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MODULAR ALGEBRAS OF BURNSIDE p-GROUPS

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Abstract

In this article we study a quotient ring (the thinned algebra introduced by S. Sidki) of the group algebra over finite fields- in characteristic p- of an infinite class of Burnside p-groups acting on trees, of Branch type. These groups are just-infinite and we prove, as has been shown by S. Sidki to hold for the Gupta-Sidki 3-group, the corresponding rings are also just-infinite. In most cases we are able to establish that they are also primitive. In the case of the Grigorchuk 2-group, we prove that its thinned algebra is either primitive or the augmentation ideal is nil, and furthermore, we show that it is abundant in nilpotent elements.

1. Introduction

Automorphism groups of rooted trees have been a source of new examples in combinatorial group theory. We are interested in some particular subgroups of $Aut(\mathcal{T})$, which are infinite finitely generated p-groups (often called *Burnside groups*).

R. Grigorchuck [4], N. Gupta and S. Sidki [2], [3] constructed examples of such groups having simple descriptions and satisfying several remarkable properties. They are *just-infinite* (all their proper quotients are finite, but they are infinite) and not finitely presented (see [8], [7]). Furthermore, these groups have attracted great interest and have motivated the recent development by Grigorchuk of the theory of *branch groups* (see [6]).

In [1], N. Gupta and S. Sidki studied the G_E*-groups*, depending on a vector of parameters E as groups of automorphisms of regular one-rooted trees. By choosing appropriately the vector E we can construct a series of Burnside groups which generalize the Gupta-Sidki 3-group and, in particular cases, we have more examples of branch groups.

In [10], S. Sidki established a general construction of a certain quotient ring of the group algebra of the automorphism group of a regular one-rooted tree.

This ring which he called the *thinned algebra* is generated by an isomorphic copy of the original group. In the same article, he considered the Gupta-Sidki 3-group in its action on the ternary tree and proved that the image of its group algebra, over finite fields of characteristic 3, in the thinned algebra is primitive and just-infinite.

In this article, we will study the behavior of quotient rings of the same type associated to certain G_E-groups. For these groups, the image of their group algebras (which we call *modular algebras*) in the thinned algebra are just-infinite and we have a dichotomy: its Jacobson radical is either equal to its augmentation ideal or it is the null ideal. In case of the G_E-groups generated by the automorphism x, which cyclically permutes the vertices of the first level of a p-ary tree, and the automorphism recursivelly defined by $y = (x, x^2, ..., x^{p-1}, y)$, we can show that the modular algebra is also a primitive ring and so generalizes Sidki's results. For the Grigorchuck 2-group, our results suggest strongly that the augmentation ideal ω of its modular algebra is nil (see Theorem 2).

2. G_E-groups

To describe a G_E-group we consider a prime number p and a set $E = \{\epsilon_0, ..., \epsilon_{p-2}\}$ with entries ϵ_i in the cyclic group $\mathbb{Z}/p\mathbb{Z}$.

An infinite p-regular one-rooted tree $\mathcal{T}$ with root ϕ and $\mathcal{A} = Aut(\mathcal{T})$ are defined in the obvious manner (see [1]). The vertices of $\mathcal{T}$ are labeled by symbols in the set $\mathcal{M}$ formed by sequences of elements in $Y = \{0, 1, ..., p-1\}$ and the subtree headed by the vertex u will be denoted by $\mathcal{T}_u$.

For an element β of $\mathcal{A}$, we use the notation $\beta^{[1]}$ for the automorphism $(\beta, \beta, ..., \beta)$, which fixes the first level of the tree by acting as β on all the subtrees $\mathcal{T}_j$, $0 \leq j \leq p-1$. Inductively, $\beta^{[i+1]} = (\beta^{[i]})^{[1]}, i \geq 1$.

We can note that if $\mathcal{A}_i$ denotes the pointwise stabilizer of words of length i in $\mathcal{M}$ then $\mathcal{A}_1 = \mathcal{F}(Y, \mathcal{A}) = \{f | f : Y \to \mathcal{A}\}$ and the group $\mathcal{A}$ is a semi-direct product $\mathcal{A} = \mathcal{A}_1 \rtimes S$, where $S = Sim(Y)$.

We define two particular elements in $\mathcal{A}$. The first of them is the rooted automorphism x which cyclically permutes the vertices of the first level of $\mathcal{T}$ and consequently, cyclically permutes the subtrees $\mathcal{T}_0, ..., \mathcal{T}_{p-1}$ and the other is the directed automorphism y recursivelly defined by $y = (x^{\epsilon_0}, x^{\epsilon_1}, ..., x^{\epsilon_{p-2}}, y)$, which belongs to $\mathcal{A}_1$.

A G_E-group is generated by x and y where $x^p = y^p = 1$ and each element α in that group is described uniquely by $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{p-1})x^{j(\phi)}$, where inductively $\alpha_m = (\alpha_{m0}, \alpha_{m1}, ..., \alpha_{m(p-1)})x^{j(m)}$, for α_m an element in the group and some function $j : \mathcal{M} \to Y$.

3. Particular Burnside groups

3.1 Gupta-Sidki *p*-groups

In [1] and [3], N. Gupta and S. Sidki studied some particular G_E-groups considering $p \geq 3$ and the vectors of parameters $E_1 = \{1, p-1, 0, ..., 0\}$ and $E_2 = \{1, p-1, ..., 1, p-1\}$, respectively obtaining $y = (x, x^{p-1}, 1, ..., 1, y)$ and $y = (x, x^{p-1}, ..., x, x^{p-1}, y)$. They proved the just-infiniteness in the first case and special properties for the case $p = 3$ were established in [2].

In our case, we consider a class of G_E-groups which also arise as generalizations of the Gupta-Sidki 3-group. We have $E = \{1, 2, ..., p-1\}$, that is, $y = (x, x^2, ..., x^{p-1}, y)$ and applying the results in [3] it follows that $\mathcal{H} =< x, y >$ is a p-group. Obviously, the subgroup $\mathcal{H}_1$ (which fixes the first level of the tree pointwisely) projects onto $\mathcal{H}$ and so $\mathcal{H}$ is infinite. From [6] we have that $\mathcal{H}$ is a just-infinite branch group.

