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ACTIONS ON ALGEBRAS AND APPLICATIONS

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Abstract

This text represents some working materials for the author's mini-course delivered at Algebra School in Brasilia within the week of July 24 - July 28, 2000. This is not a finished research or survey paper and not an account of the talks actually given. The author apologizes for any inconveniences that may arise because of such a nature of all that follows.

1. Theme One: Gradings and Actions on Associative Algebras

1.1 Introduction. Some Examples

In this lecture mini-course we are going to discuss some recent results on various classes of algebraic systems endowed with additional structure, which can be formalized as the action (or, quite often, coaction) of a Hopf algebra. In particular, we will speak about graded algebras, algebras with a fixed group of automorphisms or with a fixed Lie algebra of derivations, etc.

There is no need to put an additional effort to have these structures: they naturally exist in the whole number of situations. For example, the polynomial or free algebras are graded by the degrees of their homogeneous elements and classical simple Lie algebras are graded by their root systems. Any algebra A has the automorphism group and the derivation Lie algebra, and these act on A, along with their subgroups and subalgebras.

An important remark: in these notes we are not covering any substantial portion of the material in the area of actions of algebras. Instead, we describe some results and approaches suggested in the recent work of the author and his research colleagues. There are a number of other researches and their results are summarized in quite a few books, of which we mention just [M3, Pa2, NVO].

We start with some simple examples, which can be done "by hand".

Let us look, for instance, at the gradings by the group $\mathbb{Z}_2$. Suppose we have an associative algebra A over a field F with such a grading: $A = A_0 \oplus A_1$. By definition, A_0 and A_1 are F-subspaces and $A_0A_0 \subset A_0$, $A_0A_1, A_1A_0 \subset A_1$ and $A_1A_1 \subset A_0$. This tells us that A_0 is a subalgebra, A_1 is a left and right A_0-module and there is also a bilinear map $\mu : A_1 \times A_1 \to A_0$ which is compatible with the left and right A_0-module structures on A_0 and A_1. In other words, the role of A_0 in this interplay between A_0 and A_1 is much more important. So, the basic question is: to what extent the properties of A are defined by those of A_0? For example, if $A_0 = 0$ then $A^2 = A_1^2 \subset A_0 = 0$. If we take A_1 arbitrary and $\mu = 0$ then, in this way, we expire all $\mathbb{Z}_2$-graded algebras A with $A_0 = 0$. If $A_0^2 = 0$ then $A^4 = 0$. Indeed, it is sufficient to look only at the product of the form $abcd$ where $a \in A_{i_1}$, $b \in A_{i_2}$, $c \in A_{i_3}$, and $d \in A_{i_3}$. Here i_j is always 0 or 1. Now, if we add indices modulo 2, then $a \in A_{i_1}$, $ab \in A_{i_1+i_2}$, $abc \in A_{i_1+i_2+i_3}$, $abcd \in A_{i_1+i_2+i_3+i_4}$. Now $m_1 = i_1$, $m_2 = i_1 + i_2$, $m_3 = i_1 + i_2 + i_3$ and $m_4 = i_1 + i_2 + i_3 + i_4$ take only two values 0 and 1. If at least two of them take the same value 0, say, m_k and m_l, with $k < l$, then the product of the first k factors and the next $l - k$ factors belongs to A_0, hence their product is zero. Otherwise at least 3 values, say m_l, m_2, m_3 take value 1. Then two neighboring products of the factors from $k + 1$ to l and from $l + 1$ to n are in A_0 and, again, the whole product is zero. Actually, exactly the same argument works to show that if A is graded by a not necessarily commutative group G and $(A_0)^n = 0$ then $A^{sn} = 0$. But if we assume that A_0 is commutative then already no such simple answer exists. Still, there is an answer and this will be discussed later.

The situation will immediately change if we allow the number of grading components to be infinite. For instance, let us consider the algebra A of polynomials of positive degree. It can be given a grading by integers if we assume that the homogeneous polynomials of degree n form the graded component A_n, for any $n \in \mathbb{Z}$. Then $A_n = 0$ except when n is positive. We have $A_0 = 0$ but of course $A^n \neq 0$, for any $n \geq 1$. This is just one example showing that if we want some properties of A_0 inherited by A we should always *assume the grading finite, i.e. only finitely many A_g nonzero.*

We can also look at the algebras where $\mathbb{Z}_2$ acts by automorphisms. For instance, what are the finite-dimensional associative algebras, which are simple in the following sense. A is simple in this sense if it has no proper nonzero ideals which are stable under the action of all automorphisms in question. Let us also for simplicity assume that the base field F of coefficients is algebraically closed. If $\mathbb{Z}_2$ acts trivially then we speak about simple algebras in the usual sense, that is, $A \cong M_n(F)$, the full matrix ring of order n over F. Otherwise, let $\varphi : A \to A$ be a nontrivial automorphism of order 2, the generator of $\mathbb{Z}_2$. Then if char$F \neq 2$ it has two eigenvalues ± 1 and we have the decomposition $A = A^1 \oplus A^{-1}$ into the sum of the respective eigenspaces. If $a \in A_\alpha$, $b \in A_\beta$, we have

$$\varphi(ab) = \varphi(a)\varphi(b) = (\alpha a)(\beta b) = (\alpha\beta)(ab),$$

showing that $ab \in A_{\alpha\beta}$. This shows that we can set $A_0 = A^1$ and $A_1 = A^{-1}$ and our algebra becomes $\mathbb{Z}_2$-graded. In this case it is well-known (and will be shown below) that there are only three types of such algebras

1. $A = F[\langle s \rangle_2] \otimes M_k$, k a natural number; if we choose a multiplicative form of $\mathbb{Z}_2 \cong \langle s \rangle_2$ then the basis of A_0 is spanned by the matrix units E_{ij} and that of A_1 by sE_{ij}, with the natural product.

2. $A = M_{k,l}$ where we have that A, as an algebra, is isomorphic to the full matrix algebra M_n, $n = k + l$, and the grading is given by splitting any matrix of order n into four matrices A_{11}, A_{12}, A_{21} and A_{22} of orders $k \times k$, $k \times l$, $l \times k$, and $l \times l$, respectively. The matrices with zero blocks A_{12} and A_{21} form A_0 and those with zero blocks A_{11} and A_{22} form A_1.

Clearly, now we also have the classification of associative finite-dimensional algebras, which are simple in the sense of the action of $\mathbb{Z}_2$ by automorphisms. The action of φ on A is identity on A_0 and negative identity on A_1. By the way, in the exceptional case of $\mathrm{char} F = 2$ if we take the group algebra $F[\langle s\rangle_2]$ then this is a $\mathbb{Z}_2$-simple algebra but it is not even semisimple and has a nonzero nilpotent radical $R = \langle 1+s\rangle$. This happens because $(1+s)^2 = 1^2 + s^2 = 0$. This is one of the reasons why we will be assuming in the future that *if a finite group G acts on an algebra by automorphisms then its order is not divisible by the characteristic of the field.*

As we saw from our examples, any $\mathbb{Z}_2$-graded algebra over a field of characteristic different from 2 is an algebra with an action of the group $\mathbb{Z}_2$. The same is true for any algebra graded by a finite abelian group G over an algebraically closed field F of characteristic zero. To show this we consider *the group of characters* $\widehat{G}$ of G. The elements of this group are the homomorphisms (=*characters*) of G into the multiplicative group F^* of nonzero elements of the base field F. The product $\chi\psi$ of two characters χ, $\psi \in \widehat{G}$ is given by $(\chi\psi)(g) = \chi(g)\psi(g)$. If $A = \bigoplus_{g\in G} A_g$ is a G-graded algebra, then we may set $\chi \circ a = \chi(g)a$ as soon as $a \in A_g$. Now we have that $\widehat{G}$ acts on A, that is,

$$(\chi\psi)\circ a = (\chi\psi)(g)a = \chi(g)\psi(g)a = \chi(g)(\psi\circ a) = \chi\circ(\psi\circ a),$$

as required. Also, assume $a \in A_g$, $b \in A_h$. Then $ab \in A_{gh}$ and we have

$$\chi\circ(ab) = \chi(gh)(ab) = (\chi(g)\chi(h))(ab) = (\chi(g)a)(\chi(h)b) = (\chi\circ a)(\chi\circ b),$$

as required. It is well-known from the representation theory of finite abelian groups that the elements of $\widehat{G}$ separate the elements of G, that is, for any $g \neq h \in G$ there is a character χ such that $\chi(g) \neq \chi(h)$. It follows that any $\widehat{G}$-invariant subspace B of A is *graded* in the following sense. One has that $B = \bigoplus_{g\in G} B_g$ where $B_g = B \cap A_g$ for any $g \in G$. For, if $a = \sum_g a_g$ belongs to an invariant subspace B then for any $\chi \in \widehat{G}$ we have the elements $\chi\circ a = \sum_g \chi(g)a_g$. It is now an elementary exercise to show that any $a_g \in B$, proving the required. Conversely, any G-graded subspace B is $\widehat{G}$-invariant. Indeed, in this case we have to check that $\chi\circ a \in B$ only for homogeneous a, that is, such one that $a \in A_g$ for some $g \in G$. But such an a is an eigenvector for any $\chi \in \widehat{G}$, proving the desired.

As an example, we can classify $\mathbb{Z}_p$-graded simple finite-dimensional associative algebras, p a prime number. We set $G = \mathbb{Z}_p$. Given a finite-dimensional associative algebra A without proper G-graded ideals we can assume actually that A is acted by the group of characters $\widehat{G}$ which is, as is well-known, isomorphic to G itself. So we will classify such A with an action of $\widehat{G}$ that have no nontrivial proper invariant subspaces. Notice that the nilpotent radical N of A is invariant under all automorphisms of A, in particular, under the action of $\widehat{G}$. It follows that $N = \{0\}$. Now A is a semisimple algebra, that is the sum of ideals which are simple algebras. If I is one of such ideals then $\widehat{G}$ acts on the sets of these ideals. The stabilizer $\Lambda = \{\chi | \chi \circ I = I\}$ of I under this action is a subgroup of $\widehat{G}$, hence $\Lambda = \{\varepsilon\}$ or either $\Lambda = \widehat{G}$. In the latter case I is $\widehat{G}$-invariant, so that $A = I$ is a simple algebra in the usual sense. In the former case there are precisely p different minimal ideals in the orbit of I. Because the sum of these ideals is $\widehat{G}$-invariant, these are all the minimal ideals of A. If ψ is a generator of $\widehat{G}$, we can write

$$A = \bigoplus_{k=0}^{p-1} \psi^k(I). \tag{1}$$

Now let us construct another algebra $C = F[\widehat{G}] \bigotimes I$, where $F[G]$ is the group algebra of G with the multiplication $(\chi \otimes x)(\rho \otimes y) = 0$ in all cases except $\rho = \psi$ in which case it is $\chi \otimes (xy)$. Now $\widehat{G}$ acts on C naturally: $\chi * (\pi \otimes x) = (\chi\pi) \otimes x$. If we do all the verifications we observe that $\widehat{G}$, indeed, acts on C. Thus C is a G-graded algebra. Let us show that our algebra A of the former case is isomorphic to C. We construct a map $f : C \to A$ given by $f(\chi \otimes x) = \chi \circ x$. Because for $\chi \neq \pi$ the ideals $\chi \circ I$ and $\pi \circ I$ are different, it easily follows that we have a homomorphism of algebras. Since $\widehat{G}$ is abelian f is compatible with the action of this group. Now f is obviously surjective from (1) and the dimensions of A and C are the same. So f is a homomorphism of algebras with the action of $\widehat{G}$. A simple argument then shows that f is also a homomorphism of G-graded algebras. Actually, we can replace $C = F[\widehat{G}] \otimes I$ by another G-graded algebra $D = F[G] \otimes I$, with natural G-grading $D_g = g \otimes I$ and natural tensor product multiplication $(a \otimes x)(b \otimes y) = (ab) \otimes (xy)$. We only have to map $\psi^k \otimes x$ into $\sum_{i=0}^{p-1} \nu^{ik} g_0^i$ where g_0 is the generator of G and ν is a p-th primitive root of 1.

