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LIMITS OF GRAPHS, II

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Abstract

The family of maps of $\mathbb{P}^1$ into $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2,2)$ is parameterized by an open subset of $\mathbb{P}^5 \times \mathbb{P}^5$. We show that the parameter space of such maps admits a compactification obtained from $\mathbb{P}^5 \times \mathbb{P}^5$ by a sequence of 4 blowups along explicit, smooth centers. It yields a desingularization of the corresponding component of the Hilbert scheme of curves in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and a description of all flat limits of such graphs.

Resumo

A família de mapas de $\mathbb{P}^1$ em $\mathbb{P}^1 \times \mathbb{P}^1$ de bi-grau $(2,2)$ é parametrizada por um aberto de $\mathbb{P}^5 \times \mathbb{P}^5$. Mostramos que esse aberto admite uma compactificação obtida por uma seqüência de 4 explosões com centros lisos e explícitos. Isto fornece uma dessingularização da componente correspondente do esquema de Hilbert de curvas em $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ e uma descrição de todos os limites planos de tais gráficos.

Introduction

Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a map defined by

$$\varphi(z) = \Big((F_0(z) : F_1(z)), (G_0(z) : G_1(z))\Big)$$

where F_0, F_1, G_0 and G_1 are homogeneous polynomials of degree d in the homogeneous coordinates z_0, z_1 such that $\gcd(F_0, F_1) = \gcd(G_0, G_1) = 1$. The graph,

$$\Gamma_\varphi \ \subset \ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

is a curve defined by the of equations $F = G = 0$, with

$$\begin{cases} F := x_1 F_0(z_0, z_1) - x_0 F_1(z_0, z_1), \\ G := y_1 G_0(z_0, z_1) - y_0 G_1(z_0, z_1). \end{cases}$$

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Let $\mathbb{X}_d$ denote the parameter space of pairs (F, G). Our aim is to describe a smooth compactification of the parameter space of graphs of bidegree (2,2). In contrast with the (1,1) case (cf. 1.7), now the corresponding component of the Hilbert scheme is dominated by a sequence of four blowups along explicit, smooth centers

$$\begin{array}{lllllllll} \mathbb{X}_2^4 & \longrightarrow & \mathbb{X}_2^3 & \longrightarrow & \mathbb{X}_2^2 & \longrightarrow & \mathbb{X}_2^1 & \longrightarrow & \mathbb{X}_2 \\ \downarrow & & & & & & & & \\ \mathbb{H}\mathrm{ilb}. & & & & & & & & \end{array}$$

The vertical map is birational to its image and contracts a smooth divisor.

Hilbert schemes were introduced by A. Grothendieck [3] as a fundamental tool for the study of families of algebraic varieties. He proved that the Hilbert scheme of closed subschemes with a given Hilbert polynomial and contained in a given projective scheme X is in fact a closed subscheme of a suitable Grassmann variety. Hartshorne [4] has shown that for X=$\mathbb{P}^n$, the Hilbert scheme is also connected.

Not much is known about specific Hilb's. Basic questions such as the number and dimensions of irreducible components remain largely unanswered. Grassmannians and families of hypersurfaces of fixed degree are among the few, essentially trivial examples of well known Hilbert schemes. The Hilbert scheme of zero cycles on a smooth surface is also well understood, cf. Iarrobino [6]. Piene and Schlessinger [11] treated the first non trivial case, the family of twisted cubics. An alternative, more elementary compactification for a parameter space of twisted cubics has been constructed in [16]. The Hilbert scheme component of the complete intersections of 2 quadrics has been studied in [1]. Complete intersections of a quadric and a cubic has been considered in [12].

Our modest study of the specific Hilbert scheme component considered here was partly motivated by the first author's work [8] on the moduli space of branched superminimal immersions from the Riemann sphere to the 4-sphere. It is, to the best of our knowledge, the first example in a multi-graded setting (other than the much easier (1,1) case already mentioned).

We summarize the main idea. We wish to resolve the indeterminacies of the rational map

$$\Gamma : \mathbb{X}_d \cdots \rightarrow \mathbb{H}\mathrm{ilb}. \tag{1}$$

The locus where Γ is not defined corresponds to pairs (F, G) with gcd $\neq 1$. Such pairs are detected by the requirement that there exist relations $AF + BG = 0$ where A, B are polynomials of z-degree lower than d. This fits in happily with the following fact: flat limits along 1-parameter families $\{\Gamma_{(F_t,G_t)}\}$ with $\gcd(F_t, G_t) = 1$ for $t \neq 0$ and $\gcd(F_0, G_0) \neq 1$ are given by tri-homogeneous ideals generated by polynomials of higher multi-degrees. (See 1.7,(ii) below for a concrete example.) Thus we are led to consider certain vector bundle maps roughly defined by $(A, B) \mapsto AF+BG$ (cf. (3) below for the precise definition). On the one hand, the locus where such a map drops rank turns out to be nice. On the other hand, blowing it up produces a new parameter space whose points carry information for the choice of subspaces of the space of multi-forms of higher multi-degrees with the right dimension. These subspaces span tri-homogeneous ideals that (generically) define subschemes of $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ with the correct Hilbert polynomial.

In short, the goal is to flatten a family of curves $\{\Gamma_{(F,G)} \subset \mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1\}_{(F,G)\in\mathbb{X}_d}$ by flattening certain families of linear spaces of tri-homogeneous ideals. This mimics and slightly refines the usual general construction of $\mathbb{H}\mathrm{ilb}$. We also benefit from ideas going back to the study of complete linear maps, cf. [14], [7]. The end result describes a smooth, projective parameter space which is an explicit desingularization of a specific component of $\mathbb{H}\mathrm{ilb}$. The special cases quoted above seem to point to the following loose observation: instead of focusing only at sufficiently high degrees, above the regularity bound, one should keep track of the lower degree pieces as well. Perhaps a suitable refinement of Hilb can be represented by a "better" subscheme of a product of flag varieties. See 1.7(ii) for more precision.

In the present case ($d = 2$), it is shown that any flat degeneration of a graph is given by an ideal spanned by a vector space of forms of multi-degree (2,2,3), of dimension 24. These vector spaces are fibers of the saturation (see 1 below) of the subsheaf of multi-forms generated by the ‘ $AF + BG$ ’ construction.

Our lack of knowledge of sharp, explicit bounds for regularity, especially in the multi-degree setting, is the main technical difficulty we face in handling higher bidegree cases. Several verifications of a local nature were done using Maple [10] and Singular [2]. The general philosophy behind a choice of coordinate neighborhoods where computations are performed is simply this: any bad behavior (such as singularity or non-flatness) shows up at closed orbits of the natural linear group action. Fortunately these orbits are easy to detect in the present case. Maple is used to calculate local generators for the Fitting ideals of the loci where ranks drop. A script is available at [15]. A similar approach should also work in higher bidegrees and dimensions.

1 Notation and preliminaries

We denote by $\mathcal{S} = \mathbb{C}[x_0, x_1, y_0, y_1, z_0, z_1]$ the polynomial ring over the complex numbers. Let $\mathcal{S}_x^d$ denote the space of homogeneous polynomials of degree d in the variables x_0, x_1 and define $\mathcal{S}_y^d$, $\mathcal{S}_z^d$ likewise. We also put $\mathcal{S}_{xy}^{i,j} = \mathcal{S}_x^i \cdot \mathcal{S}_y^j$, $\mathcal{S}_{xyz}^{i,j,k} = \mathcal{S}_x^i \cdot \mathcal{S}_y^j \cdot \mathcal{S}_z^k$, etc....

The linear group

$$\mathbb{G} := \mathbb{GL}(\mathcal{S}_x^1) \times \mathbb{GL}(\mathcal{S}_y^1) \times \mathbb{GL}(\mathcal{S}_z^1)$$

acts naturally on the $\mathcal{S}_{xyz}^{i,j,k}$ as well as on various parameter spaces defined below.

Set

$$\begin{cases} F := x_1 F_0 - x_0 F_1, \\ G := y_1 G_0 - y_0 G_1, \end{cases} \tag{2}$$

where $F_i, G_i \in \mathcal{S}_z^d$. Any element of $\mathcal{S}_{xz}^{1,d}$ can be written in the form $x_1 F_0 - x_0 F_1$ for suitable $F_i \in \mathcal{S}_z^d$ and similarly for $\mathcal{S}_{yz}^{1,d}$. We also note once and for all that, for polynomials F, G as above, the $\gcd(F, G)$ must be a purely z-factor common to all F_0, F_1, G_0, G_1.

Let $\Gamma_{(F,G)}$ denote the subscheme of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to a pair (F, G) as in (2).

For a vector bundle V, we write $\mathbb{P}(V)$ for the projective bundle of lines through the origin in the fibers of V. As a notational shorthand we shall often denote by the same letter a nonzero vector and the corresponding point in the associated projective space.

Write

$$\mathbb{X}_d := \mathbb{P}(\mathcal{S}_{xz}^{1,d}) \times \mathbb{P}(\mathcal{S}_{yz}^{1,d}).$$

The parameter space of graphs of maps of $\mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d, d) is the open subset of $\mathbb{X}_d$ consisting of pairs (F, G) such that F and G are irreducible.

The space of graphs is contained in the open subset, $\mathbb{X}^\circ$ of $\mathbb{X}_d$ consisting of pairs (F, G) such that $\gcd(F, G) = 1$. Observe that $\mathbb{X}^\circ$ contains non-graph elements (e.g., $F = x_0 z_0^d$, $G = y_0 z_1^d$). Nevertheless, every element in $\mathbb{X}^\circ$ still yields a *curve* (closed subscheme of dimension one) in $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$.

We denote by

$$\mathcal{O}_{xz}(-1) \rightarrowtail \mathcal{S}_{xz}^{1,d} \otimes \mathcal{O}_{xz} \qquad \Big(\text{resp.} \quad \mathcal{O}_{yz}(-1) \rightarrowtail \mathcal{S}_{yz}^{1,d} \otimes \mathcal{O}_{yz}\Big)$$

the tautological line subbundle over the projective space $\mathbb{P}\left(\mathcal{S}_{xz}^{1,d}\right)$ (resp. $\mathbb{P}\left(\mathcal{S}_{yz}^{1,d}\right)$). Tensor the first by $\mathcal{S}_{xyz}^{i-1,j,k} \otimes \mathcal{O}_{\mathbb{X}_d}$ and the second by $\mathcal{S}_{xyz}^{i,j-1,k} \otimes \mathcal{O}_{\mathbb{X}_d}$. Taking direct sums yields a map of locally free $\mathcal{O}_{\mathbb{X}_d}$-modules induced by multiplication,

$$\begin{array}{ccc} \left(\mathcal{O}_{xz}(-1) \otimes \mathcal{S}_{xyz}^{i-1,j,k}\right) \;\oplus\; \left(\mathcal{O}_{yz}(-1) \otimes \mathcal{S}_{xyz}^{i,j-1,k}\right) & \stackrel{\mu}{\longrightarrow} & \mathcal{S}_{xyz}^{i,j,d+k} \otimes \mathcal{O}_{\mathbb{X}_d} \\ (F \otimes A \;,\; G \otimes B) & \longmapsto & AF + BG. \end{array} \tag{3}$$

For $i = 1, \ldots, d$, let

$$\mathbb{X}_{d,i} := \{(F, G) \in \mathbb{X}_d \mid F = C\overline{F},\, G = C\overline{G} \text{ with } C \in \mathbb{P}(\mathcal{S}_z^i)\,,\, (\overline{F}, \overline{G}) \in \mathbb{X}_{d-i}\}. \tag{4}$$

It can be checked that the $\mathbb{X}_{d,i}$ are the loci where a suitable multiplication map μ as above drops rank.