It is clear that $\mathcal{H}_1$ is the normal closure of $< y >$ in $\mathcal{H}$ and $\mathcal{H} = \mathcal{H}_1. < x >$.

An interesting property of the group $\mathcal{H}$ is that

$$\mathcal{H}^{(2)} \times ... \times \mathcal{H}^{(2)} \leq \mathcal{H}',$$

where $\mathcal{H}'$ denotes the commutator subgroup of $\mathcal{H}$ and $\mathcal{H}^{(2)} = [\mathcal{H}', \mathcal{H}']$.

In fact, for any $s, q = 1, ..., p-1$, we have commutator elements

$$[(y^{-x})^s, y^q] = (*, 1, ..., 1, [x^s, y^q])$$

$$[(y^{x^{p-1}})^s, y^q] = (1, 1, ..., *, [x^s, y^q])$$

where $*$ denotes some element of $\mathcal{H}'$. Since the group $\mathcal{H}$ is generated by x and y of order p, the commutator subgroup $\mathcal{H}'$ is generated by the commutators $[x^s, y^q]$, when $s, q = 1, ..., p-1$. So for any two elements a_1, a_2 in $\mathcal{H}'$ there exist elements b_1, b_2 in $[\mathcal{H}_1, \mathcal{H}_1]$ for which

$$b_1 = (*, 1, ..., 1, a_1)$$
$$b_2 = (1, 1, ..., *, a_2)$$

hence

$$[b_1, b_2] = (1, 1, ...1, [a_1, a_2])$$

which implies that $1 \times ... \times 1 \times \mathcal{H}^{(2)} \leq \mathcal{H}'$. As the group $\mathcal{H}$ acts transitively on the first level of the tree, our observation is proved.

For the group $\mathcal{H}$ we consider an automorphism δ which will be important in future results. Such automorphism fixes x and inverts y. We start with a permutation on $p-1$ symbols $t = (0, p-2)(1, p-3)...(\frac{p-3}{2}, \frac{p-1}{2})$, and we have $x^t = x^{-1}$. By taking $\beta : x \to x^{-1},\ y \to y^{-1}$, and $\gamma : x \to x,\ y \to y^{-1}$, we have $\delta = \gamma^{[2]} t^{[1]}$ the required automorphism.

We can extend the notion of *depth function* for $\mathcal{H}$ as S. Sidki has done for the Gupta-Sidki 3-group in [9]. The *length* of $\alpha \in \mathcal{H}_1$, $l(\alpha)$, is the syllabic length of α as an element in the free product $< y > * < y^x > *...* < y^{x^{p-1}} >$. By extending to an arbitrary element in $\mathcal{H}$ we have $l(\alpha x^k) = l(\alpha) + 1$.

We note that each element $h \in \mathcal{H}$ can be described by

$$h_\phi (:= h) = (h_0, h_1, ..., h_{p-1}) x^{i(\phi)}, h_s = (h_{s0}, h_{s1}, ..., h_{s(p-1)}) x^{i(s)}$$

for some function $i : \mathcal{W} \to \{0, 1, ..., p-1\}$, where $\mathcal{W}$ is the set of vertices of $\mathcal{T}$.

Thus, by induction on $l(h)$, there exists $k \geq 0$ such that $h_s \in < y > \cup < x >$ for all sequences s of length k. We call the least such k by *depth* of h and denote it by $d(h)$. We note that $d(x) = d(y) = 0$.

3.2 The Grigorchuk 2-group

The Grigorchuk 2-group was initially introduced as a group of permutations of subintervals of the interval $[0,1]$ (see [4]) but it can be interpreted as a group of automorphisms of the binary tree $\tilde{\mathcal{T}}$. It is generated by three particular automorphisms of $\tilde{\mathcal{T}}$. The first of them is a which cyclically permutes the vertices of the first level of $\tilde{\mathcal{T}}$ and the other automorphisms are d and b, recursivelly defined by $d=(1,b)$, where $b=(a,c)$ and $c=(a,d)$. We note that $a^2=d^2=b^2=c^2=1$ and $db=c$.

We also have a *depth function* for $\tilde{\mathcal{H}}$. The *length* of $\alpha\in\tilde{\mathcal{H}}$, $l(\alpha)$, is the number of occurrences of d and b in α when α is considered as an element of the free product $<d>*<b>*<a>$. If $\vartheta=(\vartheta_0,\vartheta_1)$ is an element which estabilizes the first level of $\tilde{\mathcal{T}}$ then

$$0\leq l(\vartheta_i)\leq l(\vartheta)/2,\text{ for all } i\text{,unless } \vartheta=d \text{ or } b.$$

But we note that each element $h\in\tilde{\mathcal{H}}$ can be described by

$$h_\phi(:=h)=(h_0,h_1)a^{i(\phi)}, h_s=(h_{s0},h_{s1})a^{i(s)}$$

for some function $i:\mathcal{W}\to\{0,1\}$, where $\mathcal{W}$ is the set of vertices of $\tilde{\mathcal{T}}$.

Thus, by induction on $l(h)$, there exists $k\geq 0$ such that $h_s\in<d,b>\cup<a>$ for all sequences s of length k. We call the least such k by *depth* of h and denote that by $dep(h)$. We note that $dep(d)=dep(b)=dep(a)=0$.

4. The modular algebra of tree automorphisms

The construction of a ring associated to a general automorphism group A of a specific regular tree was given by S. Sidki in [9]. It was based on the natural K-algebra homomorphism $K[G\times L]\to K[G]\oplus K[L]$, called *summation thinning*, which extends the identity map of the direct product of groups $G\times L$ into the sum of K-algebras $K[G]\oplus K[L]$.

