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VERBAL GENERALIZATIONS OF THE RESTRICTED BURNSIDE PROBLEM

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1. Introduction

It is now more than ten years since Zel'manov solved the Restricted Burnside Problem (RBP for short) [28, 29]. The solution had a profound impact on further development of group theory; an extensive research around the RBP has been carried out by many people in different places. It was discovered that the methods involved in the solution of the RBP can be very effective in treatment of other problems. In the present paper we discuss certain generalizations of the RBP. These are helpful in understanding the nature of questions that can be handled in the spirit of the RBP. To illustrate the techniques used in the proof of practically all main results presented here we prove in Section 6 Theorems 2.2, 3.5 and 3.6 (see the next sections). Most of the other results are given without proofs.

It has been known for some time that the following assertions are equivalent.

1.1 *Let m and n be positive integers. Then the order of any m-generated finite group of exponent n is $\{m, n\}$-bounded.*

1.2 *The class of locally finite groups of exponent n is a variety.*

1.3 *Any residually finite group of exponent n is locally finite.*

The Restricted Burnside Problem is exactly the question whether the first of the above assertions is true. In 1956 Hall and Higman reduced the problem to the case of prime-power exponent [9]. Basically, their results show that each of the assertions 1.1–1.3 is equivalent to that obtained by replacing the term "finite" with the term "nilpotent". Thus, the solution of the RBP is equivalent to each of the following statements.

1.4 *Let m and n be positive integers. Then the order of any m-generated nilpotent group of exponent n is $\{m, n\}$-bounded.*

1.5 *The class of locally nilpotent groups of exponent n is a variety.*

1.6 *Any residually nilpotent group of exponent n is locally nilpotent.*

All the statements 1.1 –1.6 are important for understanding the connexion between the RBP and other issues addressed here.

The following question is perhaps the best motivation for the results and problems discussed in this paper. It was posed by Mazurov in [12, question 13.34].

1.7 *Let G be a group satisfying the identity $[x, y]^n \equiv 1$. Does it follow that G' is periodic?*

The answer to the above question is now known to be negative: Deryabina and Kozhevnikov showed that for sufficiently big odd integers n there exist counter-examples [3]. Their methods are based on Ol'shanskii's techniques [16]. On the other hand, G' is periodic if $n = 2$ [14] or $n = 3$ [7, 15] (in the former case G' has exponent 4). In sharp contrast with the results of Deryabina and Kozhevnikov in the case that G is residually finite we have the following theorem.

Theorem 1.8 *Let $q = p^s$ be a prime-power, G a residually finite group such that $[x, y]^q = 1$ for all $x, y \in G$. Then G' is locally finite.*

Note that in general a periodic residually finite group need not be locally finite. The corresponding examples have been constructed in [1, 5, 6, 8, 23]. Theorem 1.8 was proved in [19] using the techniques developed by Zel'manov in his solution of the RBP. It is natural to look for a viewpoint from which both – the solution of the RBP and Theorem 1.8 – are seen as results of similar nature. Thus, this paper is an attempt to explain the relationship between Theorem 1.8 and the RBP.

2. Verbal subgroups in residually finite groups

If w is a word in variables $x_1, \dots, x_t$ we think of it primarily as a function of t variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by the values of w. The word w is commutator if the sum of the exponents of any variable involved in w is zero. Otherwise we say that w is a non-commutator word. Let us consider the following problem.

Problem 2.1 *Let n be a positive integer and w a word. Assume that G is a residually finite group such that any w-value in G has order dividing n. Does it follow that the verbal subgroup $w(G)$ is locally finite?*

Of course the RBP is precisely Problem 2.1 with $w = x$. In fact it is easy to see that the answer to Problem 2.1 is positive whenever w is any non-commutator word. Indeed, suppose $w(x_1, \dots, x_t)$ is such a word and that the sum of the exponents of x_i is $r \neq 0$. Now, given a residually finite group G, substitute the unit for all the variables except x_i and an arbitrary element $g \in G$ for x_i. We see that g^r is a w-value for all $g \in G$. Hence G satisfies the identity $x^{nr} \equiv 1$ and therefore is locally finite by the result of Zel'manov.