Remarks 1.1. For $(F, G) \in \mathbb{X}^\circ$, the Hilbert polynomial of the curve $\Gamma_{(F,G)} \subset \mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ with respect to the ample sheaf $\mathcal{O}(1, 1, 1)$ is equal to $(2d + 1)t + 1$. More generally, we have

$$\chi\big(\mathcal{O}_{\Gamma_{(F,G)}}(a, b, c)\big) = d(a + b) + c + 1.$$

In fact, for a complete intersection $\Gamma_{d,d'} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by multiforms of multidegrees $d = (d_1, d_2, d_3)$, $d' = (d'_1, d'_2, d'_3)$, the Koszul resolution yields,

$$\chi\big(\mathcal{O}_{\Gamma_{d,d'}}(a,b,c)\big) = (a+1)(b+1)(c+1) -$$
$$\Big((a-d_1+1)(b-d_2+1)(c-d_3+1) + (a-d'_1+1)(b-d'_2+1)(c-d'_3+1)$$
$$- (a-d_1-d'_1+1)(b-d_2-d'_2+1)(c-d_3-d'_3+1)\Big).$$

The family of curves $\{\Gamma_{(F,G)} \mid (F,G) \in \mathbb{X}^\circ\}$ is flat. The induced map (1) defined by $(F,G) \mapsto \Gamma_{(F,G)}$ is a monomorphism. In particular, for $(F_0, G_0) \neq (F_1, G_1)$ we have $\Gamma_{(F_0,G_0)} \neq \Gamma_{(F_1,G_1)}$.

Let $(F,G) \in \mathbb{X}^\circ$ and put $\gamma = \Gamma_{(F,G)}$. Then the usual normal bundle calculation (cf. Sernesi [13]) shows that $\mathbb{H}$ilb is smooth at γ and

$$\dim_\gamma \mathbb{H}\mathrm{ilb} = 4d + 2 = \dim \mathbb{X}_d.$$

Thus $\mathbb{X}^\circ$ is a "good" approximation to $\mathbb{H}$ilb.

Definition 1.2. Let A be an integral domain, $\mathcal{F}$ an A-module and $\mathcal{M} \subseteq \mathcal{F}$ a submodule. We define the *saturation* of $\mathcal{M}$ in $\mathcal{F}$ as

$${}^s\mathcal{M} = \{m \in \mathcal{F} \mid \exists\, a \in A,\ a \neq 0,\ am \in \mathcal{M}\}.$$

Thus ${}^s\mathcal{M}$ is just the inverse image under $\mathcal{F} \to \mathcal{F}/\mathcal{M}$ of the torsion submodule of $\mathcal{F}/\mathcal{M}$.

The following facts are easy to check.

1. ${}^{ss}\mathcal{M} = {}^s\mathcal{M}$.
2. For any multiplicative system $S \subset A$ we have $S^{-1}({}^s\mathcal{M}) = {}^s(S^{-1}\mathcal{M})$.
3. For submodules $\mathcal{M}, \mathcal{M}' \subseteq \mathcal{F}$, if $\mathcal{M}_f = \mathcal{M}'_f$ for some nonzero $f \in A$ then ${}^s\mathcal{M} = {}^s\mathcal{M}'$.
4. If $\mathcal{M} \subseteq \mathcal{F} = A^n$ is a locally split submodule then ${}^s\mathcal{M} = \mathcal{M}$.

Thus, one may define the saturation of a sheaf of modules over an integral scheme.

We now describe the main steps used at each blowup in order to produce the saturation of certain subsheaves of the free sheaf $\mathcal{S}^{ijk}_{xyz} \otimes \mathcal{O}$. Their easy proofs will be omitted.

Lemma 1.3. *Let $\mathbb{U}$ be an integral affine variety with coordinate ring A and let*

$$M = \begin{bmatrix} I_r & * & * & * \\ 0 & f_1 & \cdots & f_s \end{bmatrix}$$

be a triangular $(r{+}1) \times n$ matrix with entries in A, where I_r denotes an identity block of size r. Let $\mathcal{M} \subseteq A^n$ be the submodule spanned by the rows of M. Assume the ideal J of $(r{+}1)$-minors is non zero. Let $\rho : \mathbb{U} \cdots \to \mathbb{G}\mathrm{r}_{r+1}(\mathbb{C}^n)$ be the rational map defined by $\mathcal{M}$. Let $\mathbb{U}' \to \mathbb{U}$ be the blowup of the scheme of zeros $\mathbb{V}{=}\mathcal{Z}(J)$ and let $\mathbb{V}'$ denote the exceptional divisor. Then we have the following.

1. *The map ρ extends to a morphism $\rho' : \mathbb{U}' \to \mathbb{G}\mathrm{r}_{r+1}(\mathbb{C}^n)$.*

2. *Suppose $\mathbb{V}$ is a complete intersection of codimension t in $\mathbb{U}$ and $J = \langle f_1, \ldots, f_t \rangle$ for some $t \leq s$. Then $\mathbb{U}'$ is the closed subscheme of $\mathbb{U} \times \mathbb{P}^{t-1}$ defined by $f_i x_j = f_j x_i$, $1 \leq i, j \leq t$, where the x_i denote homogeneous coordinates for $\mathbb{P}^{t-1}$.*

3. *Let $\mathbb{U}'_0 = \mathbb{U}' \cap (\mathbb{U} \times \mathbb{C}^{t-1}) \subset \mathbb{U} \times \mathbb{P}^{t-1}$ be the affine open subset given by $x_0 = 1$ and put $\mathbb{V}'_0 = \mathbb{V}' \cap \mathbb{U}'_0$. Then there are regular functions $y_2, \ldots, y_s$ on $\mathbb{U}'_0$ such that $f_i = y_i f_1$ for $2 \leq i \leq s$ and the coordinate ring of $\mathbb{V}'_0$ is the polynomial ring $(A/J)[y_2, \ldots, y_t]$.*

4. *The restriction of ρ' to the open subset $\mathbb{V}'_0 \cong \mathbb{V} \times \mathbb{C}^{t-1}$ of the exceptional divisor is given by*

$$\rho'(z, a) = \mathcal{M}_z + \langle e_{r+1} + a_2 e_{r+2} + \cdots + a_t e_{r+t} + y_{t+1}(a) e_{r+t+1} + \cdots + y_s(a) e_{r+s} \rangle$$

where $a = (a_2, \dots, a_t) \in \mathbb{C}^{t-1}$ *and* $\mathcal{M}_z$ *denotes the span of the first* r *rows of* M *at the closed point* $z \in \mathbb{V}$, *whereas the* e_i *are the standard unit vectors in* $\mathbb{C}^n$.

5. *Put* $B := \mathcal{O}(\mathbb{U}_0')$. *The saturation of the image of* $\mathcal{M} \otimes B$ *in* B^n *is a split, free submodule with basis given by the first* r *rows of* M *together with the "new generator",*

$$e_{r+1} + y_2 e_{r+2} + \cdots + y_s e_n$$

obtained by dividing the last row of M *by* f_1, *the local equation of the exceptional divisor.*

We have the following description of $\mathbb{X}_{d,d}$, our first blowup center.

Lemma 1.4. *Let*

$$\begin{array}{rcl} \mu : \mathcal{O}_{xz}(-1) \otimes \mathcal{S}^1_y \oplus \mathcal{O}_{yz}(-1) \otimes \mathcal{S}^1_x & \longrightarrow & \mathcal{S}^{1,1,d}_{xyz} \otimes \mathcal{O}_{\mathbb{X}_d} \\ (F \otimes A \quad , \; G \otimes B) & \longmapsto & AF + BG \end{array}$$

be as in (3). *Then*

1. $\overset{4}{\wedge}\mu$ *vanishes precisely along* $\mathbb{X}_{d,d}$ (see (4)).
2. *We have a natural identification* $\mathbb{X}_{d,d} \cong \mathbb{P}(\mathcal{S}^1_x) \times \mathbb{P}(\mathcal{S}^1_y) \times \mathbb{P}(\mathcal{S}^d_z)$.

Proof. If $AF + BG = 0$ then $F = BH$ and $G = -AH$ for some $H \in \mathcal{S}^d_z$. Hence $(F, G) \in \mathbb{X}_{d,d}$. A similar verification, now with $\mathbb{C}[\epsilon]$ coefficients, shows that the scheme of zeros of $\overset{4}{\wedge}\mu$ is reduced, whence equal to $\mathbb{X}_{d,d}$. The last assertion is immediate.

□

Since $4 \geq \operatorname{rank}(\mu) \geq 3$ and $\operatorname{rank}(\mu) < 4$ precisely along $\mathbb{X}_{d,d}$, we have a rational map

$$\rho : \mathbb{X}_d \cdots \rightarrow \mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,d}_{xyz}) \tag{5}$$

given by

$$\begin{aligned} \rho(F, G) &= \operatorname{image}(\mu_{|(F,G)}) \\ &= (y_0 F, y_1 F, x_0 G, x_1 G), \end{aligned}$$

for $(F, G) \in \mathbb{X}_d \setminus \mathbb{X}_{d,d}$.

Proposition 1.5. *Let $\mathbb{X}_d^1$ be the blowup of $\mathbb{X}_d$ along $\mathbb{X}_{d,d}$. Then we have the following.*

1. *$\mathbb{X}_d^1$ embeds in $\mathbb{X}_d \times \mathbb{G}\mathrm{r}_4(\mathcal{S}_{xyz}^{1,1,d})$ as the closure of the graph of ρ.*

2. *ρ extends to a morphism $\rho' : \mathbb{X}_d^1 \to \mathbb{G}\mathrm{r}_4(\mathcal{S}_{xyz}^{1,1,d})$.*

3. *The exceptional divisor $\mathbb{X}_{d,d}^1 \to \mathbb{X}_{d,d}$ is a $\mathbb{P}^{3d-1}$–bundle with fiber over a point*
$$(\overline{F}, \overline{G}, C) \in \mathbb{P}(\mathcal{S}_x^1) \times \mathbb{P}(\mathcal{S}_y^1) \times \mathbb{P}(\mathcal{S}_z^d)$$
equal to $\mathbb{P}\left((\mathcal{S}_z^d/\langle C\rangle) \otimes (\overline{F}\mathcal{S}_y^1 + \overline{G}\mathcal{S}_x^1)\right)$.

4. *Each point in the fiber of $\mathbb{X}_{d,d}^1$ over $(\overline{F}, \overline{G}, C)$ represented by*
$$H_1 \in \mathcal{S}_z^d \cdot (\overline{F}\mathcal{S}_y^1 + \overline{G}\mathcal{S}_x^1) \bmod C \cdot (\overline{F}\mathcal{S}_y^1 + \overline{G}\mathcal{S}_x^1).$$
is mapped by ρ' to the 4-plane $U = \langle H_1\rangle + C \cdot (\overline{F}\mathcal{S}_y^1 + \overline{G}\mathcal{S}_x^1) \subset \mathcal{S}_{xyz}^{1,1,d}$.

Proof. The main point is assertion (4), which can be checked by an easy calculation of the normal bundle of the blowup center.