A much more general result can be found in [BSZ2]. We are able to describe all G-graded simple algebras for a finite abelian group G. Recall that in this case a G-graded algebra $R = \oplus_{g \in G} R_g$ is a left module over the group ring $F[\widehat{G}]$ of the group $\widehat{G}$ of all irreducible characters on G and any $\chi \in \widehat{G}$ acts on R by an automorphism. Also recall that any G-graded algebra $R = \oplus_{g \in G} R_g$ has a canonical G/H-grading where H is a subgroup of G and $R_t = \oplus_{g \in t} R_g$ for any coset $t = aG \in G/H$.

Theorem 1.1 ([BSZ2]). *Let G be a finite abelian group and $R = \oplus_{g \in G} R_g$ a finite-dimensional G-graded algebra over an algebraically closed field of characteristic zero. If R is G-graded simple then:*

1) R is semisimple with isomorphic simple components;

2) there exists a subgroup $H \subseteq G$ and a simple ideal $B \subseteq R$ such that B is G/H-homogeneous;

3) as a G-graded algebra, R is isomorphic to the left $F[\widehat{G}]$-module $A = F[\widehat{G}] \otimes_{F[\Lambda]} B$ with the multiplication

$$(\chi \otimes b)(\psi \otimes c) = \begin{cases} 0 & \text{if } \psi\chi^{-1} \notin \Lambda \\ \chi \otimes b(\lambda * c) & \text{if } \psi = \chi\lambda \text{ with } \lambda \in \Lambda \end{cases} \tag{2}$$

where Λ is a subgroup of $\widehat{G}$ of all automorphisms $\mu \in \widehat{G}$ such that $\mu(B) \subseteq B$, moreover, $H = \{g \in G | \lambda(g) = 1 \text{ for all } \lambda \in \Lambda\}$.

Note that any algebra R satisfying 1), 2), 3) is always graded simple. Clearly, *this description is up to the gradings of matrix algebras.* We mentioned a result on $\mathbb{Z}_2$ earlier. Now we are going to use actions to obtain this classification and then formulate, without proofs, the results from [BSZ2], with the description of the gradings by an arbitrary finite abelian group on matrix algebras over algebraically closed fields of characteristic 0.

So, we want to find all possible $\mathbb{Z}_2$-gradings on the matrix algebra $A = M_n$ over an algebraically closed field of characteristic different from 2. According to what was said above, given any such grading, we have an automorphism of order

2 whose eigenspaces are exactly the grading subspaces A_0 and A_1. According to the well-known Noether - Skolem Theorem any automorphism is equal to an inner automorphism induced by a non-degenerate matrix $T \in M_n$. Since the automorphism has order 2 we have that T^2 is a scalar matrix λE. By our hypothesis there is $\mu \in F$ such that $\mu^2 = \lambda$. Then $U = \frac{1}{\mu}T$ induces the same automorphism and $U^2 = E$. Thus U is a diagonalizable matrix with the diagonal elements ± 1. Using an appropriate basis, we get $U = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with k positive 1's and l negative 1's, $k + l = n$. If we take an arbitrary $n \times n$ matrix X and naturally split it into four blocks X_{11} of order $k \times k$, X_{12} of order $k \times l$, X_{21} of order $l \times k$ and X_{22} of order $l \times l$ then

$$U \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} U^{-1} = \begin{pmatrix} X_{11} & -X_{12} \\ -X_{21} & X_{22} \end{pmatrix}.$$

This shows that

$$A_0 = \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \right\} \text{ and } A_1 = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \right\}.$$

Now we give some results about the gradings of matrix algebras from [BSZ2], omitting the proofs.

1.2 Abelian gradings on matrix algebras

We start with the definition of gradings of a special type. Let G be an arbitrary group and $R = M_n(F)$ the $n \times n$ matrix ring over a field F. We consider the n-th direct power $G^n = G \times \ldots \times G$ of the group G and show that with any element $x = (g_1, \ldots, g_n) \in G^n$ one can associate a grading on the matrix ring.

First, we define a grading on the *subring* generated by the matrix units $E_{ij}, 1 \leq i, j \leq n$. Fix $x = (g_1, \ldots, g_n) \in G^n$ and let

$$E_{ij} \in R_g \iff g = g_i^{-1} g_j. \tag{3}$$

Since $E_{ij}E_{kl} = 0$ for $k \neq j$, and $g_i^{-1}g_j g_j^{-1} g_r = g_i^{-1} g_r$, the condition (3) in fact defines a grading on R. If in addition we require that all scalar matrices belong to the component R_e where e is the identity element of G (this condition holds automatically if we consider $M_n(F)$ not as a graded ring but as a graded F-algebra), that is,

$$F \subseteq R_e, \tag{4}$$

then we get a finite G-grading on $M_n(F)$. This grading will be trivial if and only if $g_1 = \ldots = g_n$.

Definition 1.2.1. *A grading on the matrix ring $M_n(F)$ is called* elementary *if it satisfies Conditions (3), (4) for some set $(g_1, \ldots, g_n) \in G^n$.*

It is well-known that a simple (left) Artinian ring is isomorphic to the matrix ring $M_n(D)$ over some division ring D. For such a ring one can obtain a grading from Condition (1) if we require in addition that

$$D \subseteq R_e. \tag{5}$$

Definition 1.2.2. *A grading on a simple Artinian ring $M_n(D)$ is called* elementary *if it satisfies Conditions (3), (5) for some set $(g_1, \ldots, g_n) \in G^n$.*

Note that all matrix units are homogeneous with respect to any elementary grading and the grading is uniquely defined by $h_1, \ldots, h_{n-1} \in G$ such that $E_{12} \in R_{h_1}, \ldots, E_{n-1,n} \in R_{h_{n-1}}$.

Recall that a map $\varphi : A = \bigoplus_{g\in G} A_g \to B = \bigoplus_{g\in G} B_g$ is called a *homomorphism (isomorphism)* of graded rings if φ is an ordinary ring homomorphism (isomorphism) and $\varphi(A_g) \subseteq B_g, g \in G$.

Next we are going to describe all abelian gradings on finite-dimensional simple algebras over an algebraically closed field F of characteristic zero. If G is a finite group then the matrix algebra $M_n(F)$ admits a lot of elementary G-gradings. But one can construct also gradings of a different type, as it will be shown below.

Let $R = M_n(F)$ be a matrix algebra over an algebraically closed field F of characteristic zero and $G = \langle a\rangle_n \times \langle b\rangle_n$ the direct product of two cyclic groups of order n. Let also ε be a primitive nth root of 1 in F. We set

$$X_a = \begin{pmatrix} \varepsilon^{n-1} & 0 & \dots & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad Y_b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then

$$X_a Y_b X_a^{-1} = \varepsilon Y_b \ , \quad X_a^n = Y_b^n = E \tag{6}$$

and all $X_a^i Y_b^j, 1 \le i, j \le n$, are linearly independent. Clearly, the elements $X_a^i Y_b^j, i, j = 1, \dots, n$, form a basis of R and all products of these basic elements are uniquely defined by (6).

Now for any $g \in G, g = a^i b^j$, we denote by R_g a one-dimensional subspace

$$R_g = \langle X_a^i Y_b^j \rangle \tag{7}$$

Then from (6) it follows that $R = \oplus_{g \in G} R_g$ is a G-grading on $M_n(F)$.

Definition 1.2.3. *The grading on $M_n(F)$ given by (6),(7) is called an ε-grading.*

Remark 1.2.1. It is not difficult to see that for different primitive roots of 1, ε and μ, the matrix algebra M_n with ε-grading is not isomorphic to M_n with μ-grading, that is, there exist no isomorphism $\varphi : M_n \to M_n$ such that $\varphi(X_a(\varepsilon)) = X_a(\mu), \varphi(X_b(\varepsilon)) = X_b(\mu)$. But one can extend the notion of graded isomorphism and say that $A = \oplus_{g \in G} A_g$ is isomorphic to $B = \oplus_{h \in H} B_h$ as a *graded algebra* if there exist a group isomorphism $\tau : G \to H$ and algebra isomorphism $\varphi : A \to B$ such that $\varphi(A_g) = B_{\tau(g)}$. In this sense the ε-grading and the μ-grading are isomorphic.

All homogeneous components of the grading defined above are one - dimensional. Hence it cannot be elementary because the identity component of any elementary grading has dimension at least n.

Another way for finding new gradings is the tensor multiplication. Let G be an abelian group and S, T two subgroups in G. If $A = \oplus_{s \in S} A_s, B = \oplus_{t \in T} B_t$

are some S- and T-gradings on A and B, respectively, then $C = A \otimes B$ is a G-graded algebra with $C_g = \oplus_{st=g} A_s B_t$ and Supp $C = ST$. In particular, one can equip $C = A \otimes B$ with a $G = S \times T$-grading if A is S-graded and B is T-graded. Moreover, if $\dim A_s = 1, \dim B_t = 1$ for any $s \in \text{Supp } A, t \in \text{Supp } B$ then $\dim C_g = 1$ for any $g \in \text{Supp } C$.

Now we can show how to construct any grading on a matrix algebra with one-dimensional components.

Theorem 1.2 ([BSZ2]). *Let F be an algebraically closed field, char $F = 0$, and $M_n(F) = R = \oplus_{g \in G} R_g$ a grading on a matrix algebra over F by an abelian group G such that $\dim R_g \leq 1$ for any $g \in G$. Then $H = \text{Supp } R$ is a subgroup of G, $H = H_1 \times \cdots \times H_k$, $H_i \simeq Z_{n_i} \times Z_{n_i}, i = 1, \dots, k$, and R is isomorphic to $M_{n_1}(F) \otimes \cdots \otimes M_{n_k}(F)$ as an H-graded algebra, where $M_{n_i}(F)$ is an H_i-graded algebra with some ε_i-grading.*

We call a G-grading $R = \oplus_{g \in G} R_g$ "fine" if $\dim R_g \leq 1$ for any $g \in G$. The previous Theorem gives us all "fine" gradings on $M_n(F)$. In fact any abelian grading on $M_n(F)$ can be constructed from the "fine" and elementary gradings in the following way.

Theorem 1.3 ([BSZ2]). *Let G be an abelian group and $M_n(F) = R = \oplus_{g \in G} R_g$ a matrix algebra over an algebraically closed field F, char $F = 0$, with a G-grading. Then there exist a decomposition $n = tq$, a subgroup $H \subseteq G$ and a tuple $(g_1, \dots, g_q) \in G^q$ such that $M_n(F)$ is isomorphic to $M_t(F) \otimes M_q(F)$ as a G-graded algebra where $M_t(F)$ is an H-graded algebra with a "fine" H-grading and $M_q(F)$ has an elementary grading defined by $(g_1, \dots, g_q)$.*

In the proof of 1.3 it is useful to use the following result:

Lemma 1.2.1. *Let $C = M_k(F)$ be a $k \times k$ matrix algebra over a field F with a G-grading $C = \oplus_{g \in G} C_g$. If $\dim C_e = 1$ then this G-grading is "fine" and any non-zero homogeneous element in C is invertible.*

Our most recent researches allow at this point stating the following result about the structure of not necessarily simple graded rings.