Hence, Problem 2.1 is essentially about commutator words. Theorem 1.8 shows that if n is a prime-power and $w = [x, y]$, the answer is positive. In this paper we will work with multilinear commutators. A word w is called a multilinear commutator of weight t if it has form of a multilinear Lie monomial in precisely t independent variables. Particular examples of multilinear commutators are the derived words, defined by the equations:

$$\delta_0(x) = x,$$

$$\delta_k(x_1, \dots, x_{2^k}) = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1} \dots, x_{2^k})],$$

and the lower central words:

$$\gamma_1(x) = x,$$

$$\gamma_{k+1}(x_1, \ldots, x_{k+1}) = [\gamma_k(x_1, \ldots, x_k), x_{k+1}].$$

The next result was obtained in [21].

Theorem 2.2 *Let $q = p^s$ be a prime-power and w a multilinear commutator. Assume that G is a residually finite group such that any w-value in G has order dividing q. Then the verbal subgroup $w(G)$ is locally finite.*

The lack of an analogue of the Hall-Higman theory corresponding to Problem 2.1 is here partially compensated by the hypothesis that q is a prime-power. However even in this case the reduction to residually nilpotent groups involves certain rather complicated tools, Thompson's classification of minimal non-soluble finite groups among others [24]. A proof of Theorem 2.2 will be given in Section 6.

3. Varieties of groups

There is another way to link the solution of the RBP and Theorem 1.8.

Problem 3.1 *Let $n \geq 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally finite. Is that a variety?*

Problem 3.2 *Let $n \geq 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally nilpotent. Is that a variety?*

In the case $w = x$ both Problem 3.1 and Problem 3.2 are equivalent to the RBP. Furthermore, similarly to the situation with Problem 2.1 both Problem 3.1 and Problem 3.2 have positive solutions whenever w is a non-commutator word. Clearly, a positive solution of Problem 3.1 would have implied that of both Problem 2.1 and Problem 3.2. In particular Theorem 1.8 could be deduced from the corresponding case of Problem 3.1. On the other hand, we have no reason to immediately expect a positive solution of Problem 3.1. What we can do now is to deduce Theorem 1.8 from the next theorem [22], which provides a positive result related to Problem 3.2.

Theorem 3.3 *Given positive integers k and n, let $\mathfrak{X} = \mathfrak{X}(k,n)$ be the class of all groups G such that $\gamma_k(G)$ is locally nilpotent and $[x_1, x_2, \ldots, x_k]^n = 1$ for any $x_1, x_2, \ldots, x_k \in G$. Then $\mathfrak{X}$ is a variety.*

To deduce Theorem 1.8 from Theorem 3.3 one notices that if Q is any finite quotient of a group G satisfying the hypothesis of Theorem 1.8 then Q' is a p-group. It follows that G' is residually nilpotent. Hence, G residually belongs to the variety $\mathfrak{X}(2,q)$, which of course implies $G \in \mathfrak{X}(2,q)$. Thus, G' is locally nilpotent and locally finite.

The free group of countable rank of the variety $\mathfrak{X}(2,q)$ seems to be a good candidate for a solution of the following problem.

Problem 3.4 (Kovács, [12, question 8.21]) *Suppose the group G is free in some variety. Can G' be periodic of infinite exponent?*

It can be extracted from the proof of Theorem 3.3 that if G is a finitely generated group in $\mathfrak{X}(k,n)$ then the exponent of $\gamma_k(G)$ is finite and bounded in terms of k, n and the number of generators of G. We conjecture that if G is infinitely generated then the exponent of $\gamma_k(G)$ can be infinite. More precisely, let $X = X(2,q)$ be a free group of infinite rank of the variety $\mathfrak{X}(2,q)$. We know that X' is locally finite. We conjecture that the exponent of X' is infinite whenever $q \neq 2$.

Problem 3.2 also has a positive solution for the derived words δ_k.

Theorem 3.5 *Given positive integers k and n, the class of all groups G in which every δ_k-value has order dividing n and $G^{(k)}$ is locally nilpotent is a variety.*

One can show (and this is not obvious at all) that if n is a prime-power then the assumption that $G^{(k)}$ is locally nilpotent can be replaced in the hypothesis of Theorem 3.5 by the weaker assumption that $G^{(k)}$ is locally finite. This gives us the following theorem.