□

Lemma 1.6. *Let*
$$\mathbf{o} = (x_1 z_1^2, y_1 z_1^2)$$
be a point on the unique closed orbit of $\mathbb{X}_2$. Put
$$U = \langle x_1 y_1 z_0 z_1,\ x_1 y_1 z_1^2,\ x_0 y_1 z_1^2,\ x_1 y_0 z_1^2\rangle,$$
$$\mathbf{o}^1 = (\mathbf{o}, U) \in \mathbb{X}_2^1 \subset \mathbb{X}_2 \times \mathbb{G}\mathrm{r}_4(\mathcal{S}_{xyz}^{1,1,2}).$$
Then $\mathbb{G}{\cdot}\mathbf{o}^1$ is the unique closed orbit of $\mathbb{G}$ in $\mathbb{X}_2^1$ and it maps isomorphically onto $\mathbb{G}{\cdot}\mathbf{o}$.

Proof. Any closed orbit in $\mathbb{X}_2^1$ lies over the unique closed orbit $\mathbb{G}{\cdot}\mathbf{o}$. Therefore, it suffices to determine closed orbits of the fiber over $\mathbf{o}$, acted on by its stabilizer. The fiber of $\mathbb{X}_2^1$ over $\mathbf{o}$ embeds in $\mathbb{G}r_4(\mathcal{S}_{xyz}^{1,1,2})$ as the family of subspaces $\langle x_1y_1z_1^2,\, x_0y_1z_1^2,\, x_1y_0z_1^2,\, H\rangle$ parametrized by

$$\mathbb{P}^5 \cong \mathbb{P}\left((\mathcal{S}_z^2\cdot(x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1))/(z_1^2(x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1))\right)$$

(cf. 1.5-3). We may write the polynomial $H := b_1x_1y_1z_0z_1 + b_2x_1y_0z_0z_1 + b_3x_0y_1z_0z_1 + b_4x_1y_1z_0^2 + b_5x_1y_0z_0^2 + b_6x_0y_1z_0^2$, where $(b_1:\cdots:b_6)\in\mathbb{P}^5$. This follows from (1.3) taking into account that the classes mod $z_1^2(x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1)$ of the six monomials occurring in H give a basis for

$$\mathcal{S}_z^2\cdot(x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1)\Big/\left(z_1^2(x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1)\right).$$

We now act with appropriate 1-parameter subgroups of the stabilizer $\mathbb{G}_{\mathbf{o}}$. First assume $b_1 = 1$. Taking $z_0\to tz_0$ and leaving the remaining variables fixed, one sees that the $\mathbb{G}_{\mathbf{o}}$−orbit closure contains a point corresponding to a 4-plane as above with no z_0^2−term in H. If, say $b_4 = 1$, we replace z_0 by $z_0 + z_1$. By a similar argument, we finally find that the $\mathbb{G}_{\mathbf{o}}$−orbit determined by $x_1y_1z_0z_1,\ x_1y_1z_1^2,\ x_0y_1z_1^2,\ x_1y_0z_1^2$ is in the closure, as asserted.

□

Remarks 1.7. (i) For any d, there is a unique closed orbit of $\mathbb{G}$ in $\mathbb{X}_d^1$, namely, the orbit of the point $(x_1z_1^d, y_1z_1^d, U)\in\mathbb{X}_{d,d}\times\mathbb{G}r_4(\mathcal{S}_{xyz}^{1,1,d})$, with $U = \langle x_1y_1z_0z_1^{d-1}\rangle + z_1^d\cdot\langle x_1\mathcal{S}_y^1+y_1\mathcal{S}_x^1\rangle$. For $d=1$, an easy verification shows that the Hilbert polynomial of the subscheme of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ defined by the 4-plane U is equal to $3t+1$. This gives a simple argument showing that $\mathbb{X}_1^1$ maps (in fact isomorphically, cf. [9]) to (a component of) $\mathbb{H}$ilb.

(ii) A calculation of the flat limit for the family $\{x_1z_1^2+tx_0z_1z_0+t^2x_0z_0^2,\, y_1z_1^2+ty_0z_0z_1+t^2y_1z_0^2\}_{t\in\mathbb{A}^1\setminus 0}$ (e.g., imitating [5], p. 260) yields the ideal

$$\left\langle \begin{array}{c} x_1z_1^2,\ y_1z_1^2,\ (x_1y_0-y_1x_0)z_1z_0,\ (x_1-x_0)y_1z_1z_0^2,\ x_0(y_1-y_0)y_1z_1z_0^2, \\ (x_1x_0y_0^2 - x_0^2y_0y_1 + x_1^2y_1^2 - 2x_1x_0y_1^2 + 2x_0^2y_1^2 - x_0x_1y_1y_0)z_0^3 \end{array}\right\rangle.$$

Examining the intersections of the latter ideal with each of the vector spaces $\mathcal{S}_{xyz}^{112},\ \mathcal{S}_{xyz}^{113},\ \mathcal{S}_{xyz}^{213},\ \mathcal{S}_{xyz}^{123}$ and $\mathcal{S}_{xyz}^{223}$, we find the respective dimensions 4,8,14,14,24.

This indicates the route to be pursued below: we will show that a suitable modification of $\mathbb{X}_d^1$, obtained by further blowups, embeds in

$$\mathbb{X}_d \times \mathbb{G}\mathrm{r}_4\left(\mathcal{S}_{xyz}^{1,1,2}\right) \times \mathbb{G}\mathrm{r}_8\left(\mathcal{S}_{xyz}^{113}\right) \times \mathbb{G}\mathrm{r}_{14}\left(\mathcal{S}_{xyz}^{213}\right) \times \mathbb{G}\mathrm{r}_{14}\left(\mathcal{S}_{xyz}^{123}\right) \times \mathbb{G}\mathrm{r}_{24}\left(\mathcal{S}_{xyz}^{223}\right).$$

2 The second blowup

We start by describing the structure of $\mathbb{X}_{2,1}^1$, the second blowup center.

Lemma 2.1. *Let $\mathbb{X}_{d,d-1}^1$ denote the strict transform of $\mathbb{X}_{d,d-1}$ under the blowup of $\mathbb{X}_d$ along $\mathbb{X}_{d,d}$, and let $\mathbb{X}_{d,d}^1$ be the exceptional divisor. Then we have the following.*

1. *There is a commutative diagram,*

$$\begin{array}{ccc} \mathbb{X}_{1,1}^1 \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right) & \xrightarrow{\sim} & \mathbb{X}_{d,d-1}^1 \cap \mathbb{X}_{d,d}^1 \\ \downarrow & & \downarrow \\ \mathbb{X}_1^1 \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right) & \xrightarrow{\sim} & \mathbb{X}_{d,d-1}^1 \end{array} \qquad (6)$$

   where the vertical arrows are embeddings.

2. *Now set $d=2$. The rational map $\Gamma : \mathbb{X}_2^1 \cdots \to \mathbb{H}\mathrm{ilb}$ fails to be defined precisely along $\mathbb{X}_{2,1}^1$.*

Proof. Let $\mathbb{X}_1 \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right) \to \mathbb{X}_d$ be defined by $(F,G,C) \mapsto (FC,GC)$. It clearly factors through $\mathbb{X}_{d,d-1}$. One checks that the multiplication map of (1.4) pulls back to the composition of maps

$$\mathcal{O}_z(-1) \otimes \left(\mathcal{O}_{zx}(-1) \otimes \mathcal{S}_y^1 \oplus \mathcal{O}_{zy}(-1) \otimes \mathcal{S}_x^1 \to \mathcal{S}_{xyz}^{1,1,1} \otimes \mathcal{O}_{\mathbb{X}_1}\right) \rightarrowtail \mathcal{S}_{xyz}^{1,1,d} \otimes \mathcal{O}_{\mathbb{X}_d}$$

where $\mathcal{O}_z(-1)$ is the tautological line subbundle on $\mathbb{P}\left(\mathcal{S}_z^{d-1}\right)$, the first arrow is the multiplication map for $d=1$ twisted by $\mathcal{O}_z(-1)$ and the last arrow is split injective. Consequently, $\mathbb{X}_{1,1} \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right)$ is the scheme theoretic pullback of $\mathbb{X}_{d,d}$, thereby inducing the bottom map in diagram (6). To check it is an isomorphism, first observe it is certainly so off the respective exceptional divisors. To complete the verification, we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{X}_1^1 \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right) & & \hookrightarrow & & \mathbb{X}_1 \times \mathbb{G}\mathrm{r}_4(\mathcal{S}_{xyz}^{1,1,1}) \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right) \\ \downarrow & & & & \downarrow p \\ \mathbb{X}_{d,d-1}^1 & \hookrightarrow & \mathbb{X}_d^1 & \hookrightarrow & \mathbb{X}_d \times \mathbb{G}\mathrm{r}_4(\mathcal{S}_{xyz}^{1,1,d}) \end{array}$$

where the horizontal arrows are embeddings and the rightmost vertical map p is given by $(F, G, U, C) \mapsto (FC, GC, C \cdot U)$. We want to show that the left vertical map is an isomorphism. To do this, we actually show that its composition with the bottom embeddings still is an embedding, although p itself is not. Let $(FC, GC, C \cdot U) = (F'C', G'C', C' \cdot U')$. Set $H = \gcd(U)$, $H' = \gcd(U')$. Now $C \cdot U = C' \cdot U'$ implies $CH = C'H'$. From (1.5)-4, with $d = 1$, we see that $H = 1$ holds. Hence we get $C = C' \in \mathbb{P}\left(\mathcal{S}_z^{d-1}\right)$ and $(F, G, U) = (F', G', U')$.

We claim further that the image of the top horizontal map is also disjoint from the ramification locus. It suffices to show that the tangent map dp is injective at points in the image of $\mathbb{X}_1^1 \times \mathbb{P}\left(\mathcal{S}_z^{d-1}\right)$, say at the point (F, G, C, U). Let (F', G', C', u') be a tangent vector at that point. With the usual identifications of tangent spaces for projective spaces and grassmannians, we have

$$F' \in \mathcal{S}_{xz}^{1,1}/\langle F \rangle, \quad G' \in \mathcal{S}_{yz}^{1,1}/\langle G \rangle, \quad C' \in \mathcal{S}_z^{d-1}/\langle C \rangle,$$
$$u' \in \mathrm{Hom}(U, \mathcal{S}_{xyz}^{1,1,1}/U).$$

By Leibnitz's rule, we may write

$$dp_{(F,G,C,U)}(F', G', C', u') = (FC' + CF' \bmod F,\ GC' + CG' \quad \bmod G,\ \theta)$$

where $\theta \in \mathrm{Hom}(C \cdot U, \mathcal{S}_{xyz}^{1,1,d}/(C \cdot U))$ is defined, for $A \in U$, by

$$\theta(CA) = C'A + C\widetilde{u'(A)} \bmod C \cdot U.$$

Here the tilde indicates a choice of a representative of a class in $\mathcal{S}_{xyz}^{1,1,1}/U$. Abusing notation, let C' denote any representative of its class $\bmod C$. Now if $\theta = 0$, we get that $C'A$ is divisible by C for any $A \in U$. Hence $\gcd(C' \cdot U) = C' \gcd(U)$ is divisible by C. Again by (1.5), we have $\gcd(U) = 1$ and hence C' is a multiple of C, thus representing the zero class. Hence, $C\widetilde{u'(A)} \in C \cdot U$ for all A, implying that $u' = 0$. It follows easily that $F' = 0 = G' = C'$ as desired, completing the verification of (1).