Theorem 1.4. *Let G be a finite abelian group and A is a finite-dimensional graded algebra over an algebraically closed field F of characteristic 0, N the nilpotent radical of A. Then there is a semisimple graded subalgebra S in A such that $A = S \oplus N$ and, given another subalgebra S' with the same property, there is an element $x \in N_e$ such that $S' = (1+x)S(1+x)^{-1}$.*

Another result is the description of simple graded modules over a finite-dimensional simple graded algebra.

2. Theme Two: Actions and Polynomial Identities on Associative Algebras

In this part we again consider graded associative algebras over an arbitrary field F which have the form

$$R = \sum_{g \in G} R_g, \tag{8}$$

where G is a finite multiplicative group and

$$R_g R_h \subseteq R_{gh} \quad \text{for all } g, h \in G.$$

Definition 2.0.4. *A PI-*algebra *is an algebra satisfying a non-trivial polynomial identity. If A is an associative algebra then this is an identical relation of the form*

$$x_1 \ldots x_n \equiv \sum_{\substack{\sigma \in S_n \\ \sigma \neq e}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

In the case of Lie algebras as well as so called Lie type algebras *(see Part 2.3) we must consider so called* left-normed *products, i.e. those where each next factor is multiplied on the product of all previous ones, say a, ab, $(ab)c$, $((ab)c)d$, etc. Then the polynomial identity takes the form*

$$x_0 x_1 \ldots x_n \equiv \sum_{\substack{\sigma \in S_n, \\ \sigma \neq e}} \beta_\sigma x_0 x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Bergen and Cohen showed in [BC] that any G-graded algebra R is a PI-algebra as soon as its identity component R_e is a PI-algebra. This is not true if

G is infinite because any free associative algebra with identity element is graded by the infinite cyclic group with identity component the base field. The proof in [BC] uses a PI-structure theorem on the centralizers of separable algebras proved by Montgomery in [M1]. This approach does not produce a bound on the degree of the polynomial identity satisfied by R except in the case when R is graded-semiprime. Our primary aim in [BGR] was to provide a new, quantitative proof of this result which bounds the minimal degree of the polynomial identity satisfied by R in terms of $|G|$ and the minimal degree of an identity satisfied by R_1, only. Namely, we prove the following:

Theorem 2.1 ([BGR]). *Let F be an arbitrary field and let G any finite group. Suppose that R is a G-graded associative F-algebra such that R_1 satisfies a polynomial identity of degree d. Then R satisfies a polynomial identity of degree n, where n is any integer satisfying the inequality*

$$\frac{|G|^n(|G|d-1)^{2n}}{(|G|d-1)!} < n!$$

It follows that R satisfies a polynomial identity of degree n as soon as

$$e|G|(|G|d-1)^2 \leq n,$$

where e is the base of the natural logarithm.

A similar result holds for algebras graded by finite semigroups.

Finally, it should be pointed out that this combinatorial technique was inspired by the work of Latyshev [Lat1] and Regev [Re2].

But before we begin our exposition of combinatorial techniques we devote some space to a shorter proof using the structure theory.

2.1 Structure Theory Proof

A simple remark is as follows.

Remark 2.1.1. Let R be a G - graded associative algebra as in (8) with R_e a PI-algebra and K be any commutative algebra over F. We consider the tensor product $R \otimes_F K$ and set $(R \otimes_F K)_g = R_g \otimes_F K$. Then $R \otimes_F K$ is a G- graded algebra with its zero component a PI-algebra.

An important consequence is that if $R \otimes_F K$ as in Remark 2.1.1 turns out to be a PI-algebra and K is an associative algebra with identity element we have that R is itself a PI-algebra.

2.1.1 Semiprime Case

We first consider the case where R is a semiprimitive associative algebra as in (8) with R_e a PI-algebra. We are going to use the following result which is an adaptation of [BMPZ, Lemma 1.5 in Chapter 5]. In that Lemma it is shown that the dimensions of all irreducible R-modules are uniformly bounded if and only if the dimensions of all graded irreducible R-modules are uniformly bounded (provided G is a finite group). Actually, it has been shown that with each irreducible R - module V one can associate an irreducible graded R - module W with dim $V \leq$ dim W. We briefly recall the construction.

Suppose that V is an irreducible R - module. Consider a collection of vector spaces W_g with attached F - isomorphisms $\tau_g : W_g \longrightarrow V$, $g \in G$. Set $W = \oplus_{g \in G} W_g$. We define the structure of a graded R - module on W as follows:

$$a \circ w = \tau^{-1}_{\alpha+g}(a \circ \tau_g(w)) \text{ where } a \in R_\alpha,\ w \in W_g,\ \alpha, g \in G.$$

The mappings τ_g, $g \in G$, can be extended to an epimorphism $\tau : W \longrightarrow V$ which is a homomorphism of (non - graded) R - modules. Pick any non - zero homogeneous element $w \in W$; let S be a graded R - module generated by w, and let T be a maximal graded R - module not containing w. Then, obviously, $U = (S+T)/T$ is a graded irreducible R - module. Since V is irreducible and $\tau(w) \neq 0$ one has $\tau(T) = \{0\}$ and $\tau(S+T) = V$. So we obtain an epimorphism $\bar{\tau} : U \longrightarrow V$ of non - graded R - modules.

This enables us to prove the following.

Proposition 2.1.1. *Let R be as in (8) with G finite. Suppose that the base field F is algebraically closed and such that its cardinality exceeds* dim R_e. *If*

R_e satisfies a non-trivial polynomial identity of degree d then the dimensions of all irreducible R - modules are bounded by $|G| \cdot (d/2)$.

If R is a semiprimitive algebra as in Proposition 2.1.1, with R_e a PI-algebra, then we can consider R as a subcartesian product in the Cartesian product

$$P = \prod_{\alpha} P_\alpha \text{ where each } P_\alpha \text{ is primitive} \tag{9}$$

It follows by Proposition 2.1.1 that each P_α satisfies a polynomial identity of degree at most $|G| \cdot d$.

Now we can prove an intermediate result.

Proposition 2.1.2. *Let a graded semiprime associative algebra R as in (8) be such that R_e satisfies a polynomial identity of degree d. Then R is itself a PI-algebra satisfying a polynomial identity of degree $|G| \cdot d$.*

2.1.2 Amitsur's trick

If R is an arbitrary algebra as in (8), with R_e a PI-algebra of degree d, let us consider an algebra

$$B = \prod_{\beta \in I} B_\beta \tag{10}$$

where I runs through the set of all p - tuples of the form $(a_1, \ldots, a_p)$ with $a_i \in R$ and $p = |G| \cdot d$, $B_\beta \cong R$ for ay β. It is obvious that B is a G - graded algebra with its zero component isomorphic to

$$B_e = \prod_{\beta \in I} (B_\beta)_e. \tag{11}$$

Clearly, B_e is a PI-algebra of still the same degree d and then, by the above section 2.1.1 , $B/\mathcal{L}(B)$ is a PI-algebra of degree p, $\mathcal{L}(B)$ being the lower nil-radical of B. If $f(x_1, \ldots, x_p) \equiv 0$ is a non-trivial identity of $B/\mathcal{L}(B)$ then for any $y_1, \ldots, y_p$ there exists n such that $(f(x_1, \ldots, x_p))^n = 0$. It remains to choose $y_1, \ldots, y_p$ so that for any selection $a_1, \ldots, a_p \in R$ there is a component $\beta \in I$ such that $x_1(\beta) = a_1, \ldots, x_p(\beta) = a_p$.

Finally we have our main result.

Theorem 2.2 ([BC]). *Let R be an associative algebra over an arbitrary field F which has the form*

$$R = \bigoplus_{g \in G} R_g \tag{12}$$

where G is an additive commutative group. If R_1 satisfies an identity of degree d and $p = |G|$ then R satisfies a power of the standard identity of order pd, that is, for some t we have

$$S_{pd}(x_1, \ldots, x_{pd})^t \equiv 0$$

identically in R.

2.2 Combinatorial Proof

We need some special techniques to work with identities in the case of graded algebras.

2.2.1 Free graded algebras

Let $F\langle Z\rangle$ be the free associative algebra over F generated by a countably infinite set Z, and let G be a finite group of order s. We represent Z in the form

$$Z = \bigcup_{g \in G} Z_g,$$

where $Z_g = \{z_1^{(g)}, z_2^{(g)}, \ldots\}$ are disjoint sets. We often abbreviate $Z_1 = Y$ and $z_i^{(1)} = y_i$ for each $i \geq 1$. The indeterminates from Z_g are said to be homogeneous of degree g. The homogeneous degree of a monomial $z_{i_1}^{(g_1)} \cdots z_{i_t}^{(g_t)}$ in $F\langle Z\rangle$ is defined to be $g_1 g_2 \cdots g_t$, as opposed to its total degree, which is defined to be t. Denote by $\mathcal{F}_g$ the subspace of the algebra $F\langle Z\rangle$ generated by all the monomials having homogeneous degree g. Notice that $\mathcal{F}_g \mathcal{F}_h \subseteq \mathcal{F}_{gh}$ for every g, h in G. Consequently, $F\langle Z\rangle = \bigoplus_{g \in G} \mathcal{F}_g$ is a G-grading, and $F\langle Z\rangle$ is the free G-graded algebra generated by the sets Z_g, $g \in G$.

2.2.2 Graded identities

Let us fix an enumeration of G: $1 = g_1, g_2, \ldots, g_s$. R is said to satisfy the graded identity

$$f = f(z_1^{(g_1)}, \ldots, z_{t_1}^{(g_1)}, z_1^{(g_2)}, \ldots, z_{t_2}^{(g_2)}, \ldots, z_1^{(g_s)}, \ldots, z_{t_s}^{(g_s)}) \equiv 0,$$

where f is a nonzero element of $F\langle Z\rangle$, if for arbitrary $r_1^{(g_i)}, \ldots, r_{t_i}^{(g_i)} \in R_{g_i}$ the equality

$$f(r_1^{(g_1)}, \ldots, r_{t_1}^{(g_1)}, r_1^{(g_2)}, \ldots, r_{t_2}^{(g_2)}, \ldots, r_1^{(g_s)}, \ldots, r_{t_s}^{(g_s)}) = 0$$

is satisfied by R. We denote by $T_G(R)$ the set of all polynomials $f \in F\langle Z\rangle$ such that the graded identity $f \equiv 0$ is satisfied by R. In other words, $T_G(R)$ is the ideal of G-graded identities of R.

If, for each $i \geq 1$, we set $x_i = \sum_{g\in G} z_i^{(g)}$, then the set $T(R)$ of polynomials in the x_i's vanishing in R coincides with the T-ideal of polynomial identities of R. It is clear that $T(R) \subseteq T_G(R)$.

For each $n \geq 1$, define a subspace of $F\langle Z\rangle$ by

$$V_n = \mathrm{Span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} |\ \sigma \in \mathcal{S}_n\}.$$

Then V_n is the space of multilinear polynomials of degree n in the variables x_i, $1 \leq i \leq n$. Let G^n denote the direct product of n copies of G. For each $a = (a_1, \ldots, a_n)$ in G^n define

$$V_n^a = \mathrm{Span}_F\{z_{\sigma(1)} \cdots z_{\sigma(n)} |\ \sigma \in \mathcal{S}_n, z_i = z_i^{(a_i)} \text{ for each } i\}.$$

Notice that V_n^a is the space of multilinear polynomials of degree n in the variables $z_1^{(a_1)}, \ldots, z_n^{(a_n)}$. Assign

$$V_n^G = \bigoplus_{a\in G^n} V_n^a.$$

From the above definitions, it follows that $V_n \subseteq V_n^G$ and

$$V_n \cap T_G(R) = V_n \cap T(R).$$

Observe as well that

$$\dim_F V_n = n! \quad \text{and} \quad \dim_F V_n^G = |G|^n n!$$

The integers

$$c_n(R) = \dim_F \frac{V_n}{V_n \cap T(R)} \quad \text{and} \quad c_n^G(R) = \dim_F \frac{V_n^G}{V_n^G \cap T_G(R)}$$

are called the nth codimension and the nth G-graded codimension of R.