Theorem 3.6 *Given a positive integer k and a prime-power n, the class of all groups G in which every δ_k-value has order dividing n and $G^{(k)}$ is locally finite is a variety.*

Theorem 3.5 and Theorem 3.6 will be proved in Section 6.

4. The Engel condition

Let n be a positive integer. If x, y are elements of a group G, we define

$$[y,_0 x] = y;\ [y,_n x] = [[y,_{n-1} x], x].$$

An element x is called a (left) n-Engel element if $[g,_n x] = 1$ for any $g \in G$. A group G is called n-Engel if all elements of G are n-Engel. It is a long-standing problem whether any n-Engel group is locally nilpotent. In [26] Wilson proved that this is true if G is residually finite. In fact the Engel condition in residually finite groups can be treated using, by and large, the same techniques as those developed in the solution of the RBP. In particular, we have the following analogues of Theorem 2.2 and Theorem 3.3.

Theorem 4.1 *Let n be a positive integer and w a multilinear commutator. Assume that G is a residually finite group such that any w-value in G is n-Engel. Then the verbal subgroup $w(G)$ is locally nilpotent.*

Theorem 4.2 *Given positive integers k and n, consider the class of all groups G such that $\gamma_k(G)$ is locally nilpotent and $[x_1, x_2, \ldots, x_k]$ is n-Engel for any $x_1, x_2, \ldots, x_k \in G$. This class is a variety.*

5. Associated Lie algebras

In this section we briefly describe some Lie-theoretic machinery necessary for the proofs of most of the discussed results. Zel'manov's Theorem 5.4 can be accurately characterized as a quintessence of the Lie-theoretic part of the solution of the RBP. Moreover, it has provided a clue for solving a number of other group-theoretic problems (see for example [20]).

Let G be residually a finite p-group. The terms of the lower central series of G will be denoted by $\gamma_j(G)$. Write $D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k}$. The subgroups D_i form a central series of G known as the Zassenhaus-Jennings-Lazard series (see [10, Chapter 8]). Set $L(G) = \oplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field $\mathbb{F}_p$ with p elements. In fact $L(G)$ even has the structure of a restricted Lie algebra (Lie p-algebra) but we shall treat it as just a Lie algebra. Let us denote by $L_p(G)$ the subalgebra of $L(G)$ generated by D_1/D_2. Fix a positive number c, and assume that G is generated by $a_1, a_2, \ldots, a_m$. Let $\rho_1, \rho_2, \ldots, \rho_s$ be the list of all commutators in $a_1, a_2, \ldots, a_m$ of weight at most c. Here s obviously is $\{c, m\}$-bounded. The following lemma is implicit in Zel'manov [30, p. 71].

Lemma 5.1 *If $L_p(G)$ is nilpotent of class c then for any $i \geq 1$ the group G can be written as a product $G = \langle\rho_1\rangle\langle\rho_2\rangle \ldots \langle\rho_s\rangle D_{i+1}$ of cyclic subgroups generated by the ρ_j's and D_{i+1}. In particular, if every ρ_j has finite order then $G = \langle\rho_1\rangle\langle\rho_2\rangle \ldots \langle\rho_s\rangle$.*

Proof. For any positive integer i the subgroup D_i is generated by D_{i+1} and elements of the form $[b_1, \ldots, b_j]^{p^k}$, where $jp^k \geq i$ and $b_1, \ldots, b_j \in \{a_1, \ldots, a_m\}$. This can be shown using for example the Collection Formula [10, p. 240].

The lemma will be proved by induction on i, the case $i = 0$ being trivial. Assume that $i \geq 1$ and

$$G = \langle\rho_1\rangle\langle\rho_2\rangle \ldots \langle\rho_s\rangle D_i.$$

Then any element $x \in G$ can be written in the form

$$x = \rho_1^{\alpha_1} \rho_2^{\alpha_2} \ldots \rho_s^{\alpha_s} y, \qquad (*)$$

where $y \in D_i$. Without any loss of generality we can assume that $D_{i+1} = 1$.