For (2), it suffices to exhibit one-parameter families $\gamma_{i,t}$ emanating from the same point in $\mathbb{X}^1_{2,1}$, such that the corresponding limit curves $\gamma_{1,0}$ and $\gamma_{2,0}$ are distinct points in $\mathbb{H}\mathrm{ilb}$. For instance, consider the one-parameter families,

$$\begin{aligned} \gamma_{1,t} &:= (x_0 z_0 z_1 + t x_0 z_1^2,\ y_0 z_0^2), \\ \gamma_{2,t} &:= (x_0 z_0 z_1,\ y_0 z_0^2 + t y_1 z_1^2). \end{aligned}$$

Of course at $t = 0$ both families yield the same point $(x_0 z_0 z_1,\ y_0 z_0^2) \in \mathbb{X}_{2,1} \smallsetminus \mathbb{X}_{2,2}$ $= \mathbb{X}^1_{2,1} \smallsetminus \mathbb{X}^1_{2,2}$. Computing the flat limits, we find

$$\begin{aligned} \gamma_{1,0} = \lim_{t\to 0} \gamma_{1,t} &= \langle x_0 z_0 z_1,\ y_0 z_0^2,\ x_0 y_0 z_1^3 \rangle \\ \gamma_{2,0} = \lim_{t\to 0} \gamma_{2,t} &= \langle x_0 z_0 z_1,\ y_0 z_0^2,\ x_0 y_1 z_1^3 \rangle. \end{aligned}$$

These are easily seen to be distinct one dimensional closed subschemes in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (e.g., check for $x_0 = y_0 = z_1 = 1$) with the correct Hilbert polynomial.

□

2.2 The second blowup center

Let $\mathcal{U}$ be the rank 4 subbundle of $\mathcal{S}^{1,1,2}_{xyz} \times \mathbb{X}^1_2$ defined by the pullback of the natural subbundle on $\mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,2}_{xyz})$. Define the map of multiplication by z's,

$$\begin{aligned} \mu_z : \mathcal{U} \otimes \mathcal{S}^1_z &\longrightarrow \mathcal{S}^{1,1,3}_{xyz} \otimes \mathcal{O}_{\mathbb{X}^1_2} \\ H \otimes C &\longmapsto HC. \end{aligned} \tag{7}$$

We describe below the second blowup center, namely, the locus where μ_z drops rank.

Lemma 2.3. *With the notation as above, the scheme of zeros of $\bigwedge^8 \mu_z$ is smooth of dimension 7 and coincides with the strict transform $\mathbb{X}^1_{2,1}$ of $\mathbb{X}_{2,1}$ (cf. (4)).*

Proof. Let $\mathbb{X}^1_{\mu_z}$ be the scheme of zeros of $\bigwedge^8 \mu_z$. It suffices to verify smoothness in a neighborhood of the unique closed orbit $\mathbb{G} \cdot \mathbf{o}^1$ (cf. 1.6). Consider the standard open affine subset $\mathbb{U}_x$ of $\mathbb{P}(\mathcal{S}^{1,2}_{xz})$ with coordinate functions $a_1, \dots, a_5$ so that $\mathcal{O}_{xz}(-1)_{|\mathbb{U}_x}$ is the trivial subbundle of $\mathcal{S}^{1,2}_{xz}{}_{|\mathbb{U}_x}$ with base

$$F := x_1 z_1^2 + a_1 x_1 z_0 z_1 + a_2 x_1 z_0^2 + a_3 x_0 z_1^2 + a_4 x_0 z_0 z_1 + a_5 x_0 z_0^2.$$

Similarly, take $\mathbb{U}_y \subset \mathbb{P}\left(\mathcal{S}^{1,2}_{yz}\right)$ with coordinate functions $a_6, \dots, a_{10}$ so that the restriction of $\mathcal{O}_{yz}(-1)$ over $\mathbb{U}_y$ is the trivial subbundle of $\mathcal{S}^{1,2}_{yz}{}_{|\mathbb{U}_y}$ with base

$$G := y_1 z_1^2 + a_6 y_1 z_0 z_1 + a_7 y_1 z_0^2 + a_8 y_0 z_1^2 + a_9 y_0 z_0 z_1 + a_{10} y_0 z_0^2.$$

Put $\mathbb{U} := \mathbb{U}_x \times \mathbb{U}_y$. This is an affine chart around $\mathbf{o} \in \mathbb{X}_d$. The map μ may be represented on $\mathbb{U}$ by a 4×12 matrix obtained by multiplying F by y's and G by x's and collecting coefficients of the monomials $x_i y_j z_r z_s$ in a suitable order. Performing elementary operations, we achieve a matrix containing a 3×3 identity block in the form,

$$\begin{bmatrix} I_3 & * & * & * \\ 0 & f_1 & \cdots & f_9 \end{bmatrix}, \tag{8}$$

where the entries

$$\begin{array}{lll} f_1 = a_6 - a_1, & f_2 = a_7 - a_2, & f_3 = a_9 - a_8 a_6, \\ f_4 = a_{10} - a_8 a_7, & f_5 = a_4 - a_3 a_6, & f_6 = a_5 - a_3 a_7 \end{array}$$

generate the ideal of $\mathbb{X}_{2,2}$ in the coordinate ring of $\mathbb{U}$.

Let $\mathbb{U}^1 \subset \mathbb{X}^1_2$ denote the open subset where $\varepsilon_1 := a_1 - a_6$ spans the exceptional ideal. This is an affine 10-space with coordinate functions $b_i = f_i/f_1$, $i = 2 \dots 6$ together with a_1, a_3, a_6, a_7, a_8. The blowup map $\mathbb{U}^1 \to \mathbb{U}$ is given by

$$\begin{gathered} a_2 = \varepsilon_1 b_2 + a_7,\ a_4 = \varepsilon_1 b_5 + a_3 a_6,\ a_5 = \varepsilon_1 b_6 + a_3 a_7, \\ a_9 = -b_3 \varepsilon_1 + a_8 a_6,\ a_{10} = -b_4 \varepsilon_1 + a_8 a_7. \end{gathered}$$

The bundle $\mathcal{U}$ trivializes over $\mathbb{U}^1$, with a basis obtained from the rows of (8) as in the recipe (1.3-5). The homomorphism μ_z may be represented by an 8×16-matrix. The latter can be put into triangular form with a 7×7 identity block,

$$\begin{bmatrix} I_7 & * & * & * \\ 0 & \star & \cdots & \star \end{bmatrix}. \tag{9}$$

The ideal of its 8×8 minors is spanned by the $\star$'s in the last row, among which we find that the following three suffice:

$$a_7 + b_2^2 - b_2 a_6,\ b_2 b_3 - b_4,\ b_5 b_2 - b_6. \tag{10}$$

This shows that $\mathbb{X}^1_{\mu_z}$ is smooth and of dimension 7. Since $\mathbb{X}^1_{2,1}$ is of the same dimension, it coincides with $\mathbb{X}^1_{\mu_z}$ provided $\mathbb{X}^1_{2,1} \subseteq \mathbb{X}^1_{\mu_z}$. As $\mathbb{X}^1_{2,1}$ is irreducible, we only need to show that the inclusion holds off the exceptional divisor. Pick $(F, G) \in \mathbb{X}_{2,1} \smallsetminus \mathbb{X}_{2,2}$. We may write $F = C\overline{F}$, $G = C\overline{G}$ for some $C \in \mathbb{P}(\mathcal{S}^1_z)$, $\overline{F} \in \mathbb{P}(\mathcal{S}^{1,1}_{xz})$, $\overline{G} \in \mathbb{P}\left(\mathcal{S}^{1,1}_{yz}\right)$ with gcd($\overline{F}, \overline{G}$)=1. Now the fiber of $\mathcal{U}$ over (F, G) is the 4-plane $C \cdot (\overline{F} \cdot \mathcal{S}^1_y + \overline{G} \cdot \mathcal{S}^1_x)$. Say $\overline{F} = f_1 z_0 + f_2 z_1$, $\overline{G} = g_1 z_0 + g_2 z_1$ with $f_i \in \mathcal{S}^1_x$ and $g_i \in \mathcal{S}^1_y$. Note that $(f_2 g_1 - f_1 g_2) \neq 0$ because gcd($\overline{F}, \overline{G}$)=1. Now μ_z maps $C(f_2 g_1 - f_1 g_2) z_1 \otimes z_0 - C(f_2 g_1 - f_1 g_2) z_0 \otimes z_1$ to zero. This shows that the rank of μ_z drops at (F, G), hence (F, G) lies in $\mathbb{X}^1_{\mu_z}$ as desired.

□

Proposition 2.4. *Let $\mathbb{X}^2_2 \to \mathbb{X}^1_2$ be the blowup along $\mathbb{X}^1_{2,1}$. Then we have the following.*

1. *$\mathbb{X}^2_2$ embeds in $\mathbb{X}^1_2 \times \mathrm{Gr}_8(\mathcal{S}^{1,1,3}_{xyz})$.*
2. *The exceptional divisor $\mathbb{X}^2_{2,1} \to \mathbb{X}^1_{2,1}$ is the $\mathbb{P}^2$–bundle with fiber over a point*

$$\left((\overline{F}, \overline{G}), U, C\right) \in \mathbb{X}^1_{2,1} = \mathbb{X}^1_1 \times \mathbb{P}(\mathcal{S}^1_z) \subset \mathbb{X}_1 \times \mathrm{Gr}_4(\mathcal{S}^{1,1,1}_{xyz}) \times \mathbb{P}(\mathcal{S}^1_z)$$

given by

$$\mathbb{P}\left((\mathcal{S}^2_z \cdot U)/(C \cdot \mathcal{S}^1_z \cdot U)\right).$$

3. *Let $\mathcal{V}$ be the rank 8 subbundle of $\mathbb{X}^2_2 \times \mathcal{S}^{1,1,3}_{xyz}$ defined by the pullback of the tautological subbundle of $\mathrm{Gr}_8(\mathcal{S}^{1,1,3}_{xyz}) \times \mathcal{S}^{1,1,3}_{xyz}$. Let a point V in the fiber of $\mathbb{X}^2_{2,1}$ over $(\overline{F}, \overline{G}, U, C) \in \mathbb{X}^1_{2,1}$ be represented by*

$$H_2 \in \mathcal{S}^2_z \cdot U \mod C \cdot \mathcal{S}^1_z \cdot U.$$

Then the fiber of the vector bundle $\mathcal{V}$ over V is the 8-plane given by $\langle H_2 \rangle + C \cdot \mathcal{S}^1_z \cdot U$.

