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FC ELEMENTS OF ALGEBRAS AND ORDERS

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Abstract

Let $\mathcal{U}(R)$ denote the group of units of an associative ring with unity R. We study elements of R which have a finite conjugacy class under the action of the elements of $\mathcal{U}(R)$. In particular, we give a survey of known results in the case when R is a group ring and state some new results for algebras and orders.

Resumo

Seja $\mathcal{U}(R)$ o grupo de unidades de um anel associativo com unidade R. Estudamos elementos de R que têm classe de conjugação finita sob a ação dos elementos de $\mathcal{U}(R)$. Em particular, descrevemos os resultados conhecidos no caso em que R é um anel de grupo e enunciamos alguns resultados novos para álgebras e ordens.

1 Introduction

Given an associative ring R with unity, we shall denote by $\mathcal{U}(R)$ the group of units of R; i.e., the set of invertible elements of R. We recall that an *FC group* is a group G such that all of its elements have finite conjugacy classes in G. More generally, we denote by $\Phi(G)$ the *FC-center* of G, that is:

$$\Phi(G) = \{g \in G \mid [G : \mathcal{C}_G(g)] < \infty\}.$$

I. N. Herstein showed in [17] that if D is a division ring then $\Phi(\mathcal{U}D)$ coincides with $\mathcal{Z}(\mathcal{U}D)$, the centre of $\mathcal{U}D$. The study of the FC-center of groups of units of group rings started with papers by S. K. Sehgal and H. J. Zassenhaus [26], C.

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Polcino Milies [21] and G. Cliff and S. K. Sehgal [8]. Also, A. Williamson [30], studied elements of a periodic group G which have finite conjugacy class in the group of units of its integral group ring. These results also follow from a paper by A. A Bovdi [3]. A more general approach was given by S. K. Sehgal and H. J. Zassenhaus in [27]. This work was followed by several papers studying group rings over fields [22], [12].

In this short survey, we recall the origins of the theory of FC groups, then give a general view of the existing results about group rings with FC unit groups and, finally, state similar results in the more general context of algebras and orders.

2 FC groups

The results in this section are now standard. They are included here, with references to the original papers where they were published, only to serve as a historical introduction to the subject.

In 1948, R. Baer [1] introduced a series of finiteness conditions for groups which we list below:

- **(FC)** *Every element in the group G possesses only a finite number of conjugates in G.*
- **(LF)** *Every element in the group G is contained in a finite normal subgroup.*
- **(FO)** *There exists only a finite number of elements of any given order in the group G.*

Exploring the relations among these concepts, he obtained the following results.

Theorem 2.1 *A group G is* **(LF)** *if and only if it is* **(FC)** *and contains no element of infinite order.*

As a consequence, it is easy to see that **(FO)** implies **(LF)** which, in turn, implies **(FC)**.

Theorem 2.2 *A group G is FC if and only if every element is contained in a finitely generated normal subgroup and $G/\mathcal{Z}(G)$ is LF, where $\mathcal{Z}(G)$ denotes the centre of G.*

Theorem 2.3 *A group G is FO if and only if $\mathcal{Z}(G)$ is FO and, for every prime p, the factor group $G/\mathcal{Z}(G)$ contains only a finite number of elements of order a power of p.*

In 1951, B. H. Neumann [19] continued the study of FC groups and gave their fundamental properties:

Theorem 2.4 *Let G be an FC group. Then, the set T of elements of finite order in G is a characteristic subgroup of G, $G' \subset T$ and thus G/T is an abelian torsion-free group. Moreover, if G is finitely generated, then T is finite.*

This result has several consequences. Among these, we quote:

- If G is generated by a set of elements of finite order, then G is periodic (i.e., G=T). Moreover, if G is finitely generated, then G is finite.
- The torsion subgroup T is locally finite.
- (Baer) G is a periodic FC group if and only if G is LF.