Lemma 2.2.1. *For every n, $c_n(R) \leq c_n^G(R)$.*

Proof. We have

$$\frac{V_n}{V_n \cap T(R)} = \frac{V_n}{V_n \cap [V_n^G \cap T_G(R)]} \cong \frac{V_n + [V_n^G \cap T_G(R)]}{V_n^G \cap T_G(R)},$$

the latter being a subspace of

$$\frac{V_n^G}{V_n^G \cap T_G(R)}.$$

□

Notice that R satisfies an (ordinary) multilinear polynomial identity of degree n precisely when $c_n(R) < n!$. Therefore, by the lemma, in order to prove that R satisfies an identity, it suffices to show that $c_n^G(R) < n!$ for some n.

2.2.3 The width of a monomial

For a monomial w in $F\langle Z\rangle$, we say w has *width* m if w contains a product of m consecutive submonomials of homogeneous degree 1, and m is maximal. We wish to demonstrate next that if a monomial has large total degree then it must also have large width.

Let us begin with a general lemma about finite groups.

Lemma 2.2.2. *Any fixed word $w = a_1 a_2 \cdots a_{|G|d}$ in a finite group G contains a product of d consecutive subwords each with trivial evaluation.*

Proof. Set $t = |G|d$. Then the number of initial subwords $w_1 := a_1, w_2 := a_1 a_2, \ldots, w_t := a_1 a_2 \cdots a_t$ of w is t. Therefore there exists g in G which is the evaluation of at least d-many of these initial subwords. If $g = e$ then there exists $i_1 < \cdots < i_d$ such that evaluating in G we have

$$e = w_{i_1} = w_{i_2} = \cdots = w_{i_d}.$$

Therefore in this case the required product of subwords of w is

$$w_{i_1}(a_{i_1+1}\cdots a_{i_2})\cdots(a_{i_{d-1}+1}\cdots a_{i_d}).$$

So, we may assume that $g \neq e$ and g is the evaluation of at least $d+1$ initial subwords. So, there exists $i_1 < \cdots < i_{d+1}$ such that evaluating in G we have

$$g = w_{i_1} = w_{i_2} = \cdots = w_{i_{d+1}}.$$

Now for each j, $1 \leq j \leq d$, write $w_{i_{j+1}} = w_{i_j} g_j$ where g_j is the appropriate subword of w. It follows that the evaluation of each g_j in G is 1 and

$$w_{i_{d+1}} = w_{i_1} g_1 g_2 \cdots g_d.$$

Thus $g_1 g_2 \cdots g_d$ is the required product of subwords of w.

□

We now apply this result to the free G-graded algebra.

Corollary 2.2.1. *Every monomial w in $F\langle Z\rangle$ of total degree $|G|d$ has width at least d.*

Proof. Again put $t = |G|d$, and write $w = v_1 v_2 \cdots v_t$, where the v_i's lie in Z. Let a_i be the homogeneous degree of each v_i. Now applying Lemma 2.2.2 to the fixed word $a_1 a_2 \cdots a_t$ in G yields the desired result.

□

We now deduce the key result of this section. Recall that we have set $Y = Z_1$ and $y_i = z_i^{(1)}$ for all $i \geq 1$.

Proposition 2.2.1. *Suppose that*

$$y_1 y_2 \cdots y_d + \sum_{\substack{\sigma\in\mathcal{S}_d\\ \sigma\neq 1}} \alpha_\sigma y_{\sigma(1)} \ldots y_{\sigma(d)} \in T_G(R),$$

where the α_σ are scalars. Then for every monomial $w = v_1 v_2 \cdots v_t$ in $F\langle Z\rangle$ with total degree $t \geq |G|d$ we have

$$v_1 v_2 \cdots v_t \in \mathrm{Span}_F\{v_{\tau(1)} \cdots v_{\tau(t)} |\, \tau \in \mathcal{S}_t, \tau \neq 1\} + T_G(R)$$

Proof. By the previous corollary, $w = w'u_1 \cdots u_d w''$, where the u_i are elements of $\mathcal{F}_1$. Thus,

$$\begin{aligned} v_1 v_2 \cdots v_t &\equiv w'u_1 \cdots u_d w'' \\ &\equiv -\sum_{\substack{\sigma \in \mathcal{S}_d \\ \sigma \neq 1}} \alpha_\sigma w' u_{\sigma(1)} \ldots u_{\sigma(d)} w'' \pmod{T_G(R)}. \end{aligned}$$

□

2.2.4 Good permutations

Let $1 \leq t \leq n$ be an integer and $\sigma \in \mathcal{S}_n$. Following [Re2], we call the permutation σ *t-bad* if there exists a sequence $1 \leq i_1 < \cdots < i_t \leq n$ such that $\sigma(i_1) > \cdots > \sigma(i_t)$. Otherwise, σ is t-good.

For each $\sigma \in \mathcal{S}_n$ and $a = (a_1, \ldots, a_n)$ in G^n, set $w_\sigma^a = z_{\sigma(1)} \cdots z_{\sigma(n)}$, where $z_i = z_i^{(a_i)}$. The monomial w_σ^a is called t-good if σ is t-good. Notice that there are $|G|^n$-many t-good monomials in V_n^G corresponding to the same t-good permutation in $\mathcal{S}_n$.

Proposition 2.2.2. *Suppose that R_e satisfies a polynomial identity of degree d, and fix integers t and n where $n \geq t \geq |G|d$. For each $a = (a_1, \ldots, a_n)$ in G^n, V_n^a is spanned, modulo $V_n^G \cap T_G(R)$, by the set*

$$\{w_\sigma^a | \sigma \in \mathcal{S}_n \text{ is } t\text{-good}\}.$$

Proof. We may assume that R_e satisfies a multilinear polynomial identity of degree d. Fix a in G^n and set $w = w_1^a = z_1 z_2 \cdots z_n$, where $z_i = z_i^{(a_i)}$ for each i. The natural basis

$$\mathcal{M}_n = \{w_\sigma^a | \sigma \in \mathcal{S}_n\}$$

of V_n^a is well-ordered by the left lexicographic order on the subscripts of the z_i's. We shall show that whenever σ is t-bad, then, modulo $V_n^G \cap T_G(R)$, w_σ^a is a linear combination of smaller monomials in $\mathcal{M}_n$.

By assumption, there exist $1 \leq i_1 < \cdots < i_t \leq n$ such that $\sigma(i_1) > \cdots > \sigma(i_t)$. Factorize w_σ accordingly:

$$w_\sigma = w_0(z_{\sigma(i_1)} \cdots)(z_{\sigma(i_2)} \cdots) \cdots (z_{\sigma(i_t)} \cdots).$$

Write $w_j = z_{\sigma(i_j)} \cdots z_{\sigma(i_{j+1}-1)}$ for each j, $1 \le j \le t-1$, and $w_t = z_{\sigma(i_t)} \cdots z_{\sigma(i_n)}$. Now, by Proposition 2.2.1, we have

$$w_1 w_2 \cdots w_t \equiv \sum_{\substack{\tau \in \mathcal{S}_t \\ \tau \ne 1}} \alpha_\tau w_{\tau(1)} \cdots w_{\tau(t)} \pmod{V_t^G \cap T_G(R)},$$

for some scalars α_τ. It follows that

$$w_\sigma \equiv \sum_{\substack{\tau \in \mathcal{S}_t \\ \tau \ne 1}} \alpha_\tau w_0 (z_{\sigma(i_{\tau(1)})} \cdots) \cdots (z_{\sigma(i_{\tau(t)})} \cdots) \pmod{V_n^G \cap T_G(R)}.$$

Because all the monomials on the right are smaller in the lexicographic ordering, the result now follows.

□

According to [Lat1], if $n \ge t$ then the number of t-good permutations in $\mathcal{S}_n$ is at most

$$\frac{(t-1)^{2n}}{(t-1)!}$$

We have, therefore, the following immediate corollary to Proposition 2.2.2.

Corollary 2.2.2. *For $n \ge t \ge |G|d$,*

$$c_n^G(R) \le \frac{|G|^n (t-1)^{2n}}{(t-1)!}$$

We are now ready to deduce our main Theorem 2.1.

Proof. For n large enough,

$$c_n(R) \le c_n^G(R) \le \frac{|G|^n (|G|d-1)^{2n}}{(|G|d-1)!} < n!$$

Therefore R satisfies a multilinear polynomial identity of degree n. It remains to estimate the minimal integer n satisfying this inequality. Consider Euler's Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The following inequality is well-known:

$$\left(\frac{x}{e}\right)^x < \frac{\Gamma(x+1)}{\sqrt{2\pi x}} < \Gamma(x+1)$$

for all $x \geq 1$ (see page 105 of [F], for example). Substituting n for x where n is the least integer greater than or equal to $e|G|(|G|d-1)^2$ yields

$$(|G|(|G|d-1)^2)^n < \left(\frac{n}{e}\right)^n < \Gamma(n+1) = n!$$

as required.

□

2.3 G-identities of associative algebras

Another direction of PI theory is identities of algebras with involution. Let A be an associative algebra with an involution $* : A \to A$. The classical S.Amitsur's theorems [AM1, AM2] say that a non-trivial $*$-identity in an associative algebra with involution $*$ implies a non-trivial ordinary identity and if the set A_+ (respectively, A_-) of symmetric (respectively, skew-symmetric) elements satisfies a non-trivial polynomial identity then the whole of A satisfies a non-trivial polynomial identity. In that proof no relation was found between the degrees of identities in A_+ or A_- and in A. The new combinatorial approach just described gives us an upper bound for the minimal degree of an identity on A depending only on the degree of $*$-identities or, say, on the degree of the identity in A_+ (or on A_-) and that in A.

In fact, a more general construction can be considered. Namely, let A be an associative algebra over a field F and $\mathrm{Aut}^*(A)$ the group of all automorphisms and anti-automorphisms of the F-algebra A and $G \leq \mathrm{Aut}^*(A)$ a finite subgroup. If $\mathrm{Aut}(A)$ is the group of all automorphisms of A then $G \cap \mathrm{Aut}(A)$ is a normal subgroup of G of index ≤ 2.

Let X be a set, G a finite group and H a normal subgroup of G; if we interpret H as automorphisms and $G \setminus H$ as anti-automorphisms, we can construct $F\langle X|G\rangle$, the free algebra on X with a G-action. Now $F\langle X|G\rangle$ is freely generated by the set $\{x^g = g(x) \mid x \in X, g \in G\}$ on which G acts in a natural way: $(x^{g_1})^{g_2} = x^{(g_2 g_1)}$. Extend this action to $F\langle X|G\rangle$: if v and w are monomials, $g \in G$, then $(vw)^g = v^g w^g$ if $g \in H$ and $(vw)^g = w^g v^g$ if $g \in G \setminus H$.

By linearity, now G acts on $F\langle X|G\rangle$ with H as automorphisms and $G \setminus H$ as anti-automorphisms.

Given any algebra R as above, by interpreting $G \le \mathrm{Aut}^*(R)$ and $H = G \cap \mathrm{Aut}(R)$, any set theoretic map $\phi : X \mapsto R$ extends uniquely to a homomorphism $\bar{\phi} : F\langle X|G\rangle \mapsto R$ such that $\bar{\phi}(x^g) = \phi(x)^g$. For a fixed R, let $\bar{\Phi}$ be the set of all such homomorphisms and set

$$I = \bigcap_{\bar{\phi}\in\bar{\Phi}} \mathrm{Ker}\bar{\phi}.$$

An element $f \in F\langle X|G\rangle$ will be called a *G-polynomial*; if $f \in I$, then f will be called a *G-identity* for R.