By the remark made in the beginning of the proof we can write

$$y = (\sigma_1^{p^{k_1}})^{\beta_1} (\sigma_2^{p^{k_2}})^{\beta_2} \dots (\sigma_t^{p^{k_t}})^{\beta_t}, \qquad (**)$$

where each σ_n is of the form $[b_1, \dots, b_j]$, with $jp^{k_n} \geq i$ and $b_1, \dots, b_j \in \{a_1, \dots, a_m\}$

Let $\tilde{a}_l$ denote $a_l D_2 \in L_p(G)$; $l = 1, \dots, m$. By the hypothesis $L_p(G)$ is nilpotent of class c, that is $[\tilde{b}_1, \dots, \tilde{b}_{c+1}] = 0$ for any $b_1, \dots, b_{c+1} \in \{a_1, \dots, a_m\}$. This implies that $[b_1, \dots, b_{c+1}] \in D_{c+2}$ for any $b_1, \dots, b_{c+1} \in \{a_1, \dots, a_m\}$ and $\gamma_{c+1} \leq D_{c+2}$. Then it follows from the Collection Formula that for any $d \geq c+1$ we have $\gamma_d \leq D_{d+1}$.

Now, if σ_n is of the form $[b_1, \dots, b_j]$ with $j \geq c+1$ then

$$\sigma_n^{p^{k_n}} \in \gamma_j^{p^{k_n}} \leq D_{j+1}^{p^{k_n}} \leq D_{(j+1)p^{k_n}} \leq D_{i+1} = 1.$$

Hence we can assume that each σ_n is of the form $[b_1, \dots, b_j]$ with $j \leq c$, in which case σ_n belongs to the list $\rho_1, \rho_2, \dots, \rho_s$.

It remains to remark that $\sigma_n^{p^{k_n}} \in Z(G)$. Comparing now (*) and (**) we obtain that

$$x \in \langle \rho_1 \rangle \langle \rho_2 \rangle \dots \langle \rho_s \rangle,$$

as required.

□

Let $x \in G$, and let $i = i(x)$ be the largest integer such that $x \in D_i$. We denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$.

Lemma 5.2 *(Lazard, [13, p. 131]) For any $x \in G$ we have $(ad\,\tilde{x})^p = ad\,(\widetilde{x^p})$. In particular, if $x^n = 1$ then $\tilde{x}$ is ad-nilpotent of index at most n.*

Any group law that holds in G implies certain polynomial identity in the algebra $L_p(G)$. Wilson and Zel'manov describe in [27, Theorem 1] an effective algorithm allowing to write explicitly the polynomial identity when the group law is given (in fact they describe an algorithm that works in a more general situation but we shall not require this). Thus, we have

Lemma 5.3 *Let G be a group satisfying an identity $w \equiv 1$. Then there exists a non-zero Lie polynomial f over $\mathbb{F}_p$ depending only on p and w such that the algebra $L_p(G)$ satisfies the identity $f \equiv 0$.*

In view of Lemma 5.1, it is extremely important to have criteria for a Lie algebra to be nilpotent of bounded class. The following result was proved in [11, Corollary of Theorem 4] using a profound theorem of Zel'manov [30, III(0.4)].

Theorem 5.4 *Let L be a Lie algebra over $\mathbb{F}_p$ generated by $a_1, a_2, \ldots, a_m$. Assume that L satisfies the identity $f \equiv 0$ and that each monomial in the generators $a_1, a_2, \ldots, a_m$ is ad-nilpotent of index at most n. Then L is nilpotent of $\{k, m, n\}$-bounded class.*

6. Some proofs

In this section we illustrate the use of Zel'manov's techniques. In particular, Theorem 2.2, Theorem 3.5 and Theorem 3.6 will be proved. The proof of Theorem 3.5 is based on the next proposition.

Proposition 6.1 *Given positive integers k and n, let G be a group such that $G^{(k)}$ is locally nilpotent and every δ_k-value has order dividing n. Let $a_1, a_2, \ldots, a_m \in G$ be δ_k-values. Then the order of $H = \langle a_1, a_2, \ldots, a_m \rangle$ is $\{k, m, n\}$-bounded.*

Proof. Since $G^{(k)}$ is locally nilpotent, it is clear that H is finite and any prime divisor of $|H|$ is a divisor of n. Hence, it is sufficient to bound the order of the Sylow p-subgroup of H for any prime p. Since $O_{p'}(G^{(k)})$ is normal in G, and we can pass to the quotient $G/O_{p'}(G^{(k)})$. Thus, we assume from now on that H is a p-group and n is a p-power. Let $L = L_p(H)$. We know that the the identity ${\delta_k}^n \equiv 1$ holds in H so it follows by Lemma 5.3 that there exists a non-zero Lie polynomial f over $\mathbb{F}_p$ depending only on p, k and n such that the algebra $L_p(G)$ satisfies the identity $f \equiv 0$.