Theorem 2.5 *If $[G : \mathcal{Z}(G)]$ is finite, then G' is finite. If G' is finite, then G is an FC group.*

FC centres of groups can be used to define a chain of subgroups and develop a theory similar to that of nilpotency. This was done in 1953 by F. Haimo [16]:

Set $\Phi_1(G) = \{g \in G \mid [G : \mathcal{C}_G(g)] < \infty\}$, the FC centre of G and, inductivelly, let $\Phi_{n+1}(G)$ be the subgroup of G such that:

$$\frac{\Phi_{n+1}(G)}{\Phi_n(G)} = \Phi_1\left(\frac{G}{\Phi_n(G)}\right).$$

The sequence of subgroups

$$\Phi_1(G) \subset \Phi_2(G) \subset \cdots \Phi_n(G) \subset \cdots$$

is called the *FC chain* of G. If there exists an integer n such that $\Phi_n(G) = G$ and $\Phi_{n-1}(G) \neq G$, we say that G is *FC-nilpotent* of *FC-class* equal to n.

The following results show the similarity of these ideas with ordinary nilpotency:

Theorem 2.6 *Let N be a normal subgroup of a group G such that $N \subset \Phi_n(G)$, for some integer n and such that there exists a positive integer k for which G/N is FC-nilpotent of class k. Then, G is FC-nilpotent of FC-class $c \leq n + k$.*

Corollary 2.7 *If $G' \subset \Phi_n(G)$ for some n, then G is FC-nilpotent of FC-class $c \leq n + k$.*

Shortly afterwards, J. Erdös [15] gave simpler proofs for some of Neumann's earlier results and also proved the following.

Theorem 2.8 *If G is a finitely generated FC group, then $[G : \mathcal{Z}(G)]$ is finite and thus G' is finite.*

He also used the theory of FC groups to porve a result of Ju. G. Fjodorov:

Theorem 2.9 *If in an infinite group G every subgroup containing at least two elements is of finite index, then G is a cyclic group.*

Aproximately at the same time, B. H. Neumann [20] studied *bounded* FC groups and proved the following.

Theorem 2.10 *The number of elements in each conjugacy class of a group G is bounded if and only if the derived group G' is finite.*

The theory of FC groups has attracted considerable attention and is well developped. The interested reader may consult the book by M. J. Tomkinson [28] or the more recent survey [29].

3 FC elements in group rings

Given a ring R and a group G, we shall denote by RG the *group ring of* G *over* R. The question of when the unit group $\mathcal{U}(RG)$ is FC was first considered by S. K. Sehgal and H. J. Zassenhaus in 1977 [26] when they characterized groups G such that $\mathcal{U}(\mathbb{Z}G)$ is FC:

Theorem 3.1 *Let G be a group and let T denote the set of elements of finite order in G. Then $\mathcal{U}(\mathbb{Z}G)$ is FC if and only if one of the following conditions holds:*

(i) T is central in G.

(ii) T is abelian, non-central and, for all $t \in T$ and all $x \in G$ we have that $xtx^{-1} = t^{\pm 1}$.

(iii) $T = E \times \mathcal{Q}_8$, where E is an elementary abelian 2-group,

$$\mathcal{Q}_8 \;=\; \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ aba^{-1} = b^{-1} \rangle$$

and conjugation by any element $x \in G$ induces an inner automorphism of $\mathcal{Q}_8$.

With this result, they were able to characterize also FC unit groups of KG, when K is a field of characteristic 0. The case when $char(K) = p > 0$ was completed in a sequence of two papers, by C. Polcino Milies [21] and G. Cliff and S. K. Sehgal [8]. The characterizations envolved the condition of every idempotent of KT being central in KG. After this condition was better understood in [9], [10] and [11], these results can be stated as follows.