Let $G^n = G \times \cdots \times G$ and $g = (g_1, \ldots, g_n) \in G^n$. Write

$$P_{n,g} = \mathrm{Span}_F\{x_{\sigma(1)}^{g_{\sigma(1)}} \cdots x_{\sigma(n)}^{g_{\sigma(n)}} \mid \sigma \in S_n\}$$

for the space of multilinear polynomials in $F\langle X|G\rangle$ in the variables $x_1^{g_1}, \ldots, x_n^{g_n}$. In particular, for $1 = (1, \ldots, 1)$ we have $P_{n,1} = P_n$.

A G-identity f is $G-$multilinear if f lies in $\sum_{g\in G^n} P_{n,g}$ and *essential* if it has the form

$$f = x_1^1 \ldots x_n^1 + \sum_{1\neq\sigma\in S_n g\in G^n} \alpha_{\sigma,g} x_{\sigma(1)}^{g_1} \cdots x_{\sigma(n)}^{g_n}.$$

The main result about G-identities we would like to formulate ([BGZ2]) is as follows.

Theorem 2.3 ([BGZ2]). *Let A be an associative algebra over a field F and G a finite subgroup of* $\mathrm{Aut}^*(A)$. *Suppose that A satisfies some G-multilinear essential identity of degree d. Then A satisfies a non-trivial polynomial identity, whose degree is a function of d entirely.*

Using this general statement we improve Amitsur's results mentioned above. In the case $G = \{1, *\}$ where $*$ is an involution, G-polynomials and G-identities are called $*$-polynomials and $*$-identities, respectively. Besides, any $*$-identity of degree d implies a multilinear essential $*$-identity of degree at most $2d$.

Corollary 2.3.1. *Let A be an associative algebra with an involution $*$ over a field F satisfying a non-trivial $*$-identity of degree d. Then A satisfies a non-trivial polynomial identity whose degree is a function of d entirely.*

Corollary 2.3.2. *Let A be an associative algebra with an involution $*$ over a field F. If a non-trivial identity of degree d is satisfied on the elements of A_+ (or A_-), the whole of A satisfies a non-trivial polynomial identity whose degree is a function of d entirely.*

Remark 2.3.1. A similar question was discussed for some non-associative algebras in [BSZ1]. In particular, some results similar to Theorem 2.3 or 2.3.1 have been proven for any Lie algebra. Theorem 2.3.2 is also true for Lie algebra with an involution but only for the skew-symmetric elements. The reason is that any Lie algebra has an involution of the form $x^* = -x$. Indeed, $[x,y]^* = -[x,y]$ and $[x^*, y^*] = [-x, -y] = [x,y]$. But then all the elements are skew-symmetric and we have that $x = 0$ is an identity satisfied by any symmetric elements. This also shows that if a $*$-identity is satisfied by L then there is no need that an ordinary identity holds as well.

We finish our discussion of polynomial identities in graded algebras by looking into the question of what characteristics of the poynomial identities of A_e preserve for A as a whole. Usually identities do not preserve. For instance, $\mathcal{G}$, the Grassmann algebra in infinitely many variables is a $\mathbb{Z}_2$-graded algebra, the homogeneous tensors of even degree forming $\mathcal{G}_0$, those of odd degree - $\mathcal{G}_1$. The component $\mathcal{G}_0$ is commutative, which is a particular case of the standard identity

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

But the whole of $\mathcal{G}$ is well known to violate any of $S_n(x_1, \ldots, x_n) = 0$. Yet the so called *symmetric identities* behave nicer. If char $F = p > 0$ then any

PI-algebra over F satisfies a symmetric identity

$$P_n = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \equiv 0$$

of some degree n (see [K2]). It was shown in [BGZ1] that there exists a dependence between the degrees of P_n on A_e and A.

Theorem 2.4 ([BGZ1]). *Let G be a cancellative semigroup with the identity element e. Consider a G-graded associative ring $R = \sum_{g \in G} R_g$ such that the set $H = \{g \in G | R_g \neq 0\}$ is finite and contains n elements. If a subring R_e satisfies an identity $P_k \equiv 0$ then the whole of R satisfies $P_{nk} \equiv 0$.*

As a consequence of this Theorem we obtain two known results:

Theorem 2.5 ([Dom]). *If a ring R satisfies $P_k \equiv 0$ then $M_n(R)$ satisfies $P_{nk} \equiv 0$.*

Here $M_n(R)$ is as usually the ring of $n \times n$ matrices over R.

Theorem 2.6 ([ZSm]). *Let G be a finite abelian group of order n and $R = \sum_{g \in G} R_g$ a G-graded algebra over a field of characteristic $p > 0$. If R_1 is commutative, then R satisfies the identity $P_{np} \equiv 0$.*

3. Theme Three: Lie Type Algebras

The proof of the main results in [BGR] does not use any structure theory but is of purely combinatorial nature. Thus it seems natural to extend the main idea of [BGR] to the case of Lie algebras and even to a much wider class of algebras including as subclasses all associative algebras, Lie algebras and color Lie superalgebras.

We introduce the notation and the notions we will need. Let G be a semigroup with 1 and $A = \sum_{g \in G} A_g$ be a G-graded algebra over an arbitrary field F. We say that the G-grading is *finite*, if there is a finite subset H in G such that

$A_g = \{0\}$ for all $g \notin H$. (Without loss of generality one may assume $1 \in H$.) In other words

$$A = \sum_{g \in H} A_g, \quad |H| = s < \infty.$$

Further we say that $A = \sum A_g$ is a *Lie type* algebra if for any $g, h, k \in G$ there exists constants $\alpha, \beta \in F$, such that

$$a(bc) = \alpha(ab)c + \beta(ac)b \tag{13}$$

for all $a \in A_g, b \in A_h, c \in A_k$, where $\alpha \neq 0$. If $\alpha = 1, \beta = 0$ for all $g, h, k \in G$, then A is an ordinary associative algebra with a G-grading and if $\alpha = 1, \beta = -1$ then A is a Lie algebra over F. The relations of the type (13) are satisfied also by Lie superalgebras with a Z_2-grading as well as more general color Lie superalgebras. So called *quantum* Lie algebras (see [Lin]) also fit into the pattern of these relations. Another class that is getting more and more important is that of so called *Leibniz* algebras, that is, algebras that satisfiy the Jacoby identity but need not be anticommutative.

We start with considering Lie superalgebras.

3.1 PI-envelopes of Lie superalgebras

We consider Lie superalgebras over a field k of characteristic 0, graded by a finite abelian group G, of the form

$$L = \bigoplus_{g \in G} L_g. \tag{14}$$

L has a bracket operation [,] satisfying the generalized anticommutativity and Jacobi identities:

$$[x, y] + \beta(x, y)[y, x] = 0, \tag{15}$$

$$[[x, y], z] = [x, [y, z]] - \beta(x, y)[y, [x, z]] \tag{16}$$

for any homogeneous $x \in L_g,\ y \in L_h,\ z \in L_k$. Here $\beta : G \times G \to k^*$ is the *commutation factor*, or *bicharacter, (see Part 4.2)* i.e. a function satisfying

$$\beta(g, h + k) = \beta(g, h)\beta(g, k), \quad \beta(g, h)\beta(h, g) = 1. \tag{17}$$

We write $\beta(x, y) = \beta(g, h)$ if $x \in L_g, y \in L_h$. If $G = \{0\}$ we have ordinary Lie algebras, while if $G = \mathbb{Z}_2 = \{0, 1\}$ and $\beta(1, 1) = -1$ we have Lie superalgebras. It is easy to see that $\beta(g, g) = \pm 1$ for any $g \in G$. We set $G_\pm = \{g \in G \mid \beta(g, g) = \pm 1\}$. We call the Lie superalgebras just defined *(G, β)-Lie algebras*. For details see [BMPZ]. If $A = \bigoplus_{g \in G} A_g$ is a G-graded associative algebra, then setting

$$[x, y] = xy - \beta(x, y)yx \tag{18}$$

for $x \in A_g,\ y \in A_h$, we make A into a (G, β)-Lie superalgebra $[A]_\beta$ or simply $[A]$.

PI-envelopes of ordinary Lie algebras were studied in [B1]; here we look at the analogous situation for Lie superalgebras. We call a G-graded associative algebra A an *envelope* of L if $L \subset [A]$ and A is generated by L. We say that A is a *PI-envelope* of L if A is an envelope of L and A satisfies a non-trivial polynomial identity. A related notion is that of special Lie superalgebras: any Lie superalgebra having a PI-envelope is called *special*. Obviously, by Ado's Theorem [Sch] any finite-dimensional Lie superalgebra is special. The universal enveloping algebra $U(L)$ is the most natural example of an envelope for L and any envelope is a homomorphic image of $U(L)$. It is shown in [BMPZ] that if char $k = p > 0$, then $U(L)$ is a PI-envelope of L and so in this case the PI-envelope does not have to be finite-dimensional. Thus if we are interested in those L for which any PI-envelope is finite-dimensional, then it is natural to restrict ourselves to the case char $k = 0$. So from now on we assume char $k = 0$. In this case if we set $L_+ = \bigoplus_{g \in G_+} L_g,\ L_- = \bigoplus_{g \in G_-} L_g$ and $\dim L < \infty$, then $U(L)$ is a PI-algebra iff L_+ is abelian, i.e. $[L_+, L_+] = \{0\}$.

A surprisingly sharp result was obtained in [B1].

Theorem 3.1 ([B1]). *Let L be a finite-dimensional semisimple Lie algebra over a field k of characteristic 0, and let A be a PI-envelope of L. Then A is finite-dimensional and semisimple.*

A less sharp but more general version of this is the following.

Theorem 3.2 ([B1]). *Let L be a finite-dimensional perfect Lie algebra (that is, $L = [L, L]$) over a field k of characteristic zero. Then any PI-envelope A of L is finite-dimensional.*

These theorems were an important tool used by Yuly Billig [Bi] to prove that no affine Kac-Moody Lie algebra is special (this gave a solution to a well-known problem of Latyshev about the speciality of a homomorphic image of a special Lie algebra).

Our main aim is to generalize Theorem 3.2 to the case of (G, β) - Lie - superalgebras. If $G \neq G_+$, that is, if we have odd elements, we exhibit a whole series of examples of simple Lie superalgebras $sl(n, m), n \neq m, n, m \geq 1$, $osp(2, 2n), n \geq 1$, $W(n), n \geq 2$, whose PI-envelopes can be infinite-dimensional. So it is only natural that we assume quite often that $G = G_+$ (the case of so called color Lie algebras). In this case we are able to prove full analogues of Theorems 3.1 and 3.2.

We define a *semisimple* (G, β)-Lie-superalgebra as one without solvable G-graded ideals. Our results are as follows. We first consider the case when $G = G_+$.

Theorem 3.3 ([BM]). *Let L be a semisimple (G, β)-Lie-superalgebra over a field k of characteristic zero with $G = G_+$ and let A be a PI-envelope of L. Then A is finite-dimensional and semisimple.*

Theorem 3.4 ([BM]). *Let L be a (G, β)-Lie-superalgebra over a field k of characteristic zero with $G = G_+$, A a PI-envelope of L. Then A is finite-dimensional if and only if L is perfect (that is, $L = [L, L]$).*

In the general case, the result is as follows.