Consider an arbitrary Lie monomial σ in the generators $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m$ of $L_p(H)$ and let ρ be the group commutator in $a_1, a_2, \ldots, a_m$ having the same bracketage as σ. The definition of $L_p(H)$ yields that either $\sigma = 0$ or $\sigma = \tilde{\rho}$. Since $\rho^n = 1$, Lemma 5.2 implies that σ is ad-nilpotent of index at most n. Theorem 5.4 now says that $L_p(H)$ is nilpotent of class depending only on k, m, n. Combining this with Lemma 5.1 we conclude that there exists a $\{k, m, n\}$-bounded number s such that H can be written as a product of at most s cyclic subgroups each of order at most n. Therefore H is of order at most n^s, as required.

□

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5

Let $\mathfrak{Y}$ be the class of all groups G in which every δ_k-value has order dividing n and $G^{(k)}$ is locally nilpotent. We want to show that the class $\mathfrak{Y}$ is a variety. Clearly, the class $\mathfrak{Y}$ is closed with respect to taking quotients and subgroups of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{Y}$ then $D \in \mathfrak{Y}$. Of course, the the identity ${\delta_k}^n \equiv 1$ holds in D so it remains only to show that $D^{(k)}$ is locally nilpotent. Let T be any finite subset of $D^{(k)}$. Clearly, there exist finitely many δ_k-values $h_1, \ldots, h_m \in D$ such that $T \leq \langle h_1, \ldots, h_m \rangle$. Thus, it is sufficient to show that the group $H = \langle h_1, \ldots, h_m \rangle$ is nilpotent. Note that $D^{(k)}$ is residually locally nilpotent. If Q is any locally nilpotent quotient of $D^{(k)}$, by Proposition 6.1, the order of the image of H in Q is finite and $\{k, m, n\}$-bounded. So it follows that actually the order of the image of H in Q does not depend on Q. We conclude that H is finite. Since H is residually locally nilpotent, it is nilpotent. The proof is complete.

Deduction of Theorem 3.6 from Theorem 3.5

We shall have to deal with finite groups in which all δ_k-values are p-elements (for a fixed positive integer k). Recall that in 1951 Ore conjectured that if G is any non-abelian finite simple group then each element x of G is a commutator, that is $x = [a, b]$ for suitably chosen $a, b \in G$ [17]. This conjecture has been confirmed for many finite simple groups (see [4] and references therein). We shall say that a group G has Ore's property if any element of G is a commutator. Obviously, if G has Ore's property then any element of G is a δ_k-value.

Proposition 6.2 *Let p be a prime, G a finite group in which all δ_k-values are p-elements. Then $G^{(k)}$ is a p-group.*

Proof. Assume that the result is false and let G be a counter-example of minimal possible order. Suppose G has a proper normal subgroup N. Then by the induction hypothesis both G/N and N are soluble and so is G. Therefore $G' \neq G$ and again by induction we conclude that the kth derived group of G' (of course this is exactly $G^{(k+1)}$) is a p-subgroup. Passing to the quotient $G/G^{(k+1)}$ we can assume that $G^{(k)}$ is abelian, in which case the claim is immediate since $G^{(k)}$ is generated by p-elements.

Hence it is sufficient to deal with case that G is simple. Since every proper subgroup of G is soluble, G belongs to the list of minimal simple groups determined by Thompson [24]. In particular it follows that G is isomorphic to either $PSL_n(q)$ or the simple group of Suzuki type $Sz(q)$ over a finite field. In either case G has Ore's property. For the groups $PSL_n(q)$ this was established in R.C. Thompson [25] while for the Suzuki groups this follows from the proof of Theorem(4.1) in Arad, Chillag and Moran [2]. Hence any element of G is a δ_k-value. We see that G consists of p-elements and so is a p-group, a contradiction.

□

Theorem 3.6 follows readily from Theorem 3.5 and the above proposition.