Theorem 3.2 *Let G be a torsion group. Then $\mathcal{U}(KG)$ is an FC group if and only if either KG is finite or G is abelian.*

Theorem 3.3 *Assume that $char(K) = p > 0$ and that G is a non-torsion group which contains p-element. Then $\mathcal{U}(KG)$ is an FC group if and only if either G is abelian or G is a non-abelian FC group, $p = 2$, $T = \langle t \rangle \times A$ where $o(t) = 2$, A is a finite group of odd order, $G' = \langle t \rangle$ and T is central.*

For a given field K we shall denote by $P(K)$ the prime subfield of K.

Theorem 3.4 *Assume that char$(K) = p > 0$ and that G is a non-torsion group which contains no p-elements. Then $\mathcal{U}(KG)$ is an FC group if and only if either G is abelian or G is a non-abelian FC group, T is abelian and one of the following conditions holds:*

(i) KT is finite and for all $t \in T$ and all $x \in G$ we have that $t^x = t^{p^r}$ for some non-negative integer $r = r(x,t)$ which is a multiple of $[K : P(K)]$.

(ii) T is finite, central.

(iii) T is central, of the form $T = \mathbb{Z}(q^\infty) \times B$ for some prime $q \neq p$, $G' \subset \mathbb{Z}(q^\infty)$ and there exists an integer k such that K does not contain roots of unity of order q^k.

Theorem 3.5 *Let char$(K) = 0$ and assume that G is a non-torsion group. Then $\mathcal{U}(KG)$ is an FC group if and only if G is either abelian or a non-abelian FC group with T central and, if T is infinite, then T and K can be described as in part (iii) of the previous theorem.*

Results on the construction of the group of units of a group ring, in general, can be found in [5]. The description of group algebras having FC unit groups was also given independently by A. A. Bovdi in [4]. See also [6].

4 The supercentre of a group

In 1978, A. Williamsom [30] studied elements of a group G which have a finite conjugacy class in the unit group of the integral group ring of G. He proved the following.

Theorem 4.1 *Let G be a periodic group. An element $x \in G$ has a finite conjugacy class in $\mathcal{U}(\mathbb{Z}G)$ if and only if either:*

(i) x is central in G, or

(ii) $o(x) = 4$ *and* x *belongs to an abelian group* H *of index* 2 *in* G*, with*

$$G = \langle H, c \mid c^2 = x^2,\ h^c = h^{-1}, \forall h \in H \rangle.$$

In 1981, S. K. Sehgal and H. J. Zassenhaus [27] defined the *FC subring* of a ring R as:

$$FC(R) = \{x \in R \mid \text{the conjugacy class of } x \text{ under } \mathcal{U}(R) \text{ is finite}\}.$$

In that same paper, they also introduced the *supercentre* of a group G as:

$$\begin{aligned} S(G) &= G \cap FC(\mathbb{Z}G) = G \cap \Phi\mathcal{U}(\mathbb{Z}G) \\ &= \{g \in G \mid \text{the conjugacy class of } g \text{ in } \mathcal{U}(\mathbb{Z}G) \text{ is finite}\}. \end{aligned}$$

Theorem 4.2 *Let* G *be a finite group. Then, the FC subring of* $\mathbb{Z}G$ *consists of all those elements* $x \in \mathbb{Z}G$ *such that, for every irreducible representation* f *of* $\mathbb{Q}G$ *over* $\mathbb{Q}$ *for which* $f(\mathbb{Q}G)$ *is not a totally definite quaternion algebra, we have that* $f(x)$ *is central in* $f(\mathbb{Q}G)$*.*

Theorem 4.3 *Let* G *be a finite group. Then;*

$$T\left(\Phi(\mathcal{U}(\mathbb{Z}G))\right) = \pm S(G).$$

This last theorem is also contained in a paper by A. A. Bovdi [2].

Shortly afterwards, C. Polcino Milies and S. K. Sehgal [22] defined, for a group G and an arbitrary ring K, the *K-supercentre* of G as:

$$S_K(G) = G \cap FC(KG) = G \cap \Phi\left(\mathcal{U}(KG)\right).$$

A complete description of supercentres of groups over fields was given by S. P. Coelho and C. Polcino Milies [12].

