formally real field K we denote by K_π the pythagorean closure of K. Let $G(K_\pi; K)$ be the corresponding Galois group. Take $1 \neq \sigma \in G(K_\pi; K)$ and let H be the closed subgroup generated by σ. If F is the fixed field of H, then K_π is also the pythagorean closure of F and there is exactly one quadratic extension of F in K_π. It follows from this that $(\Sigma\dot{F}^2 : \dot{F}^2) = 2$.

To exhibit a field F verifying $(\Sigma\dot{F}^2 : \dot{F}^2) = 2$ and also admitting a π-henselian valuation ring it is enough to pick K non-pythagorean with a π-henselian valuation ring and construct F as above.

The opposite case is much more involved. From ([16], Theorem 3.2) there exists an algebraic extension K from $\mathbb{Q}$ which has a unique ordering (archimedan), exclude $\sqrt{2}$ and has the very strong property of being "pseudo real closed." If we construct F as in the case above, it follows form ([16], Theorem 3.1) that F, and also $F(\sqrt{-1})$, are pseudo real closed. Since $F(\sqrt{-1})$ is clearly not formally real it is pseudo algebraically closed. From ([12], Theorem 10.14, p. 136) we have that every valuation ring A' of $F(\sqrt{-1})$ has residue field algebraically closed and divisible value group. We now claim that the same is true for the residue field and the value group of any valuation rings A of F. Thus the lemma below shows that F does not admit any π-henselian valuation ring.

Let A be a valuation ring of F and A' an extension of A to $F(\sqrt{-1})$. Since $F(\sqrt{-1})$ is a finite extension of F, valuation theory implies that Γ_A is also divisible. Moreover the residue field of A' is a finite extension of k_A. It follows then from Artin-Scheier theory that k_A is either algebraically closed or real closed. If k_A is formally real, so is the residue field of $A \cap K$. Since by construction K has just one archimedean ordering, no valuation ring of K has formally real residue field. Therefore k_A is algebraically closed as desired.

Lemma 3.1 *If a field F has a π-henselian valuation ring A such that char $k_A \neq 2$, k_A is quadratically closed and $\Gamma_A = 2\Gamma$, then F is pythagorean.*

Proof. Let $s \in \Sigma\dot{F}^2$. Since $\Gamma_A = 2\Gamma_A$, there is $x \in \dot{F}$ such that $sx^{-2} \in A^*$. The assumptions on k_A imply there is $y \in \dot{F}$ such that $sx^{-2}y^{-2} \in 1+m_A$. Since A is

π-henselian, it follows that $sx^{-2}y^{-2} \in \dot{F}^2$. Thus $s \in \dot{F}^2$ and F is pythagorean.

□

The restriction $(\Sigma\dot{F}^2 : B_\pi(F)) > 2$ is harder to discuss. Actually we do not know if it is possible to show the existence of R-compatible valuation rings for fields verifying $(\Sigma\dot{F}^2 : B_\pi(F)) = 2$, with just one exception.

Proposition 3.2 *For a field F such that $(\Sigma\dot{F}^2 : B_\pi(F)) = 2$ and $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$, the conclusion of Theorem 2.8 holds.*

Proof. By ([10], Proposition 5.3) we have that $B_\pi(F) = D\langle 1, 1\rangle$. For $E = F(\sqrt{b})$, where $b \in B_\pi(F) \setminus \dot{F}^2$, we know by ([10], Proposition 5.4) that $B_\pi(E) = \dot{E}^2$ and $(\Sigma\dot{E}^2 : \dot{E}^2) = 4$.

Let us write $D_E\langle 1, e\rangle$, with the subscript E, to distinguish the set of elements of $\dot{E}$ which are represented by $\langle 1, e\rangle$ from $D\langle 1, e\rangle$.

If we also define $R_E = \bigcap D_E\langle 1, -e\rangle$, for every $e \in \Sigma\dot{E}^2$, we get from Theorem 2.8 that there exists a R_E-compatible valuation ring $\mathcal{O}_1$ of E with non-formally real residue field k_1 of characteristic $\neq 2$. By Theorem 2.7, $\mathcal{O}_1$ can be chosen such that every π-henselian valuation ring of E is comparable to $\mathcal{O}_1$. Hence a standard argument in valuation theory (see the prove of Proposition 2.18 in [9]) demonstrates that $\mathcal{O}_1$ is the unique extension of $\mathcal{O} = \mathcal{O}_1 \cap F$ to E. Therefore $\mathcal{O}$ is π-henselian.