Since $A=A_1\rtimes S$, the group algebra $K[A]$ is a *crossed product* of algebras $K[A]=K[A_1].K[S]$. So it is clear that the inclusion map $\nu_1:A\to$

$\mathcal{F}(Y, K[A]).K[S]$ can be canonically extended to a K-algebra homomorphism indicated by the same symbol $\nu_1 : K[A] \to \mathcal{F}(Y, K[A]).K[S]$ which induces a K-algebra homomorphism

$$\nu_2 : \mathcal{F}(Y, K[A]).K[S] \to \mathcal{F}(Y, K[A]\nu_1).K[S].$$

We consider $\tilde{\nu}_2 = \nu_1\nu_2$ and proceeding inductively for $i \geq 3$, define $\tilde{\nu}_i = \nu_{\tilde{i-1}}\nu_i$, where $\nu_i : \mathcal{F}(Y, K[A]\tilde{\nu}_{i-2}).K[S] \to \mathcal{F}(Y, K[A]\tilde{\nu}_{i-1}).K[S]$.

This infinite iteration of summation thinning applied to $K[A]$ gives us an ascending chain $T_i = Ker(\tilde{\nu}_i)$ of ideals of $K[A]$ such that $(1 + T_i) \cap A = \{1\}$, for all i.

Definition. By considering the construction above, if $T = \bigcup_{i\geq 1} T_i$, the *thinned algebra* is defined as $\overline{K[A]} = K[A]/T$.

For $p \geq 3$ and the Gupta-Sidki p-group $\mathcal{H}$ we consider $K = \mathbb{F}_p$ the Galois field with p elements. Thus A is the automorphism group of a regular p-adic tree and we denote the image of $\mathbb{F}_p[\mathcal{H}]$ in $\overline{\mathbb{F}_p[\mathcal{A}]}$ by $\mathcal{R}$, that is, $\mathcal{R} = (\mathbb{F}_p[\mathcal{H}] + T)/T$.

We observe that since $\mathbb{F}_p < x > \leq \mathbb{F}_p[S]$, then $\mathbb{F}_p < x >$ is embedded in $\mathcal{R}$. We also have $\mathbb{F}_p < y >$ embedded in $\mathcal{R}$.

Now, the p-tuples in $\mathcal{H}$ can be added component-wise in $\mathcal{R}$ and so we have $\mathcal{R} \leq \mathcal{F}(Y, \mathcal{R}) \rtimes \mathbb{F}_p < x >$. Then, if $\mu \in \mathcal{R}$ we have $\mu = u.v$, with $u = (u_0, u_1, ..., u_{p-1})$, $u_0, u_1, ..., u_{p-1} \in \mathcal{R}$ and $v = \beta_0 + \beta_1 x + ... + \beta_{p-1}x^{p-1}$.

Therefore, $\mu = \beta_0 u + \beta_1 ux + ... + \beta_{p-1}ux^{p-1}$. In this way, each element of $\mathcal{R}$ can be written in a unique form

$$\mu = \mu_0 + \mu_1 x + ... + \mu_{p-1}x^{p-1},$$

where $\mu_i = (\mu_{i_0}, \mu_{i_1}, ..., \mu_{i_{p-1}})$, $\mu_{i_j} \in \mathcal{R}, 0 \leq j \leq p-1$.

To study the quotient ring associated to the Grigorchuck 2-group we consider $K = \mathbb{F}_2$ and the automorphism group of a regular binary tree $\tilde{A}$. Then $\tilde{\mathcal{R}} = \mathbb{F}_2[\tilde{\mathcal{H}}]/\mathbb{F}_2[\tilde{\mathcal{H}}] \cap T$.

The pairs in $\tilde{\mathcal{H}}$ can be added component-wise in $\tilde{\mathcal{R}}$ and so we have $\tilde{\mathcal{R}} \leq \mathcal{F}(Y,\tilde{\mathcal{R}}) \rtimes \mathbb{F}_2[a]$. Then, if $\mu \in \tilde{\mathcal{R}}$ we can write $\mu = u.v$, with $u = (u_0, u_1)$, $u_0, u_1 \in \tilde{\mathcal{R}}$ and $v = \beta_0 + \beta_1 a$. Therefore, $\mu = \beta_0 u + \beta_1 ua$. In this way, each element of $\tilde{\mathcal{R}}$ can be written in a unique form

$$\mu = \mu_0 + \mu_1 a,$$

where $\mu_i = (\mu_{i_0}, \mu_{i_1})$, $\mu_{i_j} \in \tilde{\mathcal{R}}, 0 \leq j \leq 1$.

We also see that $\mathbb{F}_2 < d >$ and $\mathbb{F}_2 < b >$ are embedded in $\tilde{\mathcal{R}}$.

4.1 The ring $\mathcal{R}$ associated to $\mathcal{H}$

Our goal in this section is to study the quotient ring associated to the Gupta-Sidki p-groups described in 3.1. We will see that these rings are just-infinity and primitive.

As we have done in $\mathcal{H}$, given an element $\sigma \in \mathcal{R}$, we define the diagonal elements of $\overline{\mathbb{F}_p[\mathcal{A}]}$: $\sigma^0 = \sigma$, $\sigma^{[1]} = (\sigma, \sigma, ..., \sigma)$ and inductively, $\sigma^{[i+1]} = (\sigma^{[i]})^{[1]}$, for $i \geq 1$. We observe that $\sigma^{[i]}$ commutes with x, $\forall i \geq 1$.

Other elements which play a fundamental role here are the following elements of the augmentation ideal $\varpi(\mathbb{F}_p[\mathcal{H}])$:

$$x_* = (x-1)^{p-1} = x^{p-1} + ... + x + 1,$$
$$y_* = (y-1)^{p-1} = y^{p-1} + ... + y + 1$$

When we treat it modulo T, we can see that $y_* = (x_*, x_*, ..., x_*, y_*)$.