4. *There are exactly 3 closed orbits of* $\mathbb{X}_2^2$, *each represented by an 8-plane spanned by the 7-plane* $\mathcal{S}_z^1 \cdot U$ (see 1.6) *together with one of the following monomials,*

$$x_0y_1z_0^2z_1,\ x_1y_0z_0^2z_1,\ x_1y_1z_0^3.$$

Proof. Since $\operatorname{rank}\mu_z = 8$ off $\mathbb{X}^1_{\mu_z}$ and the blowup center is the scheme of zeros of $\overset{8}{\wedge}\mu_z$, the first assertion follows. Assertions 2 and 3 will be proven by the computation of tangent/normal spaces performed in the sequel. For $(F,G) \in \mathbb{X}_{2,1} \setminus \mathbb{X}_{2,2}$, write $F = C\overline{F}$, $G = C\overline{G}$, as in the proof of (2.3). Then we have

$$\begin{aligned}
\mathcal{T}\mathbb{X}_{2,1\,(F,G)} &= \mathcal{T}\mathbb{X}^1_{2,1\,(\overline{F},\overline{G},C)},\\
\mathcal{T}\mathbb{X}^1_{2,1\,(\overline{F},\overline{G},C)} &= \mathcal{S}^{1,1}_{xz}/\langle\overline{F}\rangle \oplus \mathcal{S}^{1,1}_{yz}/\langle\overline{G}\rangle \oplus \mathcal{S}^1_z/\langle C\rangle, \qquad (11)\\
\mathcal{T}\mathbb{X}_{2\,(F,G)} &= \mathcal{S}^{1,2}_{xz}/\langle F\rangle \oplus \mathcal{S}^{1,2}_{yz}/\langle G\rangle.
\end{aligned}$$

One checks that the map

$$\mathcal{T}\mathbb{X}_{2\,(F,G)} \,/\, \mathcal{T}\mathbb{X}_{2,1\,(F,G)} \longrightarrow \left(\overline{F}\cdot\mathcal{S}^{1,2}_{yz} + \overline{G}\cdot\mathcal{S}^{1,2}_{xz}\right) / \left(C\cdot(\overline{F}\cdot\mathcal{S}^{1,1}_{yz} + \overline{G}\cdot\mathcal{S}^{1,1}_{xz})\right) \qquad (12)$$

defined by

$$\left(A+\langle F\rangle, B+\langle G\rangle\right) + \mathcal{T}\mathbb{X}_{2,1\,(\overline{F},\overline{G},C)} \mapsto \left(A\overline{G}+B\overline{F}\right) + C\cdot(\overline{F}\cdot\mathcal{S}^{1,1}_{yz} + \overline{G}\cdot\mathcal{S}^{1,1}_{xz})$$

is an isomorphism. Thus it describes the normal space $\mathcal{N}_{(F,G)}$ of $\mathbb{X}_{2,1}$ in $\mathbb{X}_2$ at a point (F,G) off $\mathbb{X}_{2,2}$.

Consider the curve γ in $\mathbb{X}_2$ given by $t \to (F+tA, G+tB)$ for $A \in \mathcal{S}^{1,2}_{xz}$, $B \in \mathcal{S}^{1,2}_{yz}$ such that $\left(A+\langle F\rangle, B+\langle G\rangle\right)$ is nonzero $\mod \mathcal{T}\mathbb{X}_{2,1\,(\overline{F},\overline{G},C)}$. The tangent vector to γ at $t=0$ is not tangent to $\mathbb{X}_{2,1}$. Hence γ is not contained in $\mathbb{X}_{2,1}$. It lifts to a unique curve γ^2 in $\mathbb{X}_2^2 \subset \mathbb{X}_2^1 \times \mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz})$. We proceed to compute the $\mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz})$−coordinate V_t of γ_t^2. Write $\mu_{z,t}$ for the restriction of the map μ_z to the fiber over γ_t. Its image contains

$$(F+tA)\overline{G} - (G+tB)\overline{F} = tA\overline{G} - tB\overline{F}.$$

Hence for all nonzero t we find that $A\overline{G} - B\overline{F}$ lies in V_t. It follow that

$$V_0 = \langle A\overline{G} - B\overline{F}\rangle + C\cdot(\overline{F}\cdot\mathcal{S}^{1,1}_{yz} + \overline{G}\cdot\mathcal{S}^{1,1}_{xz}). \qquad (13)$$

We now use this to give an alternative description of the projectivization of the normal bundle of $\mathbb{X}^1_{2,1} \subset \mathbb{X}^1_2$.

Denote by $\mathcal{U}^1 \subset \big(\mathcal{S}^{1,1,1}_{xyz}\big)_{\mathbb{X}^1_1}$ the restriction to $\mathbb{X}^1_1 \subset \mathbb{X}_1 \times \mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,1}_{xyz})$ of the tautological rank 4 subbundle over $\mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,1}_{xyz})$. It follows from 1 (with $d_1 = 1$, $d_2 = 0$, $d_3 = 1$, $d'_1 = 0$, $d'_2 = 1$, $d'_3 = 1$) that $\mathcal{S}^2_z \cdot \mathcal{U}^1$ (resp. $\mathcal{O}_z(-1) \cdot \mathcal{S}^1_z \cdot \mathcal{U}^1$) is a subbundle of $\big(\mathcal{S}^{1,1,3}_{xyz}\big)_{\mathbb{X}^1_1 \times \mathbb{P}()(\mathcal{S}^1_z)}$ of rank 10 (resp. 7). We have that the target space in (12) is the fiber at $(\overline{F}, \overline{G}, C)$ of the vector bundle

$$\mathcal{N} \;=\; \big(\mathcal{S}^2_z \cdot \mathcal{U}^1\big) \Big/ \big(\, \mathcal{O}_z(-1) \cdot \mathcal{S}^1_z \cdot \mathcal{U}^1\big) \;\to\; \mathbb{X}^1_{2,1}.$$

We get a natural embedding of bundles over $\mathbb{X}^1_{2,1}$,

$$\mathbb{P}(\mathcal{N}) \quad \hookrightarrow \quad \mathbb{X}^1_{2,1} \times \mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz}).$$

From (13) we see that $\mathbb{P}(\mathcal{N})$ coincides with the projectivized normal bundle of $\mathbb{X}^1_{2,1} \subset \mathbb{X}^1_2$ over the open dense set complement of $\mathbb{X}_{2,2}$ in $\mathbb{X}_{2,1}$. This proves assertions 2 and 3.

For closed orbits, we proceed as in (1.6). Put

$$U' = \langle x_1 y_1 z_0 \rangle + z_1 \cdot (x_1 \mathcal{S}^1_y + y_1 \mathcal{S}^1_x).$$

The fiber of $\mathbb{X}^2_{2,1}$ over $\mathbf{o}^1$ is the projective plane $\mathbb{P}\big((\mathcal{S}^2_z \cdot U')/(z_1 \cdot \mathcal{S}^1_z \cdot U')\big)$ embedded in the grassmannian $\mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz})$ as the family of 8–planes given by

$$\langle (c_1 x_0 y_1 + c_2\, x_1 y_0) z_0^2 z_1 + c_3\, x_1 y_1 z_0^3 \rangle + z_1 \cdot \mathcal{S}^1_z \cdot U',$$

where c_1, c_2, c_3 denote homogeneous coordinates for the plane. Acting with suitable 1-parameter subgroups of the stabilizer of $\mathbf{o}^1$, we find precisely the 3 closed orbits as stated.

□

Remark 2.5. The subscheme of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by $x_1 z_1^2$, $y_1 z_1^2$ together with $x_1 y_1 z_0^3$ (the representative of the third closed orbit described above), gives a point in $\mathbb{H}\mathrm{ilb}$. In other words, the rational map Γ (cf. 2.1(2)) extends to a neighborhood of this closed orbit. The other two closed orbits of course do not define curves as of yet. In fact, they lie in the next blowup center.

3 Third blowup

Proposition 3.1. *Let $\mathcal{V}$ be the rank 8 subbundle of $\mathcal{S}^{1,1,3}_{xyz}$ as in* (2)*. Let*

$$\mu_{xy} : \mathcal{V} \otimes \mathcal{S}^{1,1}_{xy} \longrightarrow \mathcal{S}^{2,2,3}_{xyz}, \qquad \mu_x : \mathcal{V} \otimes \mathcal{S}^{1}_{x} \longrightarrow \mathcal{S}^{2,1,3}_{xyz}, \qquad \mu_y : \mathcal{V} \otimes \mathcal{S}^{1}_{y} \longrightarrow \mathcal{S}^{1,2,3}_{xyz}$$

be defined by multiplication. Then we have the following.

1. *The generic ranks of μ_x, μ_y and μ_{xy} are 14, 14 and 24, respectively.*
2. *The rank of μ_{xy} drops to 22 over a smooth subvariety $\mathbb{X}^2_{\mu_{xy}} \subset \mathbb{X}^2_{2,1}$.*
3. *$\mathbb{X}^2_{\mu_{xy}}$ consists of two connected components of dimension 5, denoted $\mathbb{X}^2_{\mu_x}$, $\mathbb{X}^2_{\mu_y}$, cf.* (17)*.*
4. *The scheme of zeros of $\overset{14}{\wedge}\mu_x$ (resp. $\overset{14}{\wedge}\mu_y$, $\overset{23}{\wedge}\mu_{xy}$), is equal to $\mathbb{X}^2_{\mu_x}$, (resp. $\mathbb{X}^2_{\mu_y}$, $\mathbb{X}^2_{\mu_{xy}}$).*
5. *The closed orbit of $\mathbb{X}^2_2$* (cf. 2) *determined by $x_0y_1z_0^2z_1$ (resp. $x_1y_0z_0^2z_1$) is contained in $\mathbb{X}^2_{\mu_x}$, (resp. $\mathbb{X}^2_{\mu_y}$).*

Proof. Recall from 2.1 that $\mathbb{X}^1_{2,1} \cong \mathbb{X}^1_1 \times \mathbb{P}(\mathcal{S}^1_z)$. Put

$$\mathbb{K}_x := \{(\alpha, \gamma, f) \in \mathbb{P}(\mathcal{S}^1_x) \times \mathbb{P}(\mathcal{S}^1_z) \times \mathbb{P}(\mathcal{S}^{1,1}_{xz}) \mid f \in \alpha\mathcal{S}^1_z + \gamma\mathcal{S}^1_x\}. \tag{14}$$

This is the total space for the family of conics, i.e., (1,1)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Clearly $\mathbb{K}_x$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}(\mathcal{S}^1_x) \times \mathbb{P}(\mathcal{S}^1_z)$. Consider the map

$$\mathbb{K}_x \times \mathbb{P}(\mathcal{S}^1_y) \overset{\psi}{\longrightarrow} \mathbb{X}^1_{2,1} = \mathbb{X}^1_1 \times \mathbb{P}(\mathcal{S}^1_z) \subset \mathbb{X}_1 \times \mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,1}_{xyz}) \times \mathbb{P}(\mathcal{S}^1_z) \tag{15}$$

defined by

$$\big(\alpha(x), \gamma(z), f(x,z), \beta(y)\big) \mapsto \big(f,\ \beta\gamma,\ U(\alpha,\beta,\gamma,f),\ \gamma\big)$$

where $U(\alpha,\beta,\gamma,f) := f\mathcal{S}^1_y + \beta\gamma\mathcal{S}^1_x + \alpha\beta\mathcal{S}^1_z \subset \mathcal{S}^{1,1,1}_{xyz}$. One checks that $U(\alpha,\beta,\gamma,f)$ yields a legitimate point in the fiber of $\mathbb{X}^1_1 \to \mathbb{X}_1$ over $(f,\ \beta\gamma) \in \mathbb{X}_1$. Indeed, first assume gcd$(f,\beta\gamma)$=1. Then the $U-$coordinate of the fiber of $\mathbb{X}^1_1 \to \mathbb{X}_1$

over $(f,\, \beta\gamma)$ is the 4–plane $U := f \cdot \mathcal{S}^1_y + \beta\gamma \cdot \mathcal{S}^1_x$. In view of (1.5)-4, we are required to check that U contains $\alpha\beta\cdot\mathcal{S}^1_z$. For this, let $\mathcal{S}^1_z = \langle\gamma, \gamma'\rangle$, $\mathcal{S}^1_x = \langle\alpha, \alpha'\rangle$. We clearly have $\alpha\beta\gamma \in U$. Since $f \in \langle\alpha, \gamma\rangle$ and γ does not divide f, we may write $f = \alpha\gamma' + \beta'\gamma$ for some $\beta' \in \mathcal{S}^1_x$. Now $\alpha\beta\gamma' = f\beta - \beta'\beta\gamma \in U$ as desired. Finally, if $\gcd(f, \beta\gamma) \neq 1$, we have $f = \gamma\overline{f}$ for some $\overline{f} \in \mathcal{S}^1_x$ and we certainly have $U(\alpha, \beta, \gamma, f) \subset \mathcal{S}^1_z \cdot (\overline{f}\mathcal{S}^1_y + \beta\mathcal{S}^1_x)$ as required by the description of the exceptional divisor (cf. 1.5, 4).