Let k be the residue field of $\mathcal{O}$. Clearly $char\, k \neq 2$. From valuation theory it follows that $[k_1 : k] \leq 2$. We claim that k is not formally real. Observe first that ([9], Proposition 2.14) implies $(\dot{k}_1 : \dot{k}_1^2) \leq 2$ and $-1 \in \dot{k}_1^2$. The claim is true if $-1 \in \dot{k}^2$. Otherwise $k_1 = k(\sqrt{-1})$. In the last case, however, ([14], Theorem 3.4, p. 202) implies either $D_k\langle 1, 1\rangle =$ image of the norm map $= \dot{k}^2$ or $(\dot{k} : \dot{k}^2) = 2$. The equality $D_k\langle 1, 1\rangle = \dot{k}^2$ implies either k formally real and pythagorean or $\dot{k} = \dot{k}^2$. The last case is ruled out by the assumption $-1 \notin \dot{k}^2$. On the other side, k formally real and pythagorean also cannot happen by ([10], Proposition 2.3 (a)), because F is not pythagorean. Therefore $(\dot{k} : \dot{k}^2) = 2$ is in fact the only possibility. As we have just seen that k cannot be formally real and pythagorean we must conclude $\Sigma\dot{k}^2 = \dot{k}$ and so k is not formally real, proving the claim.

Next, let m_1 be the maximal ideal of $\mathcal{O}_1$ and let $m = m_1 \cap F$ be the maximal ideal of $\mathcal{O}$. By Proposition 2.5 $R \subset (1+m)\dot{F}^2$. We also have $1+m = (1+m_1) \cap F \subset R_E \cap F$. Thus $R \subset R_E \cap F$. We claim that $R_E \cap F = R \cup bR$. Since $b \in \dot{E}^2$ the inclusion $R \cup bR \subset R_E \cap F$ is clear.

Take now $s \in \Sigma\dot{F}^2 \backslash B_\pi(F)$. Since s and sb are rigid in F, by ([5], Lemma 5.3, p. 87), s is rigid in E. Therefore, by Lemma 2.2 (2), $D_E\langle 1, -s\rangle = R_E \cup -sR_E$. Consider now the set $D_E\langle 1, -s\rangle \cap F = (R_E \cap F) \cup -s(R_E \cap F)$ (recall that $s \in F$). By ([3], Lemma 3.5) $D_E\langle 1, -s\rangle \cap F = D\langle 1, -s\rangle D\langle 1, -sb\rangle$. On the other side, as $s, sb \notin B_\pi(F)$, $D\langle 1, -s\rangle = R \cup -sR$ and $D\langle 1, -sb\rangle = R \cup -sbR$. Hence $D_E\langle 1, -s\rangle \cap F = R \cup -sbR \cup -sR \cup bR = (R \cup bR) \cup -s(R \cup bR)$. Comparing the two presentations of $D_E\langle 1, -s\rangle$, the equality $R_E \cap F = R \cup bR$ is then clear if one bears in mind the inclusion of the previous paragraph.

Going back to the proof of the proposition we have then $R \subset (1+m)\dot{F}^2 \subset R \cup bR$. In order to finish the proof we must show $(1+m)\dot{F}^2 \cap bR = \emptyset$. Heading for a contradiction, we assume that there is $r \in R$ such that $br \in (1+m)\dot{F}^2$. Thus $b \in (1+m)\dot{F}^2$, because $R \subset (1+m)\dot{F}^2$. On the other side, our assumptions imply $B_\pi(F) = \dot{F}^2 \cup b\dot{F}^2$. Hence Lemma 2.2 ((3) and (5)) implies either $-\Sigma\dot{F}^2 \cap R = -\dot{F}^2$ and $\Sigma\dot{F}^2 = D\langle 1, 1\rangle$, or $-\Sigma\dot{F}^2 \cap R = -b\dot{F}^2$ and $\Sigma\dot{F}^2 = D\langle 1, b\rangle$. As the first case cannot occur because $D\langle 1, 1\rangle = B_\pi(F) \neq \Sigma\dot{F}^2$, the second case is true. By Lemma 2.4 (1), we may assume $b \in -(1+m)$. Thus $b \in -(1+m)\dot{F}^2 \cap (1+m)\dot{F}^2$ implying $-1 \in (1+m)\dot{F}^2$. It follows from this that $-1 \in \dot{k}^2$ and then Lemma 5.6 of [10] implies $\Sigma\dot{F}^2 = D\langle 1, 1\rangle = B_\pi(F)$, a contradiction.

□

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IMECC - UNICAMP
Caixa Postal 6065
13083-970 Campinas - SP - Brasil
E-mail: engler@ime.unicamp.br