We can extend to $\mathcal{R}$ the depth function $d : \mathcal{H} \to \mathbb{Z}_{\geq 0}$ as follows. For $\lambda = \lambda_0 + \lambda_1 x + ... + \lambda_{p-1} x^{p-1} \in \mathcal{R}$, where $\lambda_i = \left(\lambda_{i_0}, \lambda_{i_1}, ..., \lambda_{i_{p-1}}\right)$ and $\lambda_{i_j} \in \mathcal{R}$ $(j = 0, 1, ..., p-1)$, define:

$$\begin{aligned} d(\lambda) &= 0, \text{ if } \lambda \in \sum_{i=0}^{p-1} \mathbb{F}_p < y > x^i \\ d(\lambda) &= \max\{d(\lambda_i)\}, \text{ otherwise;} \end{aligned}$$

where for those $\lambda_i \notin \mathbb{F}_p < y >$, we have $d(\lambda_i) = \max\{d(\lambda_{i_j})\} + 1$.

We observe that for $\lambda = \lambda_0 + \lambda_1 x + ... + \lambda_{p-1} x^{p-1} \in \mathcal{R} \backslash \mathbb{F}_p < y >$, we have $d(\lambda) \geq d(\lambda_i)$, $0 \leq i \leq p-1$ and $d(\lambda^x) = d(\lambda)$.

4.1.1. On ideals of $\mathcal{R}$

In this section we will see some results which allow us to have more information about the structure of $\mathcal{R}$. The first of them guarantees the existence of a special element in an ideal of $\mathcal{R}$ which does not involve x_* from other element of the ideal.

Lemma 1. *Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{p-1})x_* \in \mathcal{I}$. Then, for all $c \in \mathcal{H}^{(2)}$, we have $((c-1)\lambda_i(c-1)^2, 0, ..., 0) \in \mathcal{I}, \quad i = 0, 1, ..., p-1$.*

Proof. It is enough to consider a sequence of particular elements in $\mathcal{I}$:

$$\begin{aligned}
\vartheta_1 &= \lambda^{(c,1,...,1)} - \lambda \\
&= (\lambda_0^c - \lambda_0, 0..., 0) + ((c^{-1}-1)\lambda_0, 0, ..., 0, \lambda_{p-1}(c-1)))\,x \\
&\qquad + ((c^{-1}-1)\lambda_0, 0, ..., \lambda_{p-2}(c-1), 0))\,x^2 \\
&\qquad +... + ((c^{-1}-1)\lambda_0, \lambda_1(c-1), 0, ..., 0)\,x^{p-1}
\end{aligned}$$

$$\begin{aligned}
\vartheta_2 &= \vartheta_1^{(1,c,1,..,1)} - \vartheta_1 \\
&= ((c^{-1}-1)\lambda_0(c-1), 0, ..., 0))\,x + (0, (c^{-1}-1)\lambda_1(c-1), 0..., 0)x^{p-1}
\end{aligned}$$

$$\begin{aligned}
\vartheta_3 &= \vartheta_2 x^{p-1} \\
&= ((c^{-1}-1)\lambda_0(c-1), 0, ..., 0)) + (0, (c^{-1}-1)\lambda_1(c-1), 0, ..., 0)x^{p-2}
\end{aligned}$$

$$\begin{aligned}
\vartheta_4 &= \vartheta_3^{(1,...,1,c)} - \vartheta_3 \\
&= (0, (c^{-1}-1)\lambda_1(c-1)^2, 0, ..., 0))\,x^{p-2}
\end{aligned}$$

$$\begin{aligned}
\vartheta_5 &= x\vartheta_4 x \\
&= ((c^{-1}-1)\lambda_1(c-1)^2, 0, ..., 0)
\end{aligned}$$

$$\begin{aligned}
\vartheta_6 &= -c^{[1]}\vartheta_5 \\
&= ((c-1)\lambda_1(c-1)^2, 0, ..., 0)
\end{aligned}$$

Remark. *We note that if $\lambda = \lambda_0 + \lambda_1 x + ... + \lambda_{p-1}x^{p-1} \in \mathcal{R}$, with $\lambda_i \in \mathcal{F}(Y, \mathcal{R})$, $i = 0, 1, ..., p-1$ then for some $j = 0, 1, ..., p-1$ there exists a non-zero linear combination β of the λ_i such that $\lambda(x-1)^j = \beta x_*$.*

In fact, we have

$(a) If$ $\lambda_0 = \lambda_1 = ... = \lambda_{p-1}$ then $\lambda = \lambda_0 x_*$,

$(b) If\ \lambda_0 + \lambda_1 + ... + \lambda_{p-1} = 0$ we have

$$\begin{aligned} \lambda &= \lambda_1(x-1) + \lambda_2(x^2+1) + ... + \lambda_{p-1}(x^{p-1}-1) \\ &= (x-1)[\lambda_1 + \lambda_2(x+1) + ... + \lambda_{p-1}(x^{p-2} + ... + x + 1)] \\ &= (x-1)[\lambda_1 + 2\lambda_2 ... + (p-1)\lambda_{p-1} + \\ &\qquad (\lambda_2 + ... + \lambda_{p-1})(x-1) + ... + \lambda_{p-1}(x^{p-2}-1)]. \end{aligned}$$

If the independent sum of $\lambda's$ is different from zero we have

$$\lambda = (x-1)^2[\lambda_2 + ... + \lambda_{p-1} + (\lambda_3 + ... + \lambda_{p-1})(x+1) + ... + \lambda_{p-1}(x^{p-3} + ... + x + 1)]$$

and so an iteration of the argument used before will give us some $j = 0, 1, ..., p-1$ such that $\lambda(x-1)^j = \beta x_*$, where β is a non-zero linear combination of the λ_i.
$(c) If\ \lambda_0 + \lambda_1 + ... + \lambda_{p-1} \neq 0$ then $\lambda x_* = (\lambda_0 + \lambda_1 + ... + \lambda_{p-1}) x_*$.

The next lemma establishes a relation between the ideals of $\mathcal{R}$ and the normal subgroups of $\mathcal{H}$. This relation is very important because gives us a certain control of the ideals of $\mathcal{R}$ from what is already known about the normal subgroups of $\mathcal{H}$.