Theorem 3.5 ([BM]). *Let $L = L_+ \oplus L_-$ be a finite-dimensional (G, β)-Lie-superalgebra over any field k of characteristic zero. Then the following are equivalent:*

1. *Any PI-envelope of L is finite-dimensional;*

2. Any (graded) PI-homomorphic image of $U(L)$ is finite-dimensional;

3. L_+ is a perfect color Lie algebra.

The following subsection presents some of the techniques necessary for proving the results just formulated. The proof of Theorem 3.5 is not included because of the lack of space.

3.1.1 Scheunert's trick and some applications

We will need later the following version of the Poincaré-Birkhoff-Witt (PBW) Theorem.

Theorem 3.6 ([BMPZ]). *Let $L = L_+ \oplus L_-$ be a (G, β)-Lie-superalgebra, $S(L_+)$ the symmetric algebra of the vector space L_+ and $\Lambda(L_-)$ the Grassmann algebra of the vector space L_-. Then $U(L_+)$, the universal enveloping algebra of L_+, is an associative subalgebra of $U(L)$ generated by L_+ and we have the vector space isomorphisms*

$$U(L_+) \cong S(L_+) \text{ and } U(L) = U(L_+) \otimes \Lambda(L_-). \tag{19}$$

In the remaining part of this section we consider M. Scheunert's approach enabling one to pass from (G, β)-Lie-superalgebras to ordinary Lie superalgebras with a number of properties preserved. If $\sigma \in Z^2(G, k^*)$ is a multiplicative 2-cocycle, and L a (G, β)-Lie-superalgebra satisfying (15), (16) with commutation factor β, then replacing multiplication in L on homogeneous elements $x, y \in L$ by

$$[x, y]^\sigma = \sigma(x, y)[x, y] \tag{20}$$

we arrive at a (G, β')-Lie-superalgebra L^σ satisfying (15), (16) with commutation factor $\beta' = \beta\delta$, where $\delta(x, y) = \sigma(x, y)/\sigma(y, x)$. Let $\beta_0 : G \times G \to k^*$ be the ordinary superalgebra commutation factor, i.e. $\beta_0(g, h) = 1$ except $\beta_0(g, h) = -1$ for $g, h \in G_-$.

Theorem 3.7 ([Sch]). *Let G be a finitely generated abelian group, and $\beta : G \times G \to k^*$ a commutation factor with $G = G_+$. Then there exists a 2-cocycle $\sigma \in Z^2(G, k^*)$ such that $\beta\delta = 1$, where δ is as above.*

In other words, if L is a color Lie algebra, then L^σ is an ordinary Lie algebra (with the G-grading still preserved!). In the general case when $G \neq G_+$ note that $\beta' = \beta\beta_0$ satisfies the conditions of the theorem. Thus we can find σ with $\beta\beta_0\delta = 1$, whence $\beta\delta = \beta_0^{-1} = \beta_0$. It follows that from any (G,β)-Lie-superalgebra by a change of the form (20) we can switch to an ordinary Lie superalgebra $L^\sigma = L_0^\sigma \oplus L_1^\sigma$ with $L_0^\sigma = L_+, L_1^\sigma = L_-$.

Scheunert also shows that we can pass from a (G,β)-Lie-superalgebra to an ordinary Lie superalgebra by tensoring L with a twisted group algebra $k_\sigma G$ (σ as before). Here we give another approach (Proposition 3.1.1) to this switch which is more appropriate to our main aim. Along the way, we generalize Theorem 2.2 to arbitrary abelian groups (this was done independently by H. Pop [Po] in the case of algebraically closed fields).

Theorem 3.8 ([BM]). *Let G be an arbitrary abelian group, and k an arbitrary commutative ring with 1 and with group of units k^*. Then for any commutation factor $\beta : G \times G \to k^*$ there exists $\sigma \in Z^2(G,k^*)$ such that if we set $\delta(g,h) = \sigma(g,h)/\sigma(h,g)$, then $\beta\delta = \beta_0$.*

Proof. By the above remarks, it suffices to prove our statement when $G = G_+$. In this case we have to find a 2-cocycle σ with $\sigma(g,h)/\sigma(h,g) = \delta(g,h)$ for all $g,h \in G$ where δ is the commutation factor $\delta = \beta^{-1}$. We recall that σ is a 2-cocycle with values in k^* if for all $x,y,z \in G$ we have

$$\sigma(x,y)\sigma(x+y,z) = \sigma(x,y+z)\sigma(y,z). \tag{21}$$

Let $\mathcal{S}$ be the partially ordered set of all pairs $(H,\sigma), H$ a subgroup of $G, \sigma \in Z^2(H,k^*)$ with $\beta\delta = 1, \delta$ as before, with $(H,\sigma) \leq (H',\sigma')$ iff $H \subset H'$ and $\sigma'|_H = \sigma$. We can apply Zorn's Lemma to $\mathcal{S}$ because it is obvious that it is non-empty and there is a maximal element for any chain $\{(M_\alpha,\sigma_\alpha)\}$, namely (M,σ), where $M = \bigcup_\alpha M_\alpha$ and $\sigma(g,h) = \sigma_\alpha(g,h)$ if $g,h \in M_\alpha$. Then we have a maximal element (B,σ_0) in $\mathcal{S}$. If $G = B$, we are done. Otherwise there is $a \in G\backslash B$. We set $A = \langle a, B\rangle$. If $\langle a\rangle \cap B = \{0\}$, then for arbitrary

$u = s_1 a + b_1, v = s_2 a + b_2 \in A$, $s_1, s_2 \in \mathbb{Z}, b_1, b_2 \in B$ we set (following Scheunert)

$$\sigma(s_1 a + b_1, s_2 a + b_2) = \delta(b_1, a)^{s_2} \sigma_0(b_1, b_2).$$

Then

$$\sigma \in Z^2(A, k^*),\ \delta(u, v) = \sigma(u, v)\sigma(v, u)^{-1} \text{ and } \sigma|_{B \times B} = \sigma_0.$$

This contradicts the maximality of (B, σ_0). The verification of this will be included while considering the more general case $\langle a \rangle \cap B \neq \{0\}$ and so we pass to this case.

Thus we may assume that $c = pa \in B$ for a suitable prime number p. Any element in A can be uniquely written in the form $sa + b,\ 0 \leq s < p,\ b \in B$. Also, for brevity, we write σ in place of σ_0.

For $u = s_1 a + b_1, v = s_2 a + b_2$, s_1 and s_2 as s just above, $b_1, b_2 \in B$ and $s_1 + s_2 = pq + r, 0 \leq r < p$ we set

$$\sigma(u, v) = \delta(b_1, a)^{s_2} \sigma(c, b_1 + b_2)^q \sigma(b_1, b_2). \tag{22}$$

It is easily seen that our formula extends σ to A and

$$\begin{aligned} &\sigma(u, v)\sigma(v, u)^{-1} \\ &= \delta(b_1, a)^{s_2} \sigma(c, b_1 + b_2)^q \sigma(b_1, b_2) \delta(b_2, a)^{-s_1} \sigma(c, b_1 + b_2)^{-q} \sigma(b_2, b_1)^{-1} \\ &= \delta(s_2 b_1 - s_1 b_2, a) \delta(b_1, b_2) = \delta(s_1 a + b_1, s_2 a + b_2) = \delta(u, v), \end{aligned}$$

following from $\delta(g, g) = \varepsilon(g, g) = 1$ for all $g \in G$. It remains to check the cocycle identity (21) for σ. This can be done by direct verification which we omit.

□

Proposition 3.1.1. *If A is a G-graded associative algebra and $[A]_\beta$ the respective (G, β)-Lie-superalgebra given by (18), then A^σ with multipication*

$$(ab)^\sigma = \sigma(a, b)ab, \quad a \in A_g, \quad b \in A_h, \tag{23}$$

is a G-graded associative algebra and we have

$$[A^\sigma]_{\beta'} = ([A]_\beta)^\sigma,$$

where σ, δ, and $\beta' = \beta\delta$ are as described before 3.8.

Proof. This is a simple computation. Take homogeneous $a, b \in A$ and compare the β'-bracket in A^σ and the σ-twisted β-bracket in $[A]$. That is,

$$\begin{aligned} [a,b]_{\beta'} &= (ab)^\sigma - \beta'(a,b)(ba)^\sigma \\ &= \sigma(a,b)ab - \beta'(a,b)\sigma(b,a)ba \\ &= \sigma(a,b)(ab - (\beta'\delta^{-1})(a,b)ba) \\ &= \sigma(a,b)(ab - \beta(a,b)ba) \\ &= \sigma(a,b)[a,b]_\beta = ([a,b]_\beta)^\sigma, \end{aligned}$$

proving our claim.

□

We close this section with some general results about graded algebras; we do not require that G be abelian.

Lemma 3.1.1. *Let A be a G-graded associative algebra over a field k, with G finite, and let $\sigma \in Z^2(G, k^*)$. Let A^σ be the (associative) algebra with multiplication*

$$(ab)^\sigma := \sigma(a,b)ab$$

for a, b homogeneous. Then

1. *A is a* PI-*algebra if and only if A^σ is a* PI-*algebra;*
2. *A is semiprime if and only if A^σ is semiprime, provided $|G|^{-1} \in k$;*
3. *A is semiprimitive if and only if A^σ is semiprimitive, provided $|G|^{-1} \in k$.*

Proof. First note that A^σ is also G-graded, with $(A^\sigma)_g = A_g$ as vector spaces, and that since $\sigma(e,g) = 1$, e the identity element of G, it is clear that $A_e \cong (A^\sigma)_e$.

1. If A is a PI-algebra, then so is A_e and $(A^\sigma)_e$. Applying a theorem of Bergen and Cohen [BC] (for a different proof holding even if G is an infinite group, and A is a Lie algebra; see [BZ1]), it follows that A^σ is a PI-algebra. Similarly the converse holds.
2. Assume that A is semiprime. By [CM], Cor. 5.4, $N(A_e) = N(A) \cap A_e$, where $N(A)$ is the prime radical of A. Since $N(A) = \{0\}$ also $N(A_e) =$

$\{0\} = N(A^\sigma)_e$. Now by a theorem of [CR] we must have $N(A^\sigma)$ nilpotent. Since A has no $|G|$-torsion, $N(A^\sigma)$ is a graded ideal by [CM], Cor. 5.5. But then $(N(A^\sigma))^{\sigma^{-1}}$ is a nilpotent graded ideal of A, a contradiction, unless $N(A^\sigma) = \{0\}$.

3. The proof is very similar to that of 2: replace the prime radical by the Jacobson radical $J(A)$ and [CM], Cor. 5.4 and 5.5, by [CM], Cor 4.2 and Theorem 4.4(3).

□

3.1.2 Some proofs

In this section we prove Theorem 3.4 and Theorem 3.3.

We start with the proof of Theorem 3.4.

Proof. We first assume that L is perfect. Without any loss of generality suppose that k is algebraically closed (otherwise apply standard procedures of extending the ground field of coefficients). Using Proposition 3.1.1 of section 3.1.1, we find $\sigma \in Z^2(G, k^*)$ such that in A^σ, the ordinary bracket $[a, b] = ab - ba$ differs from the β-bracket $[a, b] = ab - \beta(a, b)ba$ on homogeneous a, b by a nonzero scalar $\sigma(a, b)$. So the same vector space L under the new multiplication of A becomes a Lie algebra under the ordinary bracket. Since $[a, b]^\sigma = \sigma(a, b)[a, b], \sigma(a, b) \in k^*$, the span of all $[a, b]^\sigma$ is the same as that of all $[a, b]$. So we have $L^\sigma = [L^\sigma, L^\sigma]^\sigma$. Similarly we observe that A^σ is generated by L^σ as an associative algebra. By Lemma 3.1.1, part 1, A^σ is a PI-algebra.