5 FC centres of algebras and orders

Let D be an integral domain, K its field of fractions and A an algebra over K. We recall that an unital subring Λ of A is said to be a *D-order* in A if it has a finite basis when considered as a D-module and $K\Lambda = A$. The results that follow, concerning the FC-subring of algebras and orders were announced in [13] and the proofs will appear in [14].

The key remark for the results on FC-subrings of algebras that will be given below is the following.

Theorem 5.1 *Let G be a connected algebraic group. Then, every element with a finite conjugacy class is central.*

As a consequence, we obtain.

Theorem 5.2 *Let A be an algebra with unity over an infinite field K.*

(i) If A is finite dimensional, then $\mathcal{U}A$ is a connected linear algebraic group and, consequently,

$$\Phi(\mathcal{U}A) = \mathcal{Z}(\mathcal{U}A).$$

Moreover, A is generated by its units, as a vector space over K and, therefore, $\mathcal{U}A$ is FC if and only if A is commutative.

(ii) Every torsion unit of $\Phi(\mathcal{U}A)$ commutes with each algebraic unit of A and, consequently, $\Phi(\mathcal{U}A)$ is solvable of length at most 2.

(iii) Every element of $\Phi(\mathcal{U}A)$ commutes with each nilpotent element of A.

The results above were obtained using the fact that the group of units of a finite dimensional algebra can be viewed as a connected algebraic group. A more general version of these results, from another point of view, appears in [7].

For an order Λ in an algebra A, we have the following.

Theorem 5.3 *Let D be an infinite domain, K its field of fractions, A a finite dimensional $K-$algebra, Λ a $D-$order in A, $\mathcal{J} = \mathcal{J}(A)$ the Jacobson Radical of A and $\overline{A} = A/\mathcal{J}$. Assume that $Hom_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If every minimal ideal of $\overline{A}$ which is a division ring is isomorphic to K, then*

$$\Phi(\mathcal{U}\Lambda) \subset \mathcal{Z}(A),$$

Corollary 5.4 *Let D and K be as above, A be a finite dimensional $K-$algebra and Λ an order in A. Assume that $Hom_A(P_i, P_j) = 0$ for every pair of non-isomorphic principal modules P_i, P_j of multiplicity 1 in A. If K is a splitting field for A, then*

$$\Phi(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

Corollary 5.5 *Let D and K be as above, A be a semisimple finite dimensional $K-$algebra and Λ a $D-$order in A. If A has no minimal ideal which is a non-commutative division ring then*

$$\Phi(\mathcal{U}\Lambda) \subset \mathcal{Z}(A).$$

Theorem 5.6 *Let D be an infinite domain and R a D-algebra.*

(i) If R is torsion free as a D-module then

$$\Phi(GL_n(R)) = \Phi(\mathcal{U}R)I,$$

where I is the identity matrix of $M_n(R)$.

(ii) If $char(D) = 0$ and $n > 1$ then

$$\Phi(GL_n(R)) = \mathcal{Z}(GL_n(R)).$$

Applying this results in the context of group rings, we may obtain the following.

Lemma 5.7 *Let K be a field and let G be a subgroup of $GL(2, K)$. Then*

(i) if $a \in GL(2, K)$ is noncentral, its centralizer in $GL(2, K)$ is abelian and

(ii) either $\Phi(G) = \mathcal{Z}(G)$ or G is abelian-by-finite.

Proposition 5.8 *Let $G = K_8 \times \langle c \rangle$, where c is an element of order p, an odd prime, and $K_8 = \langle a, b \rangle$ is the quaternion group of order 8. Then*

$$\Phi(\mathcal{U}(\mathbb{Z}G)) = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)).$$

Proposition 5.9 *Let G be a finite group and assume that $T\Phi(\mathcal{U}(\mathbb{Z}G))$ is non-abelian. Then G is a 2-group.*

The theorem of Williamson quoted above follows easily from these results.

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