Proposition 1. *Let $\mathcal{I}$ be a 2-sided non-zero ideal in $\mathcal{R}$. Then, there exists $c \in \mathcal{H}^{(2)}$, $c \neq 1$ such that $\mathcal{I} \supseteq N - 1$, where $N = < c >^{\mathcal{H}}$.*

Proof. Let $\lambda = \lambda_0 + \lambda_1 x + ... + \lambda_{p-1} x^{p-1} \in \mathcal{I}$, $\lambda \neq 0$. The proof depends on the depth of λ, $d(\lambda)$.
(i) Suppose $d(\lambda) = 0$. In this case, $\lambda_i \in \mathbb{F}_p < y >$, $i = 0, ..., p-1$. According to the previous remark, we have an element $\beta x_* \in \mathcal{I}$, with $\beta \in \mathbb{F}_p < y >$, $\beta \neq 0$. Writing $\beta = (\beta_0, \beta_1, ..., \beta_{p-1})$, we have $\beta_i = p(x^i)$, $i = 0, ..., p-2$, and $\beta_{p-1} = p(y)$ for some polynomial $p(s) = a_0 + a_1 s + ... + a_{p-1} s^{p-1} \in \mathbb{F}_p[s]$ and $\beta_i \neq 0$, $\forall i$.

By the previous lemma, for all $c \in \mathcal{H}^{(2)}$

$$((c^{[1]} - 1)\beta_1(c^{[1]} - 1)^2, 0, ..., 0) \in \mathcal{I}$$

that is, $((c^{[1]} - 1)^3 \beta_1, 0, ...0) \in \mathcal{I}$.

Then, $((c^{[1]}-1)^3\beta_1,0,...0)(y^x-1)^{p-1}\in\mathcal{I}$, that is, $((c^{[1]}-1)^3\beta_1 x_*,0,...0)=((c^{[1]}-1)^3 x_*,0,...,0)\in\mathcal{I}$. By using the lemma again,

$$(((c-1)^6,0,...,0),0,...,0)\in\mathcal{I}\Rightarrow(((c-1)^{p^t},0,...0),0,...0)\in\mathcal{I},$$

where p^t is the smaller p-power such that $p^t\geq 6$.

Consequently, $((c,1,...1),1,...,1)^{p^t}-1\in\mathcal{I}$ and since $\mathcal{H}^{(2)}$ has unbounded exponent, this case is solved.

(ii) Suppose now $d(\lambda)>0$. Using the remark previously done we can produce $0\neq\beta x_*\in\mathcal{I}$, where $\beta=(\beta_0,\beta_1,...,\beta_{p-1})$ is a linear combination of λ_i and $\beta_i\neq 0$, for some i. Since $d(\beta)>0$, we have $d(\beta_i)\leq d(\lambda)-1$. Let $c\in\mathcal{H}^{(2)}$ and consider $c_{t+1}=c^{[t+1]}$. In $\mathcal{I}$, we have $((c_t-1)\beta_0(c_t+1)^{p-1},0,...0)\in\mathcal{I}$. Using similar argumentation as the previous case, we obtain in some stage j

$$(...((c_{t-j}+1)^{p^j},0,...,0),0,...,0),0,...,0)\in\mathcal{I}.$$

So the proof is concluded.

As a consequence of that, we have an important property of $\mathcal{R}$.

Corollary 1. *$\mathcal{R}$ is a prime ring.*

Proof. Suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-zero ideals of $\mathcal{R}$ with $\mathcal{I}_1\mathcal{I}_2=\{0\}$. Let N_1 and N_2 be nontrivial normal subgroups of $\mathcal{H}$ such that $N_1-1\subseteq\mathcal{I}_1$ and $N_2-1\subseteq\mathcal{I}_2$. If we consider $L=N_1\cap N_2$, we have $[\mathcal{H}:L]<\infty$.

But, $\{0\}=\mathcal{I}_1\mathcal{I}_2\supseteq(N_1-1)(N_2-1)\supseteq(L-1)^2$ and so, L is abelian. Consequently, L is finitely generated and therefore is finite, a contradiction. We conclude that $\mathcal{R}$ is prime.

As a consequence follows that all non-trivial ideals of $\mathcal{R}$ have finite codimension. In fact, any non-trivial ideal $\mathcal{I}$ contains $N-1$ for some nontrivial normal subgroup N of $\mathcal{H}$ which has finite index n and so $\mathcal{I}$ has codimension n. We can announce:

Corollary 2. *$\mathcal{R}$ is a just-infinite ring.*

Consider $\varpi(\mathcal{R})$ the image of the augmentation ideal $\varpi(\mathbb{F}_p[A])$ modulo D and $J(\mathcal{R})$ the *Jacobson radical* of $\mathcal{R}$.

Corollary 3. $J(\mathcal{R}) = \{0\}$ *or* $J(\mathcal{R}) = \varpi(\mathcal{R})$.

Proof. Since $\varpi(\mathcal{R})$ is maximal right-ideal, we have $J(\mathcal{R}) \leq \varpi(\mathcal{R})$. Suppose that $J(\mathcal{R}) \neq \{0\}$. Then, by previous lemma $J(\mathcal{R})$ contains $N-1$, for some non-trivial normal subgroup N of $\mathcal{H}$. And so, modulo $J(\mathcal{R})$, $\mathcal{H}$ is a finite p-group and since $K = \mathbb{F}_p$, we have $J(\mathcal{R}) = \varpi(\mathcal{R})$.

4.1.2 Primitivity of $\mathcal{R}$

Now we are able to prove that $\mathcal{R}$ is a primitive ring. The main fact is that the jacobson ideal $J(\mathcal{R})$ is the zero ideal. We will do it from the previous result by exhibiting an element in $\varpi(\mathcal{R})$ which is not invertible in $\mathcal{R}$.