By Theorem 3.2 it follows that $\dim A^\sigma < \infty$; hence $\dim A < \infty$, as required.

Conversely, we now prove that the condition $L = [L, L]$ is not only sufficient but also necessary for having any PI-envelope finite-dimensional.

We will show that if $L \neq [L, L]$, then L has a PI-envelope which is not finite-dimensional. First consider the case when L is an ordinary (graded) Lie algebra. Let $M = L/[L, L]$. The quotient map $\phi : L \to M$ induces a homomorphism of associative algebras

$$\psi : U(L) \to U(M) \cong k[X_1, \dots, X_s], s = \dim M.$$

Let $\chi : L \to \mathrm{End}(V)$ be any faithful finite-dimensional representation of L; such exists by the extension of Ado's Theorem obtained by Scheunert [Sch]. Letting $\tau := \psi \otimes \chi$, the Lie tensor product of these maps, we obtain a homomorphism of L into $Q = U(M) \otimes \mathrm{End}(V)$.

Now τ is an imbedding since χ is faithful, and it is not difficult to see that $A := \tau(U(L))$ is infinite-dimensional (since if $X_i = \psi(x_i)$, the images under τ of the distinct ordered monomials in the x_i are linearly independent). Moreover Q is a PI-algebra, since $U(M)$ is commutative and $\mathrm{End}(V)$ is finite-dimensional. Thus $A \subset Q$ is an infinite-dimensional PI-envelope of L.

Finally if β is non-trivial, we may use Scheunert's construction as before and pass to L^σ, an ordinary Lie algebra. If A is the PI-envelope constructed as above, then $A^{\sigma^{-1}}$ is an infinite-dimensional PI-envelope of L (where we have used Lemma 3.1.1 for PI as above).

□

Remark 3.1.1. Applying M. Scheunert's procedure enables us to generalize V. Kac's Theorem [Kac] about solvability of Lie superalgebras to general (G, β)-Lie-superalgebras and to show that a (G, β)-Lie-superalgebra $L = L_+ \oplus L_-$ is solvable iff L_+ is solvable.

We recall that a (G, β)-Lie-superalgebra L is called *semisimple* if it has no G-graded solvable ideals.

Proposition 3.1.2. *Let L be a G-graded Lie algebra, G a finite abelian group. If L is graded semisimple, then it is semisimple in the ordinary sense.*

Proof. We start with the case where k is an algebraically closed field. Suppose L has a solvable ideal M which is not graded. Consider the dual group $\hat{G}$. Then $\hat{G}$ acts on L by automorphisms in a natural way: if $\chi \in \hat{G}$, $x \in L_g$, then $\chi * x = \chi(g)x$. An ideal of L is G-graded if and only if it is $\hat{G}$-invariant. Now for any

$\chi \in \hat{G}$ we have that $\chi * M$ is an ideal of L and thus that $\tilde{M} = \sum_{\chi \in \hat{G}} \chi * M \supset N$ is a graded ideal. Since the sum of two solvable ideals is always solvable, we conclude that $\tilde{M}$ is a non-trivial G-graded solvable ideal of L. Hence $\tilde{M} = \{0\}$ and then $M = \{0\}$, as stated.

If k is not algebraically closed, then the standard procedure of extending the field of coefficients works because then the extended algebra is (graded) semisimple (respectively, solvable) if and only if this is true for the original algebra.

□

Now we pass to the proof of Theorem 3.3.

Proof. Let $\sigma \in Z^2(G, k^*)$ be a 2-cocycle such that L^σ is an ordinary Lie algebra and A^σ is a PI-envelope of L^σ, as in the proof of Theorem 3.4. Obviously L^σ is a G-graded algebra without G-graded solvable ideals. It follows by Proposition 3.1.2 that L^σ is an ordinary semisimple Lie algebra, hence by Theorem 3.1 we have that A^σ, as a PI-envelope of L^σ, is finite-dimensional and Jacobson semisimple. Now we have shown that A is a finite-dimensional algebra. By Lemma 3.1.1, part 3, A is also semisimple, as required.

□

3.2 Identities of Lie Type Algebras

We start with a result, which is easy to remember.

Theorem 3.9 ([BZ1]). *Let $L = \sum_{g \in G} L_g$ be a Lie algebra over an arbitrary field F, G a finite group. If the component L_1 is a Lie algebra with a non-trivial identity then such is also L.*

A question about the validity of such theorem even in the case of $|G| = 2$ was asked by A. E. Zalesskii. This result can be applied to Lie algebras with an action of a finite automorphism group G. The most correct formulation is in the case of Lie algebras with actions of finite-dimensional Hopf algebras and we have results on such algebras in Part 3.3.

Theorem 3.10 ([BZ1]). *Let L be a Lie algebra over a field F, G a finite solvable subgroup in the automorphism group* Aut G, char F *does not divide* $|G|$. *If the invariant subalgebra L^G is a PI-algebra then also L is a PI-algebra.*

In the case of associative algebras this result belongs to V. K. Kharchenko [Kh] and S. Montgomery [M2] (no need to assume G solvable).

Now the main result of this section can be formulated as follows.

Theorem 3.11 ([BZ1]). *Let*

$$L = \sum_{g \in G} L_g$$

be a Lie type algebra over an arbitrary field F with a finite G-grading where G is a semigroup with identity element and cancellation. Suppose in the identity homogeneous component L_1 we have a non trivial identity of the form

$$x_0 x_1 \cdots x_{d-1} \equiv \sum_{\sigma \in Sym(d-1), \sigma \neq e} \alpha_\sigma x_0 x_{\sigma(1)} \cdots x_{\sigma(d-1)}, \tag{24}$$

in which $x_0 x_1 \cdots x_{d-1}$ and all $x_0 x_{\sigma(1)} \cdots x_{\sigma(d-1)}$ are left-normed monomials and $\alpha_\sigma \in F$. Then also in L a nontrivial identity of the same form (24) is satisfied.

From this theorem we easily derive the result of [BC] about the associative algebras graded by a finite group. But more important here are the applications to Lie algebras and their generalizations.

Theorem 3.12 ([BZ1]). *Let $L = \oplus_{q \in Q} L_q$ be a color Lie superalgebra, over a field F with a finite grading by a semigroup G with identity element and cancellation and such that Q is a finite abelian group and the Lie algebra $L_0^{(1)}$ satisfies a non-trivial Lie identity. Then also L satisfies a non-trivial non-graded identity.*

A standard example of color Lie superalgebras is an ordinary Lie superalgebra where $Q = Z_2, \beta(0,0) = \beta(0,1) = 1, \beta(1,1) = -1$. In this case our theorem takes the following form.

Theorem 3.13 ([BZ1]). *We consider a Lie superalgebra*

$$L = L_0 \oplus L_1 = \sum_{g \in G} (L_0^{(g)} \oplus L_1^{(g)}),$$

with a finite grading by a semigroup G with identity element and with cancellation. If $L_0^{(1)}$ satisfies a non-trivial Lie algebra identity then also L is a Lie superalgebra with a non-trivial identity.

If in the formulation of Theorem 3.12 we take Q as the grading semigroup then we arrive at a corollary as follows. Let Q be a finite abelian group and

$$L = \bigoplus_{q \in Q} L_q$$

be a color Lie superalgebra over a field F. If L_0 satisfies a non-trivial Lie identity then L has a non-trivial non-graded identity.

Finally, the simplest example of a non-associative algebra satisfying (24) is a Lie algebra. Thus Theorem 3.9 is an immediate consequence of Theorem 3.11.

3.2.1 Main Techniques

For the proof of Theorem 3.11 we will need some constructions similar to those in Part 1.2.

For each g in G we consider a countable set

$$Z_g = \{z_1^{(g)}, \ldots, z_m^{(g)}, \ldots\}$$

and denote by Z the union of all $Z_g, g \in G$. Then the free non-associative algebra $F\langle Z\rangle$ over F, generated by the set Z, is naturally endowed by a G-grading. Let us fix a finite subset H in G and set

$$x_i = \sum_{g \in H} z_i^{(g)}, \quad i = 1, 2, \ldots.$$

The subalgebra $F\langle X\rangle$ generated by $X = \{x_1, x_2, \ldots\}$ in $F\langle Z\rangle$ is not G-graded but it is also a free non-associative F-algebra.

We denote by V_n the linear span of all products of the form $[x_{\sigma(1)} \cdots x_{\sigma(n)}]$ in $F\langle X\rangle$ with all possible arrangements of brackets, where σ runs through the whole of the symmetric group $Sym(n)$. Similarly, for any selection $g = (g_1, \dots, g_n)$ in which all $g_i \in H$, $i = 1, \dots, n$, by V_n^g we shall denote the subspace in $F\langle Z\rangle$, spanned by all products

$$[z_{\sigma(1)}^{(g_{\sigma(1)})} \dots z_{\sigma(n)}^{(g_{\sigma(n)})}].$$

Finally, set

$$V_n^G = \bigoplus_{g \in H^n} V_n^g.$$

By $T(L)$ we will denote the ideal of identities of algebra L in $F\langle X\rangle$, and by $T^G(L)$ the ideal of graded identities of L in $F\langle Z\rangle$. We consider two sequences of numbers

$$c_n(L) = \dim \frac{V_n}{V_n \cap T(L)},$$

$$c_n^G(L) = \dim \frac{V_n^G}{V_n^G \cap T^G(L)},$$

characterizing numerically the sets of non-graded and graded identities of L. As before, in Lemma 2.2.1, we have

Lemma 3.2.1. $c_n(L) \le c_n^G(L)$.

To compute the products of elements in a graded algebra we will need Lemma 2.2.2.

Definition 3.2.1. *The sequence $(a_1, \dots, a_n)$ composed of pairwise distinct integers is called m-decomposable if there exist indices $1 \le i_1 < i_2 < \dots < i_m \le n$ such that $a_{i_k} > a_j$ for all $j = i_k + 1, \dots, i_{k+1}$ where $k = 1, \dots, m-1$. From m-decomposability it follows that $a_{i_1} > a_{i_2} > \dots > a_{i_m}$. If the sequence $(a_1, \dots, a_n)$ has no m-decompositions, then we call it m-indecomposable*

The following is true

Lemma 3.2.2. *Let $L = \sum L_g$ be a Lie type algebra with a finite G-grading and s the number of non-zero components L_g. We consider all homogeneous in the G-grading elements $y_0, y_1, \ldots, y_n$ and denote by P the linear span of all products of the form $y_0(y_{i_1} \cdots y_{i_n})$ with all possible arrangements of brackets in the right factor $(y_{i_1} \cdots y_{i_n})$, such that $(i_1, \ldots, i_n)$ is a permutation of $1, \ldots, n$. Suppose that the identity component L_1 satisfies a multilinear identity of the form (24). Then any element in P is a linear combination of left-normed products of the form $y_0 y_{j_1} \cdots y_{j_n}$ for which $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$ and the sequence $(j_1, \ldots, j_n)$ is m-indecomposable for any $m \geq sd$.*

Proof. It follows from (13) that P is a linear span of left-normed products of elements $x_0, x_1, \ldots, x_n$, starting with x_0. Everywhere in the sequel we will omit brackets in the expression of left-normed products of the elements of L.

Remark, that we easily derive from (24) the existence of a graded identity of the form

$$z_0 z_1 \cdots z_d \equiv \sum_{\sigma \in Sym(d), \sigma \neq e} \beta_\sigma z_0 z_{\sigma(1)} \ldots z_{\sigma(d)}, \tag{25}$$

in which $z_i = z_i^{(1)}, \beta_\sigma \in F$, and z_0 is an arbitrary element. Just multiply z_0 by the left and the right sides of (24), in which x_i are replaced by z_{i+1}, and reduce all products to the left-normed expressions using (13). In doing so the product $z_0 z_1 \cdots z_d$ appears only once with coefficient $\alpha^{d-1} \neq 0$, while the other summands form the right hand side in (25).