Deduction of Theorem 2.2 from Theorem 3.6

Recall that Theorem 2.2 says that if $q = p^s$ is a prime-power and w is a multilinear commutator then, for any residually finite group G satisfying the identity $w^q \equiv 1$, the corresponding verbal subgroup $w(G)$ is locally finite. This will be deduced from Theorem 3.6 using the following lemmas.

Lemma 6.3 *Let G be a group, w a multilinear commutator of weight t. Then every δ_t-value in G is a w-value.*

Proof. The case $t = 1$ is quite obvious so we assume that $t \geq 2$ and use induction on t. Write $w = [w_1, w_2]$, where w_1 and w_2 are multilinear commutators of weight t_1 and t_2 respectively, $t_1 + t_2 = t$, and the variables involved in one of the words w_1, w_2 do not occur in the other. Let t_0 be the maximum of t_1, t_2. By the induction hypothesis any δ_{t_0}-value in G is a w_1-value as well as a w_2-value. Since $w = [w_1, w_2]$, it follows that the set of w-values contains the set of elements of the form $[x, y]$, where x, y range independently through the set of δ_{t_0}-values. Hence any commutator of δ_{t_0}-values represents a w-value. It remains to remark that $t_0 + 1 \leq t$ so the lemma follows.

□

Let w be a multilinear commutator of weight t. In the next lemma we shall require the concept of a subcommutator of weight $s \leq t$ of w. This can be defined by backward induction on s in the following way. The only subcommutator of weight t of w is w itself. If $s \leq t-1$ a multilinear commutator v of weight s is a subcommutator of w if and only if there exists a subcommutator u of weight $> s$ of w and a multilinear commutator v_1 such that either $u = [v, v_1]$ or $u = [v_1, v]$. It is quite obvious that if v is a subcommutator of w then $w(G) \leq v(G)$ for any group G.

Lemma 6.4 *Let w be a multilinear commutator, G a soluble group in which all w-values have finite order. Then the verbal subgroup $w(G)$ is locally finite.*

Proof. Let G be a counter-example whose derived length is as small as possible, and let T be the last non-trivial term of the derived series of G. Passing to the quotient over the subgroup generated by all normal locally finite subgroups of G we can assume that G has no non-trivial normal locally finite subgroups. Since T is abelian, it follows that no w-value lies in $T \setminus \{1\}$. Let $s = s(w, G)$ be the smallest number such that any subcommutator of weight s of w has no values in $T \setminus \{1\}$. Obviously, $s \geq 2$ since $T \neq 1$. We can choose a subcommutator $v = [v_1, v_2]$ of weight $\geq s$ of w such that both v_1 and v_2 are subcommutators of weight $< s$, at least one of which having non-trivial values in $T \setminus \{1\}$. Let H_i be the subgroup of T generated by the v_i-values contained in T; $i = 1, 2$. By the choice of v at least one of these subgroups is non-trivial. Since v has no values in $T \setminus \{1\}$, it follows that $H_1 \le C_G(v_2(G))$ and $H_2 \le C_G(v_1(G))$. Taking into account that $w(G) \le u(G)$ for any subcommutator u of w we conclude that H_1 and H_2 centralize the verbal subgroup $w(G)$. Hence both subcommutators v_1 and v_2 have no non-trivial value in the image of T in $G/C_G(w(G))$. This shows that $s(w, G/C_G(w(G))) \le s - 1$. The induction on $s(w, G)$ now shows that $w(G)/Z(w(G))$, the image of $w(G)$ in $G/C_G(w(G))$, is locally finite. Then, by Schur's Theorem, the derived group of $w(G)$ is locally finite [18, Part 1, Corollary to Theorem 4.12]. But then $w(G)$, being a group generated by elements of finite order, must be locally finite.

□

Theorem 2.2 can now be proved as follows. By Lemma 6.3 there exists $k \ge 1$ such that any δ_k-value in G is a w-value. Hence any δ_k-value in G has order dividing q. It follows that G belongs to the variety of all groups satisfying the identity ${\delta_k}^q \equiv 1$ and having the kth derived group locally finite (this is a variety by Theorem 3.6). Thus $G^{(k)}$ is locally finite. It is straightforward from Lemma 6.4 that $w(G)/G^{(k)}$ is likewise locally finite and the theorem follows.

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