Obviously ψ is $\mathbb{G}$-invariant. Thus in order to check that ψ is an embedding it suffices to do so in a neighborhood of the point (x_1, z_1, x_1z_1, y_1) representing the closed orbit. Clearly f, β, γ are uniquely determined by $\psi(\alpha, \gamma, f, \beta)$. Now if $\psi(\alpha, z_1, x_1z_1, y_1) = \psi(x_1, z_1, x_1z_1, y_1)$ then $\alpha y_1 z_0 \in x_1z_1\mathcal{S}^1_y + y_1z_1\mathcal{S}^1_x + x_1y_1\mathcal{S}^1_z$. It follows easily that $\alpha = x_1$. This shows that ψ is injective at closed points. One can readily see that ψ is also unramified, hence an embeding.

Next we show that ψ lifts to the map

$$\begin{array}{rcl} \widetilde{\psi} : \mathbb{K}_x \times \mathbb{P}\left(\mathcal{S}^1_y\right) & \longrightarrow & \mathbb{X}^2_{2,1} \subset \mathbb{X}^1_{2,1} \times \mathbb{G}\mathrm{r}_8\left(\mathcal{S}^{1,1,3}_{xyz}\right) \\ \eta = \big(\alpha(x), \gamma(z), f(x,z), \beta(y)\big) & \mapsto & \big(\psi(\eta),\, \gamma \cdot V(\beta, f)\big) \end{array}$$

with

$$V(\beta, f) := f \cdot \mathcal{S}^{1,1}_{yz} + \beta \cdot \mathcal{S}^{1,2}_{xz} \subset \mathcal{S}^{1,1,2}_{xyz}. \tag{16}$$

The vector space $V(\beta, f)$ is of the correct dimension, 8, since it is the space of forms of tri-degree (1,1,2) vanishing on the conic defined by $f(x,z)$, $\beta(y)$ (see 1). Furthermore, $V(\beta, f)$ contains the 7–plane $\mathcal{S}^1_z \cdot U(\alpha, \beta, \gamma, f)$ and $\gamma \cdot V(\beta, f)$ is an 8-plane of the form prescribed by (2.4-3). Therefore $\widetilde{\psi}$ factors through $\mathbb{X}^2_{2,1}$ as stated. The space of forms of tri-degree (2,2,2) (resp. (2,1,2)) vanishing on that conic is of dimension 22 (resp. 13). Hence the rank of μ_{xy} (resp. μ_x) is 22 (resp. 13) over the image of $\widetilde{\psi}$. Arguing in the same way one checks that the rank of μ_y is 14 over the image of $\widetilde{\psi}$. We set

$$\mathbb{X}^2_{\mu_x} = \widetilde{\psi}(\mathbb{K}_x \times \mathbb{P}\left(\mathcal{S}^1_y\right)) \tag{17}$$

and define $\mathbb{X}^2_{\mu_y}$ similarly, interchanging the roles of x, y.

Note that the (unique) closed orbit of $\mathbb{K}_x \times \mathbb{P}\left(\mathcal{S}_y^1\right)$ is $\mathbb{G}\cdot(x_1, z_1, x_1z_1, y_1)$. It is mapped by $\widetilde{\psi}$ to $\mathbb{G}\cdot\left((x_1z_1^2, y_1z_1^2), U, V\right)$ with U as in (1) and $V = z_1\cdot(\langle x_0y_1z_0^2\rangle + U\mathcal{S}_z^1)$. Hence $\mathbb{X}_{\mu_x}^2$ contains the closed orbit $\mathbf{o}_1^2$ of $\mathbb{X}_2^2$ determined by $x_0y_1z_0^2z_1$ and likewise $\mathbb{X}_{\mu_y}^2$ contains the one given by $x_1y_0z_0^2z_1$.

We claim that μ_{xy} drops rank to 22 precisely over $\mathbb{X}_{\mu_x}^2 \cup \mathbb{X}_{\mu_y}^2$.

To prove the claim, we compute a local matrix expression for μ_{xy} around representatives of the three closed orbits (2.4) of $\mathbb{G}$ in $\mathbb{X}_2^2$. The reader may check at once that $\operatorname{rank}\mu_{xy} = 24$ over the orbit given by $x_1y_1z_0^3$. (This corresponds to the curve defined by $x_1z_1^2$, $y_1z_1^2$, $x_1y_1z_0^3$. It actually yields a point in $\mathbb{H}\mathrm{ilb}$.) We also have $\operatorname{rank}\mu_x = \operatorname{rank}\mu_y = 14$ over that same orbit. However, $\operatorname{rank}\mu_x = 13$ and $\operatorname{rank}\mu_y = 14$ over the orbit $\mathbf{o}_1^2$ given by $x_0y_1z_0^2z_1$, whereas $\operatorname{rank}\mu_x = 14$ and $\operatorname{rank}\mu_y = 13$ over the orbit given by $x_1y_0z_0^2z_1$. We work in a neighborhood $\mathbb{U}^2$ of $\mathbb{X}_2^2$ around a representative of the closed orbit $\mathbf{o}_1^2$ and lying over the neighborhood $\mathbb{U}^1$ employed in the proof of (2.3). The map $\mathbb{U}^2 \to \mathbb{U}^1$ is given by

$$\begin{cases} a_7 = -c_2\varepsilon_2 - {b_2}^2 + b_2a_6, \\ b_4 = c_3\varepsilon_2 + b_2b_3, \end{cases}$$

where

$$\varepsilon_2 := b_6 - b_5b_2$$

is a local generator for the ideal of the exceptional divisor. Using MAPLE, we find a 16×24 (resp. 32×36)–matrix representation for μ_x (resp. μ_{xy}). A local basis of $\mathcal{V}$ is obtained from the eight rows of the matrix representing the pullback of μ_z. We substitute the above two relations in the matrix (9). The last row becomes divisible by ε_2. The resulting eight rows yield a basis for $\mathcal{V} \subset \mathcal{S}_{xyz}^{1,1,3}$ over $\mathbb{U}^2$ (see 1.3). Multiplying by x_i (resp. x_iy_j) and collecting coefficients with respect to a suitable basis of $\mathcal{S}_{xyz}^{2,1,3}$ (resp. $\mathcal{S}_{xyz}^{2,2,3}$) yields a matrix representation for μ_x (resp. μ_{xy}). Performing elementary operations we reduce μ_x (resp. μ_{xy})

to the form

$$[\mu_x] = \begin{bmatrix} I_{13} & * & * & * \\ 0 & \star & \cdots & \star \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \Big(\text{resp. } [\mu_{xy}] = \begin{bmatrix} I_{22} & * & * & * \\ 0 & \star & \cdots & \star \\ 0 & \star & \cdots & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}\Big) \tag{18}$$

where I_{13} (resp. I_{22}) denotes an identity block and the row with $\star$'s generate the ideal

$$\langle b_3, c_2, c_3, a_6 - 2\,b_2, b_6 - b_5 b_2 \rangle.$$

Proceeding similarly for μ_{xy} we find the two rows with $\star$'s generate *exactly* the same ideal. This shows that the scheme of zeros of $\overset{23}{\wedge}\mu_{xy}$ and $\overset{14}{\wedge}\mu_x$ coincide along $\mathbb{U}^2$ with a smooth, irreducible variety of dimension 5. Hence it coincides with $\mathbb{X}^2_{\mu_x} \cap \mathbb{U}^2$.

□

Remark 3.2. If we blow up $\mathbb{X}^2_2$ along the zeros of $\overset{24}{\wedge}\mu_{xy}$ (instead of $\overset{23}{\wedge}\mu_{xy}$), the resulting variety embeds in $\mathbb{X}^2_2 \times \mathbb{G}\mathrm{r}_{24}(\mathcal{S}^{2,2,3}_{xyz})$ but is singular. Instead, we prefer to take $\mathbb{X}^3_2$ as the blowup of $\mathbb{X}^2_2$ along the smooth center $\mathbb{X}^2_{\mu_{xy}} = \mathbb{X}^2_{\mu_x} \cup \mathbb{X}^2_{\mu_y}$ described above. We only get a *rational* map $\mathbb{X}^3_2 \cdots\to \mathbb{G}\mathrm{r}_{24}(\mathcal{S}^{2,2,3}_{xyz})$. Its indeterminacies will be resolved by further blowing up along a smooth center. This is done in the next section.

Proposition 3.3. *With notation as above, let* $\mathbb{X}^3_2 \to \mathbb{X}^2_2$ *be the blowup along* $\mathbb{X}^2_{\mu_{xy}} = \mathbb{X}^2_{\mu_x} \cup \mathbb{X}^2_{\mu_y}$. *Then we have the following.*

1. $\mathbb{X}^3_2$ *embeds in* $\mathbb{X}^2_2 \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz}) \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{1,2,3}_{xyz})$.

2. *The exceptional divisor* $\mathbb{X}^3_{\mu_{xy}} \to \mathbb{X}^2_{\mu_{xy}}$ *is the* $\mathbb{P}^4$*–bundle whose fiber over a point*

$$(\alpha(x), \gamma(z), f(x,z), \beta(y)) \in \mathbb{X}^2_{\mu_x}\ \big(= \widetilde{\psi}(\mathbb{K}_x \times \mathbb{P}\,(\mathcal{S}^1_y))\big)$$

is given by

$$\mathbb{P}\left((\mathcal{S}^{1,1}_{xz} \cdot V)\big/(\gamma \cdot \mathcal{S}^1_x \cdot V)\right)$$

with $V = V(\beta, f) \subset \mathcal{S}^{1,1,2}_{xyz}$ *as in* (16). *Similarly for fibers over* $\mathbb{X}^2_{\mu_y}$.

3. There are exactly two closed orbits in the exceptional divisor $\mathbb{X}^3_{\mu_{xy}}$:
- *one is represented by $x_1^2 y_0 z_0^2 z_1$ and lies over the closed orbit (cf. 3.1(5)) of $\mathbb{X}^2_{\mu_x}$;*
- *the other is represented by $x_0 y_1^2 z_0^2 z_1$ and lies over the closed orbit of $\mathbb{X}^2_{\mu_y}$.*

Proof. Since the blowup center is a disjoint union of two smooth pieces, we have that $\mathbb{X}^3_2$ is the same as first blowing up $\mathbb{X}^2_{\mu_x}$ say, and then blowing up the (strict=total) transform $\mathbb{X}^{2\,\prime}_{\mu_y}$ of $\mathbb{X}^2_{\mu_y}$. The first blowup, call it $\mathbb{X}'$, embeds in $\mathbb{X}^2_2 \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz})$ whereas the blowup $\mathbb{X}^3_2$ of $\mathbb{X}'$ along $\mathbb{X}^{2\,\prime}_{\mu_y}$ embeds in $\mathbb{X}' \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{1,2,3}_{xyz})$. Assertion 1 now follows easily.

For assertion 2 we show by explicit computation (with the help of Maple) that the image in $\mathbb{X}^2_{\mu_x} \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz})$ of the projective bundle described in the statement coincides with the image of the exceptional divisor over an open dense subset of $\mathbb{X}^2_{\mu_x}$.