Lemma 2. *Suppose that $\varphi = 1 + \lambda x_*$ is an invertible element in $\overline{\mathbb{F}_p[A]}$, where $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{p-1}) \in \mathcal{F}(Y, \overline{\mathbb{F}_p[A]})$. Then $\varphi^{-1} = 1 + \lambda' x_*$, for some $\lambda' = \left(\lambda'_0, \lambda'_1, ..., \lambda'_{p-1}\right) \in \mathcal{F}(Y, \overline{\mathbb{F}_p[A]})$.*

Furthermore,

$$\begin{array}{l} \lambda(1+\lambda'_0+\lambda'_1+...+\lambda'_{p-1})^{[1]} = -\lambda', \\ \lambda'(1+\lambda_0+\lambda_1+...+\lambda_{p-1})^{[1]} = -\lambda, \\ (1+\lambda_0+\lambda_1+...+\lambda_{p-1})^{-1} = 1+\lambda'_0+\lambda'_1+...+\lambda'_{p-1}. \end{array}$$

Proof. If $\varphi^{-1} = \mu_0 + \mu_1 x + ... + \mu_{p-1}x^{p-1}$, where $\mu_i \in \mathcal{F}(Y, \overline{\mathbb{F}_p[A]})$ then

$$\begin{array}{rcl} 1 & = & \varphi\varphi^{-1} = \varphi^{-1} + \varphi^{-1}\lambda x_* \\ & = & (\mu_1 + \mu_0\lambda + ... + \mu_{p-1}\lambda^x) + (\mu_0 + \mu_0\lambda + ... + \mu_{p-1}\lambda^x)x \\ & & +... + (\mu_{p-1} + \mu_0\lambda + ... + \mu_{p-1}\lambda^x)x^{p-1}. \end{array}$$

It follows that $\mu_1 = \mu_2 = ... = \mu_{p-1} = \mu_0 - 1$ and writing $\lambda' = \mu_0 - 1$ we have $\varphi^{-1} = \lambda' x_* + 1$. Since $\varphi^{-1}\varphi = (1+\lambda' x_*)(1+\lambda x_*) = 1$ we obtain $\lambda'(1+\lambda_0+\lambda_1+...+\lambda_{p-1})^{[1]} = -\lambda$. From $\varphi\varphi^{-1} = 1$ we also have $\lambda(1+\lambda'_0+\lambda'_1+...+\lambda'_{p-1})^{[1]} = -\lambda'$.

We obtain the equality $(1+\lambda_0+\lambda_1+...+\lambda_{p-1})^{-1}=1+\lambda'_0+\lambda'_1+...+\lambda'_{p-1}$ by multiplying the previous equation on both sides by x_*.

Theorem 1. *The element $\eta=1+yx_*$is not invertible in $\mathcal{R}$.*

Proof. Suppose we have the contrary. The element $\mu=1+y^{-1}x_*$ is also invertible in $\mathcal{R}$ since the application $x\mapsto x,\ y\mapsto y^{-1}$ is an automorphism of $\mathcal{H}$. The inverses of these elements can be written as indicated in the previous lemma

$$\eta^{-1}=1-\rho x_* \text{ and } \mu^{-1}=1-\sigma x_*$$

with $\rho,\sigma\in\mathcal{F}(Y,\overline{\mathbb{F}_p[A]})$ and $\rho(y+x_*)^{[1]}=y,\ \sigma(y^{-1}+x_*)^{[1]}=y^{-1}$. We can obtain two (ρ,σ)-dependence equations $\rho=y(1-\sigma x_*)^{[1]}(y^{-1})^{[1]}$ and $\sigma=y^{-1}(1-\rho x_*)^{[1]}y^{[1]}$ and so

$$\rho=y(y^{-1})^{[1]}-y(y^{-1})^{[1]}(1-\rho x_*)^{[2]}y^{[2]}x_*^{[1]}(y^{-1})^{[1]}.$$

If we admit $\rho\in\mathcal{F}(Y,\mathcal{R})$ and $\rho=(\rho_0,\rho_1,...,\rho_{p-1})$ then $d(\rho)\geq d(\rho_0)$. Furthermore

$$\begin{aligned}\rho_0 &= 1-(1-\rho x_*)^{[1]}y^{[1]}x_*y^{-1}\\ &= 1-(1-\rho x_*)^{[1]}y^{[1]}(1+x+...+x^{p-1})y^{-1}\\ &= 1-(1-\rho x_*)^{[1]}y^{[1]}(y^{-1}+(y^{-1})^{x^{-1}}+...+(y^{-1})^{x}).\end{aligned}$$

Moreover since $\rho_0\in\mathcal{R}$ we write $\rho_0=\alpha_0+\alpha_1x+...+\alpha_{p-1}x^{p-1}$, where $\alpha_i\in\mathcal{F}(Y,\mathcal{R})$.

Then $d(\rho_0)\geq d(\alpha_0)$ and

$$\begin{aligned}\alpha_0 &= 1-(1-\rho x_*)^{[1]}y^{[1]}y^{-1}\\ &= (1,...,1)-(1-\rho x_*)^{[1]}y^{[1]}(y^{-1},x^{-1},...,x)\\ &= (\rho x_*,\rho x_*yx^{-1},...,\rho x_*yx).\end{aligned}$$

If $\alpha_{00}=\rho x_*=\rho+\rho x+...+\rho x^{p-1}$, we have $d(\alpha_{00})=d(\rho)$ that implies $d(\rho)\geq d(\rho_0)\geq d(\alpha_0)\geq d(\alpha_{00})=d(\rho)$. Then $\rho\in\mathbb{F}_p<y>$ and $\rho=\rho_0=\alpha_0=\alpha_{00}=\rho x_*$, that is, $\rho=0$, a contradiction.

4.2 The ring $\tilde{\mathcal{R}}$ associated to $\tilde{\mathcal{H}}$

As we have done in $\tilde{\mathcal{H}}$, given an element $\sigma \in \tilde{\mathcal{R}}$, we define the diagonal elements of $\overline{\mathbb{F}_2[\tilde{\mathcal{A}}]}$: $\sigma^0 = \sigma$, $\sigma^{[1]} = (\sigma, \sigma)$ and inductively, $\sigma^{[i+1]} = (\sigma^{[i]})^{[1]}$, for $i \geq 1$. We observe that $\sigma^{[i]}$ commutes with a, $\forall i \geq 1$.

We also define some important elements in the augmentation ideal $\varpi \left(\mathbb{F}_p[\tilde{\mathcal{H}}]\right)$:

$$a_* = a + 1, \quad d_* = d + 1, \quad b_* = b + 1, \quad c_* = c + 1.$$

When we treat it modulo T, we can see that $d_* = (0, b_*)$, $b_* = (a_*, c_*)$ and $c_* = d_* b_* + d_* + b_*$. Furthermore

$$1 + d + b + c = (0, 1 + d + b + c).$$

So $1 + d + b + c = 0$ and consequently $c_* = d_* + b_*$. Then $d_* b_* = 0$.

Later we will see some results about the nilpotency of elements in $\tilde{\mathcal{R}}$, particularly about monomial in a_*, c_*, d_* and b_*.

We can extend to $\tilde{\mathcal{R}}$ the depth function $dep : \tilde{\mathcal{H}} \to \mathbb{Z}_{\geq 0}$ as follows. For $\lambda = \lambda_0 + \lambda_1 a \in \tilde{\mathcal{R}}$, where $\lambda_i = (\lambda_{i_0}, \lambda_{i_1})$ and $\lambda_{i_j} \in \tilde{\mathcal{R}}$ $(j = 0, 1)$, define:

$$\begin{aligned} dep(\lambda) &= 0, \text{ if } \lambda \in \mathbb{F}_2 < d, b > + \mathbb{F}_2 < d, b > a \\ dep(\lambda) &= \max\{dep(\lambda_i)\}, \end{aligned}$$

where for those $\lambda_i \notin \mathbb{F}_2 < a >$, we have $dep(\lambda_i) = \max\{dep(\lambda_{i_j})\} + 1$.

We observe that for $\lambda = (\lambda_0, \lambda_1) \in \tilde{\mathcal{R}} \backslash \mathbb{F}_2 < a >$, we have $dep(\lambda) > dep(\lambda_i)$, $0 \leq i \leq 1$ and $dep(\lambda^a) = dep(\lambda)$.

4.2.1 Properties of $\tilde{\mathcal{R}}$

The behavior of the modular algebra of the Grigorchuck 2-group $\tilde{\mathcal{H}}$ is very similar to that of the G_E-groups $\mathcal{H}$. In fact, for the ring $\tilde{\mathcal{R}}$ we can prove similar results to Lemma 1, Proposition 1 and Corollaries 1,2 from Section 4.1.1 as we can see below:

Lemma 3. *Let $\mathcal{I}$ be an ideal of $\tilde{\mathcal{R}}$ and $\lambda = (\lambda_0, \lambda_1)\, a_* \in \mathcal{I}$. Then, for all $c \in \gamma_3\left(\tilde{\mathcal{H}}\right)$, we have $((c+1)^2\lambda_i(c+1), 0) \in \mathcal{I}$, $i = 0, 1$.*

Proof. We observe that given an element $c \in \gamma_3\left(\tilde{\mathcal{H}}\right)$, then $(1, c), (c, 1) \in \gamma_3\left(\tilde{\mathcal{H}}\right)$ (see [5]). Thus if $\lambda = (\lambda_0, \lambda_1)\, \tau_* \in \mathcal{I}$ we can consider a sequence $\{\vartheta_i\}$ of elements in $\mathcal{I}$, constructed in the following way:

$$\begin{aligned} \vartheta_1 &= \lambda^{(c,1)} + \lambda \\ \vartheta_2 &= \vartheta_1^{(1,c)} + \vartheta_1 \\ \vartheta_3 &= \vartheta_2 a \\ \vartheta_4 &= (c, 1)\vartheta_3 + \vartheta_3 \\ \vartheta_5 &= (c, 1)\vartheta_4 = ((c+1)^2\lambda_0(c+1), 0). \end{aligned}$$

If we consider $\vartheta_4 = (1, c)\vartheta_3 + \vartheta_3$, we have $\vartheta_5 = (0, (c+1)^2\lambda_1(c+1))$ and the lemma is proved.

It is remarkable that if $\lambda = (\lambda_0, \lambda_1)\, a_* \in \tilde{\mathcal{R}}$, with $\lambda_i \in \mathcal{F}(Y, \tilde{\mathcal{R}})$, $i = 0, 1$, then for some $j = 0, 1$, there exists a linear combination β of the λ_i such that $\lambda a_*^j = \beta a_*$. In fact, if $\lambda_0 = \lambda_1$ then $\lambda = \lambda_0 a_*$, otherwise we have $\lambda a_* = (\lambda_0 + \lambda_1)\, a_*$.

Proposition 2. *Let $\mathcal{I}$ be a 2-sided non-zero ideal in $\tilde{\mathcal{R}}$. Then, there exists $c \in \gamma_3\left(\tilde{\mathcal{H}}\right)$, $c \neq 1$ such that $\mathcal{I} \supseteq N + 1$, where $N =< c >^{\tilde{\mathcal{H}}}$.*

Proof. It is not necessary to explicit the proof of this result because it is very similar to the proof of Proposition 1. It depends on the depth $depth(\alpha)$, where $\alpha = \alpha_0 + \alpha_1\tau \in \mathcal{I}$, $\alpha \neq 0$ and the previous lemma and remark are strongly used.

As a consequence, we have:

Corollary 4. *$\tilde{\mathcal{R}}$ is a just-infinite ring.*

Then we have a dichotomy to decide:

$$J\left(\tilde{\mathcal{R}}\right) = \{0\} \text{ or } J\left(\tilde{\mathcal{R}}\right) = \varpi(\tilde{\mathcal{R}}).$$

In this case the question is open but in the modular algebra $\tilde{\mathcal{R}}$ we observe an interesting phenomenon as we can see in the next section.

4.2.2 Nil elements in $\varpi(\tilde{\mathcal{R}})$

In contrast to the fact that in the ring associated to the Gupta-Sidki 3-group, $\mathbb{F}_3[y_*x_*]$ is a polynomial algebra, where $x_* = x^2 + x + 1$ and $y_* = y^2 + y + 1$ (see [10]), we can prove that the elements d_*, b_*, c_* and a_* generate a nil multiplicative semigroup showing that $\tilde{\mathcal{H}}$ has a thinning algebra of a different nature from $\mathcal{H}$.

The following simple identities are useful in our calculations.