We consider an affine chart $\mathbb{U}^3$ of $\mathbb{X}^3_2$ produced by the prescription (1.3), lying over the neighborhood $\mathbb{U}^2$ used in the proof of (3.1). Presently c_3 is a local generator for the ideal of the exceptional divisor and

$$a_1,\ a_3,\ a_7,\ a_8,\ b_2,\ b_4,\ b_5,\ c_3,\ d_2,\ d_3,\ d_4,\ d_5$$

are coordinate functions on $\mathbb{U}^3$. The map $\mathbb{U}^3 \to \mathbb{U}^2$ is given by

$$a_6 = 2, b_2 + d_3c_3,\ b_3 = -d_4c_3,\ b_6 = b_5b_2 - d_5c_3,\ c_2 = d_2c_3. \qquad (19)$$

We will compute the fiber of the exceptional divisor $\mathbb{X}^3_{\mu_x} \longrightarrow \mathbb{X}^2_{\mu_x}$ over a point $(\alpha, \gamma, f, \beta) \in \mathbb{X}^2_{\mu_x}$. Each point in that fiber may be represented by a 14-plane $W \subset \mathcal{S}^{2,1,3}_{xyz}$, obtained from the first matrix in (18). For this we substitute the relations (19) in that matrix. The 14th row becomes divisible by the local equation (c_3) of the exceptional divisor. Dividing that row by c_3 and then setting $c_3 = 0$, we find that the resulting row yields a polynomial $H_3 \in \mathcal{S}^{2,1,3}_{xyz}$ such that $V \cdot \mathcal{S}^1_x + \langle H_3 \rangle$ is a 14-plane. Here V is short for the tautological rank 8 subbundle $\mathcal{V} \subset \mathcal{S}^{1,1,3}_{xyz}$ (cf. 2.4) restricted to the present affine chart. We find that H_3 can be written in the form, $H_3 := q_1 f + q_2\beta$, where

$$\begin{aligned} q_1 &= (x_1 + (b_5 + d_4)x_0)y_0 z_0^2, \\ f &= \alpha\gamma' + \alpha'\gamma, \\ \beta &= y_1 + a_8 y_0, \\ q_2 &= ((d_5 + d_3 b_5 + d_2 b_5^2)x_0^2 + d_2 x_1^2 + (d_3 + 2d_2 b_5)x_0 x_1)z_0^3 \end{aligned}$$

with

$$\alpha := x_1 + b_5 x_0, \quad \gamma := z_1 + b_2 z_0, \quad \gamma' = (a_1 - 2b_2)z_0, \quad \alpha' = x_1 + a_3 x_0.$$

Thus we get that H_3 lies in $\mathcal{S}^{1,1}_{xz} \cdot V = \mathcal{S}^{1,1,2}_{xyz} \cdot f + \mathcal{S}^{2,3}_{xz} \cdot \beta$ with $V = V(\beta, f)$ as in (16). By construction, according to 1.3, we have $W = \langle H_3 \rangle + \mathcal{S}^1_x \cdot \gamma \cdot V$. This proves assertion 2.

Acting with suitable 1-parameter subgroups of the stabilizer of the point in $\mathbf{o}^2_1$ (which corresponds to $\alpha = x_1$, $\beta = y_1$, $\gamma = z_1$, $f = x_1 z_1$), one sees that the orbit given by the 14-plane

$$W = x_1^2 y_0 z_0^2 z_1 + \mathcal{S}^1_x \cdot z_1 \cdot V(y_1, x_1 z_1) \subset \mathcal{S}^{2,1,3}_{xyz}$$

is the unique closed orbit of $\mathbb{X}^3_{\mu_{xy}}$ lying over the closed orbit $\mathbf{o}^2_1 \subset \mathbb{X}^2_{\mu_x}$ (3.1(5)). The verification for the orbit represented by $x_0 y_1^2 z_0^2 z_1$ is similar.

□

4 Final step

Our goal is to show that the saturation of the image of $(\mu_{xy})_{|\mathbb{X}^4_2}$ is a locally split submodule of $\mathcal{S}^{223}_{xyz} \otimes \mathcal{O}_{\mathbb{X}^4_2}$ of rank 24. Here

$$\mathbb{X}^4_2 \subset \mathbb{X}^3_2 \times \mathbb{G}\mathrm{r}_{24}(\mathcal{S}^{2,2,3}_{xyz})$$

is the blowup of $\mathbb{X}^3_2$ along a smooth center, denoted $\mathbb{X}^3_{xy}$, to be described below.

Once this is accomplished, the variety $\mathbb{X}^4_2$ will parametrize a family of subschemes of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by 24 linearly independent polynomials of multidegree (2,2,3). Further, the construction yields an explicit list of the closed orbits of $\mathbb{X}^4_2$. We then check that the closed subscheme of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to a representative of each of these orbits has the correct Hilbert polynomial.

Hence $\mathbb{X}_2^4$ is the promised desingularization of the component of $\mathbb{H}$ilb of curves in $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ given by graphs of maps $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (2,2).

Let

$$\mathcal{M} \subset \mathcal{S}_{xyz}^{223} \otimes \mathcal{O}_{\mathbb{X}_2^3}$$

denote the saturation of the image of $\mu_{xy|\mathbb{X}_2^3}$. Let

$$\mathcal{M}_{14}^{213} \subset \mathcal{S}_{xyz}^{213} \otimes \mathcal{O}_{\mathbb{X}_2^3} \qquad (\text{resp. } \mathcal{M}_{14}^{123} \subset \mathcal{S}_{xyz}^{123} \otimes \mathcal{O}_{\mathbb{X}_2^3})$$

be the saturation of the image of $\mu_{x|\mathbb{X}_2^3}$ (resp. $\mu_{y|\mathbb{X}_2^3}$). By construction, the sheaves $\mathcal{M}_{14}^{123}$, $\mathcal{M}_{14}^{213}$ are locally split subbundles of $\mathcal{S}_{xyz}^{123} \otimes \mathcal{O}_{\mathbb{X}_2^3}$, $\mathcal{S}_{xyz}^{213} \otimes \mathcal{O}_{\mathbb{X}_2^3}$ of rank 14.

The saturations of the images of the multiplication maps

$$\mathcal{M}_{14}^{123} \otimes \mathcal{S}_x^1 \xrightarrow{\widetilde{\mu}_x} \mathcal{S}_{xyz}^{223} \otimes \mathcal{O}_{\mathbb{X}_2^3} \qquad \text{and} \qquad \mathcal{M}_{14}^{213} \otimes \mathcal{S}_y^1 \xrightarrow{\widetilde{\mu}_y} \mathcal{S}_{xyz}^{223} \otimes \mathcal{O}_{\mathbb{X}_2^3} \qquad (20)$$

are both equal to $\mathcal{M}$ since they all coincide over an open dense subset of $\mathbb{X}_2^3$. We shall employ these two multiplication maps in order to compute local presentations of

$$\mathcal{Q} := \operatorname{coker}\left(\mathcal{M} \hookrightarrow \mathcal{S}_{xyz}^{223} \otimes \mathcal{O}_{\mathbb{X}_2^3}\right).$$

The last blowup center, $\mathbb{X}_{xy}^3$, is defined by the Fitting ideal of $\mathcal{Q}$ given by the $24{\times}24$ minors of local matrix representations of the maps (20). The generic ranks of $\mathcal{M}$, $\mathcal{Q}$ are respectively 24, $36 - 24 = 12$.

Recall there are 3 closed orbits in $\mathbb{X}_2^3$, namely, one coming from below, i.e., disjoint from the exceptional divisor of $\mathbb{X}_2^3 \longrightarrow \mathbb{X}_2^2$ (cf. (2.4), (2.5)) and two more lying in the exceptional divisor (cf. 3.3(3)).

One checks that both $\widetilde{\mu}_x$, $\widetilde{\mu}_y$ are of the maximal rank 24 in a neighborhood of the first closed orbit, whence $\mathcal{M}$ is locally split there. As we have already observed (cf. (2.5)), the rational map Γ (cf. 2.1(2)) extends to a neighborhood of this closed orbit.

We compute a local matrix representation for $\widetilde{\mu}_y$ in a neighborhood of the closed orbit given by $x_1^2 y_0 z_0^2 z_1$ as in 3.3(3). The rank of $\widetilde{\mu}_y$ drops to 23. Proceeding by explicit calculation as in the previous cases, we find that the ideal of $24{\times}24$ minors is generated by

$$d_2,\ d_3,\ d_5,\ 2b_2 - a_1. \qquad (21)$$

Hence $\mathbb{X}^3_{xy}$ is a smooth subvariety of dimension six. We proceed to give first a global description of it and finally, check that blowing it up yields the desired flat family.

The local expressions for the various aspects, $F, G, \dots$ corresponding to a point in $\mathbb{X}^3_{xy} \subset \mathbb{X}^3_2 \subset \mathbb{X}^2_2 \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz}) \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{1,2,3}_{xyz})$ are as follows:

$$\begin{cases} F = (z_1 + b_2 z_0)^2 (a_3 x_0 + x_1), \\ G = (z_1 + b_2 z_0)^2 (a_8 y_0 + y_1), \\ H_1 = z_0 (z_1 + b_2 z_0)\big(x_1 y_1 + (a_8 - d_4 c_3) x_1 y_0 + (a_8 b_5 - a_3 d_4 c_3) x_0 y_0 + b_5 x_0 y_1\big), \\ H_2 = z_0^2 (z_1 + b_2 z_0)\big((a_8 + a_3 c_3) y_0 x_0 + c_3 x_1 y_0 + x_0 y_1\big), \\ H_3 = z_0^2 (z_1 + b_2 z_0)\big(x_1 + (d_4 + b_5) x_0\big)(a_3 x_0 + x_1) y_0. \end{cases} \tag{22}$$

Here $\langle H_1 \rangle + F \cdot \mathcal{S}^{1,2}_{yz} + G \cdot \mathcal{S}^{1,2}_{xz}$ produces a 4-plane U, and likewise, $V = \langle H_2 \rangle + U \cdot \mathcal{S}^1_z$ is an 8-plane and $W = \langle H_3 \rangle + V \cdot \mathcal{S}^1_x$ is a 14-plane.