Lemma 4.
(i) $a_*(u_0, u_1) = (u_0, u_1) + (u_1, u_0)a$;
(ii) $a_*(u_0, u_1)a_* = (u_0 + u_1)^{[1]}a_*$.

Proof. Since that $a_* = 1 + a$, then $a_*(u_0, u_1) = (u_0, u_1) + a(u_0, u_1) = (u_0, u_1) + (u_1, u_0)a$ and *(i)* is proved. To prove *(ii)*, it is sufficient to observe that $aa_* = a_*$ and the action of a on (u_0, u_1).

We want to know about the nilpotency of words $w = u_1a_*u_2a_*...u_ka_*$, where $u_i \in \{d_*, b_*, c_*\}$. Before proving our main result, we will establish some notations and definitions. The *length* of the monomial w is defined as $\mid w \mid= k$. (the number of occurrences of $a_*'s$).

We define the maps ϕ_0 and ϕ_1 on the words w of length even k by:

$$\phi_0(b_*) = \phi_0(c_*) = a_*, \ \phi_0(d_*) = 0$$
$$\phi_0(a_*b_*a_*) = c_*, \ \phi_0(a_*c_*a_*) = d_*, \ \phi_0(a_*d_*a_*) = b_*$$

and

$$\phi_1(b_*) = c_*, \ \phi_1(c_*) = d_*, \ \phi_1(d_*) = b_*$$
$$\phi_1(a_*b_*a_*) = \phi_1(a_*c_*a_*) = a_*, \ \phi_1(a_*d_*a_*) = 0$$

By the previous lemma, we have that $w = (\phi_0(w), \phi_1(w))a_*$, if $\mid w \mid$ is even and $w^2 = (\phi_0(w^2), \phi_1(w^2))a_*$, if $\mid w \mid$ is odd.

Then, we have

$$w^{2^n} = \begin{cases} \left((\phi_0(w))^{2^n}, (\phi_1(w))^{2^n}\right) a_*, \text{if } k \text{ is even,} \\ \left((\phi_0(w^2))^{2^{n-1}}, ((\phi_1(w^2))^{2^{n-1}}\right) a_*, \text{ if } k \text{ is odd.} \end{cases}$$

We call *d-term* the monomial which contains at least an occurrence of d_* and now we are ready to prove the next result.

Theorem 2. *The multiplicative semigroup generated by d_*, b_*, c_* and a_* is infinite and nil of exponent 8.*

Proof. In fact, the monomial $w = u_1a_*u_2a_*...u_ka_*$ always has degree of nilpotency less than or equal to 8, where $a_i \in \{d_*, b_*, c_*\}$. To prove that, we consider 2 cases:

Case 1. $|w|$ odd

If w is a d-term then $w^2 = 0$ since for $i = 0, 1$, we have that $\phi_i(w^2)$ contains either $\phi_i(d_*)$ or $\phi_i(a_*d_*a_*)$ and one of them is 0.

Otherwise $u_t \neq d_*$, $\forall t$. Thus, $|\phi_i(w^2)|$ is odd, for $i = 0, 1$, and contain only occurrences of $d_*'s$ and $c_*'s$. If it is a d-term, then $\phi_i(w^2)^2 = 0$ and so $w^4 = 0$. If it contains just $c_*'s$ then $\phi_j(\phi_i(w^2)^2)$ will be a d-term, so $w^8 = 0$.

Case 2. $|w|$ even

In this case, we have $w^8 = ((\phi_0(w))^8, (\phi_1(w))^8)\, a_*$ and so we can use induction on the length $|w|$ to prove that $w^8 = 0$.

Clearly we have this semigroup is infinite since we can consider the elements $\delta^n(a_*b_*)$ where δ is defined by

$$b_* \mapsto d_* \mapsto c_* \mapsto b_* \text{ and } a_* \mapsto a_*c_*a_*$$

and notice that $\delta(a_*b_*) = (0, a_*b_*)$, $\delta^2(a_*b_*) = (*, *, *, a_*b_*)$, where $*$ represents some element in the semigroup. By observing these elements we have that the n-fold decomposition of $\delta^n(a_*b_*)$ has the element a_*b_* in its 2^nth position and so they are all distinct.

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References

[1] Gupta, N.; Sidki, S., *On the Burnside Problem for periodic groups,* Math. Z. 182 (1983), 385-388.

[2] ________________, *Some infinite p-groups*, Algebra i Logica 22 (1983), 584-589.

[3] ________________, *Extensions of groups by tree automorphisms,* Contemp. Math. 33 (1984), 232-246.

[4] Grigorchuk, R. I., *Burnside's Problem on periodic groups,* Funktsional Anal. i Prilozhen 14 (1) (1980), 53-54; English transl. in Functional Anal. Appl. 14 (1980), 41-43.

[5] ______________, *On the Hilbert-Poincaré Series of Graded Algebras Associated with Groups,* Mat. Sb. 180 (2) (1989), 207-225; English transl. in Math. USSR Sbornik 66 (1) (1990), 211-229

[6] ______________, *Just infinite branch groups*, New horizons pro-p groups, (M. P. F. du Santoy, A. Shalev, D. Segal, eds) Progress in Mathematics, Vol. 184; Birkhauser, Boston (2000), 121-179.

[7] Lysenok, I. G., *A system of defining relations for the Grigorchuck group,* Mat. Zametki 38 (1985), 503-516; English transl. in Math. Notes 38 (1985), 784-792.

[8] Sidki, S., *On a* 2*-generated infinite* 3*-group: the presentation problem,* Journal of Algebra 110 (1987), 13-23.

[9] Sidki, S., *On a* 2*-generated infinite* 3*-group: subgroups and automorphisms,* Journal of Algebra 110 (1987), 24-55.

[10] ______, *A primitive ring associated to a Burnside 3-group,* J. London Math. Soc. 55 (2) (1997), 55-64.

[11] Vovkivsky, T., *Infinite torsion groups arising as generalizations of the second Grigorchuck group*, Proc. Internat. Algebra Conference in memory of G. Kurosh (1999), 357-377 .

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