Our present task is to give a global description for $\mathbb{X}^3_{xy}$ matching the data just described. Fix

$$(f, g) \in \mathbb{P}(\mathcal{S}^1_x) \times \mathbb{P}\left(\mathcal{S}^1_y\right).$$

Let $\mathcal{E}$ be the rank 3 vector bundle over $\mathbb{P}(\mathcal{S}^1_x) \times \mathbb{P}\left(\mathcal{S}^1_y\right)$ with fiber

$$\mathcal{E}_{(f,g)} = f\mathcal{S}^1_y + g\mathcal{S}^1_x.$$

Put

$$\mathcal{F} = \mathcal{E}/\mathcal{O}_{\mathcal{E}}(-1).$$

Pick

$$h_1 \in \mathbb{P}\left(\mathcal{E}_{(f,g)}\right),$$
$$h_2 \in \mathbb{P}\left((\mathcal{E}_{(f,g)})/\langle h_1 \rangle\right) = \mathbb{P}\left(\mathcal{F}_{h_1}\right).$$

Thus, h_1, h_2 define a 2-plane $\langle h_1, h_2 \rangle$ in $\mathcal{E}_{(f,g)}$. Since the latter space is three dimensional, we must have

$$\langle h_1, h_2 \rangle \cap g\mathcal{S}^1_x \supseteq \langle g h_3 \rangle$$

for some $h_3 \in \mathbb{P}(\mathcal{S}^1_x)$. This h_3 is unique unless $\langle h_1, h_2 \rangle = g\mathcal{S}^1_x$. The natural embedding

$$\mathbb{P}^1 \cong \mathbb{P}(g\mathcal{S}^1_x) \hookrightarrow \mathbb{P}\left(f\mathcal{S}^1_y + g\mathcal{S}^1_x\right) = \mathbb{P}\left(\mathcal{E}_{(f,g)}\right) \cong \mathbb{P}^2$$

lifts to the embedding,

$$\begin{array}{ccc} \mathbb{P}(g\mathcal{S}^1_x) & \hookrightarrow & \mathbb{P}\left(\mathcal{E}_{(f,g)}/\mathcal{O}_{\mathcal{E}_{(f,g)}}(-1)\right) \\ gh & \mapsto & (gh\,,\, g\mathcal{S}^1_x/gh). \end{array}$$

These are restrictions to a fiber of the embeddings of bundles,

$$\begin{array}{ccccc} & & & & \mathbb{P}(\mathcal{F}) \\ & & \nearrow & & \downarrow \\ \mathbb{P}\left(\mathcal{O}_{\mathcal{S}^1_y}(-1)\otimes\mathcal{S}^1_x\right) & & \hookrightarrow & & \mathbb{P}(\mathcal{E}) \\ & & \searrow \quad \swarrow & & \\ & & \mathbb{P}(\mathcal{S}^1_x)\times\mathbb{P}\left(\mathcal{S}^1_y\right). & & \end{array}$$

Blowing up the image of the top slant embedding yields the smooth 5-fold,

$$\begin{aligned} \mathbb{X}_x &:= \widetilde{\mathbb{P}(\mathcal{F})} \\ &= \left\{ \begin{array}{ll} (f,g,h_1,h_2,h_3) \mid & f\in\mathbb{P}(\mathcal{S}^1_x)\,,\ g\in\mathbb{P}\left(\mathcal{S}^1_y\right), \\ & h_1\in\mathbb{P}\left(f\mathcal{S}^1_y+g\mathcal{S}^1_x\right), \\ & h_2\in\mathbb{P}\left((f\mathcal{S}^1_y+g\mathcal{S}^1_x)/\langle h_1\rangle\right), \\ & h_3\in\mathbb{P}(\mathcal{S}^1_x),\ \text{with } gh_3\in\langle h_1,h_2\rangle \end{array} \right\}. \end{aligned}$$

We define a map,

$$\iota : \mathbb{X}_x\times\mathbb{P}(\mathcal{S}^1_z) \longrightarrow \mathbb{X}\times\mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,2}_{xyz})\times\mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz})\times\mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz})$$

by assigning to $\left((f,g,h_1,h_2,h_3),\gamma\right)\in\mathbb{X}_x\times\mathbb{P}(\mathcal{S}^1_z)$

the pair $(f\gamma^2,g\gamma^2)\in\mathbb{X}_2$,

the 4-plane $U=(f\mathcal{S}^1_y+g\mathcal{S}^1_x)\gamma^2+h_1\gamma\mathcal{S}^1_z$,

the 8-plane $V=U\mathcal{S}^1_z+h_2\gamma\mathcal{S}^1_z$ and

the 14-plane $W=V\mathcal{S}^1_x+h_3f\gamma\mathcal{S}^1_y\mathcal{S}^2_z$.

We now show that the map ι is an embedding that factors through $\mathbb{X}^3_2$. The assertion for embedding is straightforward and details will be omitted. To check that it factors through $\mathbb{X}^3_2$, one verifies first that the spaces U,V fit their prescriptions as in (1.5)-(4), (2.4). Now the main point is to see that $W = V \cdot \mathcal{S}^1_x$ holds in fact over an open dense subset of $\mathbb{X}_x \times \mathbb{P}(\mathcal{S}^1_z)$. This clearly implies, by continuity, that ι factors as asserted.

Acting with the group $\mathbb{G}$ on $\mathbb{X}_x \times \mathbb{P}(\mathcal{S}^1_z)$, we may set

$$\begin{cases} f := x_1, \qquad g := y_1, \qquad \gamma := z_1 \\ h_1 = x_1y_1 + \alpha_1 x_1 y_0 + \alpha_2 x_0 y_1, \\ h_3 = x_1 + \alpha_3 x_0, \quad \text{and} \\ h_2 = h_1 - g h_3. \end{cases}$$

If $\alpha_1 \neq 0$ or $\alpha_2 \neq \alpha_3$ one checks easily that

- the above assignments yield legitimate choices for a point in $\mathbb{X}_x \times \mathbb{P}(\mathcal{S}^1_z)$;
- these points fill an open dense subset of the fiber over (f, g, γ).

It is also easy to see that U as defined above fits (1.5). Now for the 8-plane, note that $V = U \cdot \mathcal{S}^1_z + \langle z_0^2 \gamma h_2 \rangle$. Thus we see that our $H_2 = z_0^2 \gamma h_2$ meets the requirement in (2.4). For $\alpha_1 \neq 0$ we have that $V \cdot \mathcal{S}^1_x$ is a 14-plane. Moreover, $a_1 h_3 f \gamma y_0 z_0^2$ is equal to the expression $(\alpha_3 - \alpha_2)\gamma z_0^2 x_0 h_1 + \gamma z_0^2(\alpha_2 x_0 + x_1)h_2$. It follows that $h_3 f \gamma \mathcal{S}^1_y \mathcal{S}^1_z \subset V \mathcal{S}^1_x$ holds over an open dense subset, as claimed.

Finally, we check the assertion about closed orbits.

The description of $\mathbb{X}_x$ shows that $\mathbf{o}_x = (x_1, y_1, x_1y_1, x_2y_1, x_1)$ represents its unique closed orbit. Likewise, $(\mathbf{o}_x, z_1)$ works for $\mathbb{X}_x \times \mathbb{P}(\mathcal{S}^1_z)$. Its image in $\mathbb{X}^3_2$ corresponds to the point $\mathbf{o}^3$ obtained in (22) by setting all a's,…,d's coordinates to zero. The fiber of the exceptional divisor over that representative of the closed orbit of $\mathbb{X}^3_2$ is a $\mathbb{P}^3$. For $(e_1 : \cdots : e_4)$ in that fiber, we find that its image in

$$\mathbb{X}_2 \times \mathbb{G}\mathrm{r}_4(\mathcal{S}^{1,1,2}_{xyz}) \times \mathbb{G}\mathrm{r}_8(\mathcal{S}^{1,1,3}_{xyz}) \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{2,1,3}_{xyz}) \times \mathbb{G}\mathrm{r}_{14}(\mathcal{S}^{1,2,3}_{xyz}) \times \mathbb{G}\mathrm{r}_{24}(\mathcal{S}^{2,2,3}_{xyz})$$

is given by

$$F = x_1 z_1^2,\ G = y_1 z_1^2,\ H_1 = x_1 y_1 z_0 z_1,\ H_2 = x_0 y_1 z_0^2 z_1,\ H_3 = x_1^2 y_0 z_0^2 z_1,$$
$$H_4 = y_1 z_0^3 (y_1(e_1 x_0^2 + e_2 x_1^2 + e_4 x_0 x_1) + e_3 y_0 x_1^2).$$

Acting on that $\mathbb{P}^3$ with the stabilizer of $\mathbf{o}^3$, namely, the subgroup of $\mathbb{G}$ (cf. §1) fixing the rays $\langle x_1 \rangle, \langle y_1 \rangle, \langle z_1 \rangle$, we find exactly one fixed point. It is given by

$e_1 = 0$, $e_3 = 0$, $e_4 = 0$. This corresponds to the choice $H_4 = x_1^2y_1^2z_0^3$. It can be checked that the subscheme of $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$defined by this choice for $F, \ldots, H_4$ has the correct Hilbert polynomial. It is in fact equal to the subscheme defined by $\langle y_1z_1, x_1z_1^2, x_1^2z_1, x_1^2y_1^2\rangle$.

Similarly, there is another closed orbit, corresponding now to the choice, $F = x_1z_1^2,\ G = y_1z_1^2,\ H_1 = x_1y_1z_0z_1,\ H_2 = x_1y_0z_0^2z_1,\ H_3 = x_0y_1^2z_0^2z_1,\ H_4 = x_1^2y_1^2z_0^3$. We remark that these two orbits are indeed distinct in $\mathbb{X}_2^4$, as well as their images in $\mathbb{H}$ilb. Indeed, the latter one is given by the ideal $\langle x_1z_1, y_1z_1^2, y_1^2z_1, x_1^2y_1^2\rangle$.

Summarizing, these are the closed orbits in $\mathbb{X}_2^4$:

1. $\langle x_1z_1^2,\ y_1z_1^2,\ x_1y_1z_0^3\rangle$ from (2.5);
2. $\langle x_1z_1^2,\ y_1z_1^2,\ x_1y_1z_0z_1,\ x_0y_1z_0^2z_1,\ x_1^2y_0z_0^2z_1,\ x_1^2y_1^2z_0^3\rangle$;
3. $\langle x_1z_1^2,\ y_1z_1^2,\ x_1y_1z_0z_1,\ x_1y_0z_0^2z_1,\ x_0y_1^2z_0^2z_1,\ x_1^2y_1^2z_0^3\rangle$.

A resolution of the corresponding ideals can be easily written down. A routine check at these closed orbits shows that the corresponding sheaf of ideals $\mathcal{I}$ on $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ is (2,2,3)-regular in the following sense: $H^1(\mathcal{I}(2,2,3)) = 0$ and $\mathcal{I}(a,b,c)$ is spanned by global sections for $(a,b,c) \geq (2,2,3)$ and $H^0(\mathcal{I}(a,b,c))$ is in fact equal to the image of $H^0(\mathcal{I}(2,2,3)) \otimes \mathcal{S}_{xyz}^{a-2,b-2,c-3}$.

By upper-semicontinuity, these assertions survive in a neighborhood of the closed orbit. It follows that the component $\mathbb{H}$ of $\mathbb{H}$ilb we are studying embeds in $\mathbb{G}\mathrm{r}_{24}(\mathcal{S}_{xyz}^{2,2,3})$. It is equal to the image of $\mathbb{X}_2^4$. We have checked that the map $\mathbb{X}_2^4 \to \mathbb{H}$ shrinks a smooth divisor (equal to the strict transform of the first exceptional divisor cf. 1.5), and the contracted fibers are isomorphic to $\mathbb{P}^1$ as schemes. This looks pretty much like a blowup along a codimension 2 locus in $\mathbb{H}$. So we suspect in fact that this component is smooth.

The terminal orbit corresponds to the subscheme of $\mathbb{P}^1{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ defined by $y_1z_1^2, x_1z_1^2, x_1y_1$. It is Cohen-Macaulay, with three components, $\langle y_1, z_1^2\rangle$, $\langle x_1, z_1^2\rangle$ and $\langle x_1, y_1\rangle$.

The second and third orbits give isomorphic (but distinct) subschemes with 4 primary components. They share the three componentes $\langle z_1, y_1^2\rangle$, $\langle z_1, x_1^2\rangle$,

$\langle x_1, y_1\rangle$, but differ at the embedded point $\langle z_1^2, y_1, x_1^2\rangle$ on the 2nd and $\langle z_1^2, x_1, y_1^2\rangle$ on the 3rd.

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