Now we introduce a lexicographic ordering on the integer tuples by comparing them from the left to the right. Let $u = y_0 y_{j_1} \cdots y_{j_n}$ be a left-normed product. If $j_1 = 1, j_2 = 2, \ldots, j_n = n$, then $(j_1, \ldots, j_n)$ is m -indecomposable. Arguing by induction, we may assume that all products $y_0 y_{k_1} \cdots y_{k_n}$ with $(k_1, \ldots, k_n)$ strictly less than $(j_1, \ldots, j_n)$ are linear combinations of left-normed products $y_0 y_{i_1} \cdots y_{i_n}$ with m-indecomposable tuples $(i_1, \ldots, i_n)$.

We assume that the tuple $(j_1, \ldots, j_n)$ is m-decomposable and the indices

$t_1, \ldots, t_m$ determine an m-decomposition, that is,

$$j_{t_1} > j_{t_1+1}, \ldots, j_{t_2}; \ldots; j_{t_{m-1}} > j_{t_{m-1}+1}, \ldots, j_{t_m}$$

and let us show that u can also be expressed through m-indecomposable left-normed products. We denote by $v_i, i = 1, \ldots, m$, a left-normed product of elements y_k, with k running through values from j_{t_i} to $j_{t_{i+1}} - 1$, and by v_0 the product of the first y_k with $k < j_{t_1}$. We denote by B the operator of subsequent multiplication by the remaining elements y_k where $k = j_{t_m} + 1, \ldots, j_n$, if $j_{t_m} < j_n$. Then the element

$$u' = v_0 v_1 \cdots v_m B$$

with the use of (13) can be written as a linear combination of u with a non-zero coefficient and the products $y_0 y_{q_1} \cdots y_{q_n}$, with $(q_1, \ldots, q_n) < (j_1, \ldots, j_n)$. Since all y_i were assumed homogeneous, we have $v_0 \in L_{g_0}, v_1 \in L_{g_1}, \ldots, v_m \in L_{g_m}$ for some $g_0, g_1, \ldots, g_m \in G$. Owing to the choice of m by Lemma 2.2.2, there exist $0 \leq l_0 < l_1 < \ldots < l_d \leq m$ such that $g_{1+l_{i-1}} \cdots g_{l_i} = 1$ are in G for all $i = 1, \ldots, d$. We introduce notation for new left-normed products:

$$w_0 = v_0 \cdots v_{l_0}, \; w_i = v_{1+l_{i-1}} \cdots v_{l_i}, \quad i = 1, \ldots, d.$$

Then $w_0, \ldots, w_d \in L_1$ and the element

$$u'' = w_0 w_1 \cdots w_d C,$$

in which C stands for the operator of right multiplications by $v_{l_d+1}, \ldots, v_m$, if $l_d < m$, and then the action of the operator B, differs from u by a scalar factor modulo a linear combination of some $y_0 y_{q_1} \cdots y_{q_n}$ with $(q_1, \ldots, q_n) < (j_1, \ldots, j_n)$. Applying (25) we arrive at

$$u'' = \sum_{\sigma \in Sym(d), \sigma \neq e} \beta_\sigma w_0 w_{\sigma(1)} \cdots w_{\sigma(d)} C \tag{26}$$

Now let us express $w_0 w_{\sigma(1)} \cdots w_{\sigma(d)} C$ on the right side of (26) through left-normed monomials $y_0 y_{q_1} \cdots y_{q_n}$, by expanding brackets with the help of (13) first on w_i, and then on v_i. Since $\sigma \neq e$ in $Sym(d)$, we are going to obtain only the summands with $(q_1,\ldots,q_n)<(j_1,\ldots,j_n)$. This gives that u'' and then also u' and u are of the form required, and the proof of Lemma 3.2.2 is complete.

□

3.2.2 Proof of Main Theorem 3.11

We start with a lemma about codimensions.

Lemma 3.2.3. *If the subalgebra L_1 satisfies an identity of the form (24), then for any b for sufficiently large n the following inequality is satisfied:*

$$c_n^G(L)<\frac{n!}{b^n}.$$

Proof. Let $g=(g_1,\ldots,g_n)$ be a tuple in which $g_1,\ldots,g_n\in H$. Then $V_n^G=\oplus_{g\in H^n}V_n^g$ and

$$V_n^G/V_n^G\cap T^G(L)=\sum_{g\in H^n}V_n^g/V_n^g\cap T^G(L). \tag{27}$$

The number of summands on the right side of (27) is equal to s^n. For the proof of the lemma it is sufficient to show that the dimension of each is less than $\frac{n!}{(bs)^n}$. For convenience we replace n by $n+1$. We fix a tuple $g=(g_0,g_1,\ldots,g_n)$ and set $z_i=z_i^{(g_i)}$. Modulo $T^G(L)$ the space $V_{n+1}^G=V_{n+1}(z_0,z_1,\ldots,z_n)$ is equal to the sum $Q_0+Q_1+\cdots+Q_n$, in which Q_i is a linear span of the left- normed products $z_0,z_1,\ldots,z_n$, starting with z_i. It is sufficient to show that the dimension of each summand is asymptotically less than $\frac{n!}{(2bs)^n}$. We consider, for example, $Q=Q_0$, that is, the linear span of left-normed products $z_0z_{\sigma(1)}\cdots z_{\sigma(n)}$. By Lemma 3.2.2 the tuple $(\sigma(1),\ldots,\sigma(n))$ can be assumed m-indecomposable, where $m=sd$.

We denote by $a_m(n)$ the number of m-indecomposable tuples composed of the numbers $1,2,\ldots,n$. Then

$$\lim_{n\to\infty}\sqrt[n]{\frac{a_m(n)}{n!}}=0 \tag{28}$$

(see, e.g., [Raz, Chapter 1, section 2.1],). From (28) it follows that for all sufficiently large n the following inequality is satisfied:

$$\sqrt[n]{\frac{a_m(n)}{n!}} < \frac{1}{2bs},$$

from which it follows that for $n \geq 2bs$ we have

$$\dim Q \leq a_m(n) < \frac{n!}{(2bs)^n} < \frac{(n+1)!}{(2bs)^{n+1}},$$

and the proof of Lemma 3.2.3 is complete.

□

Proof. [Proof of Theorem 3.11]. From Lemmas 3.2.1 and 3.2.3 it follows that the left-normed products $x_0 x_{\sigma(1)} \cdots x_{\sigma(n)}, \sigma \in Sym(n)$, are linearly dependent modulo the ideal $T(L)$ of identities of algebra L in the free non-associative algebra $F\langle X\rangle$. This means that also L satisfies a non-trivial identity of the same type as (24), and the proof of Theorem 3.11 is complete.

□

3.3 Applications to Group Theory

The applications in question are based on some of Zelmanov's work on Burnside's Problem. To formulate one of the results we need some definitions. If G is a group then we can speak about the Lie algebra $L(G)$ associated with G. We have to look at the *lower central series* $\{\gamma_n(G) | n = 1, \ldots \infty\}$ of G. If $(a,b) = aba^{-1}b^{-1}$ is the notation for the group commutator then the lower central series is given by $\gamma_1(G) = G$ and, by induction, $\gamma_{n+1}(G) = (\gamma_n(G), G)$. We have the so called "central filtration property"

$$(\gamma_n(G), \gamma_m(G)) \subset \gamma_{n+m}(G).$$

This property enables us to define a Lie ring $L(G)$ associated with G.

Definition 3.3.1. *Set* $L(G) = \bigoplus_{n=1}^{\infty} L_n(G)$ *where* $L_n(G) = \gamma_n(G)/L_{n+1}(G)$. *This abelian group can be given a Lie operation (bracket) if we set*

$$[x\gamma_{n+1}(G), y\gamma_{m+1}(G)] = (x,y)\gamma_{m+n+1}(G) \text{ for } x \in \gamma_n(G) \text{ and } y \in \gamma_{m+1}(G).$$

We call $L(G)$ *the Lie ring associated with* G.

For example, if we start with a free group $F = F(x_1, \dots, x_n)$ then, in the associated Lie ring $L(F)$ the elements $x_1\gamma_2(F), \dots, x_n\gamma_2(F)$ generate a free Lie ring.

In the case of p-groups, that is, the groups where each element is periodic with period a p-power, p a prime number, it is more appropriate "refine" $L(G)$ by inserting intermediate terms so that the corresponding quotient groups become elementary abelian p-groups, e.g. inserting normal subgroups of the form $(G^{p^k} \cap \gamma_n(G))\gamma_{n+1}(G)$, $k = 1, 2, \dots$. In the case of using this centaral filtration the additive group of $L(G)$ becomes an elementary abelian p-group, that is, a vector space over $\mathbb{Z}_p$. If K is an arbitrary field of characteristic p then we can form $\mathcal{L}(G) = L(G) \otimes_{\mathbb{Z}_p} K$ and this is a Lie algebra over K associated with G.

In a number of important cases $\mathcal{L}(G)$ reflects the properties of G quite satisfactorily, yet in working with $\mathcal{L}(G)$ one can use the full force of linear algebra. This is similar to the situation with the correspondence "Lie groups"$\leftrightarrow$"Lie algebras" but in the latter case people often speak about the groups meaning their Lie algebras, which never happens in the case of the just defined correspondence.

To formulate some recent results we need some more definitions. Given a class $\mathcal{C}$ of groups we say that a group G is *residually* a group from $\mathcal{C}$ if for any nonzero $g \in G$ there exists a normal subgroup N such that $x \notin N$ and $G/N \in \mathcal{C}$. We also say that G is *locally* (belongs to) $\mathcal{C}$ if any finitely generated subgroup of G is in $\mathcal{C}$.

It is well-known (Golod Example) that there exist finitely generated p-groups which are infinite. So the following result of Efim Zelmanov is quite important.

Theorem 3.14 ([Zel]). *Let G be a finitely generated periodic residually p-group. Suppose that the Lie algebra $\mathcal{L}(G)$ satisfies a non-trivial polynomial identity. Then G is finite.*

Now in our previous material we considered the situation where a group acts on a Lie algebra and the fixed point subalgebra satisfies a non-trivial polynomial identity. Then we have concluded that the whole Lie algebra enjoys the same property. An important lemma is due to Rocco and Shumyatsky. Suppose that

G is a torsion group with no 2-torsion and Q is a finite 2-group acting on G. Let $C = C_G(Q) = \{g \in G | q * g = g\}$ be the centralizer of Q in G.

Lemma 3.3.1. *Let N be a normal subgroup of G, which is Q-invariant. Then*

$$G_{G/N}(Q) = C_G(Q)N/N.$$

Now suppose G is residually a p-group for some prime $p \geq 2$ and that some centralizer $C_G(Q)$ satisfies a non-trivial group identity. Then we construct $\mathcal{L}(G)$ and Lemma 3.3.1 enables us to transfer the action of Q on $\mathcal{L}(G)$ with $C_{\mathcal{L}(G)}(Q)$ satisfying a non-trivial polynomial identity. This is exactly the situation where our results of Subsection 3.2 work so that Theorem 3.14 applies enabling us to establish the finiteness of the group in the appropriate cases.

Without going into further details I formulate some recent results based on this circle of ideas. The first two results are due to A. Shalev.

Proposition 3.3.1 ([Sha]). *Let G be a residually finite p-group acted on by a finite 2-group Q. Suppose that $p \neq 2$ and $C_G(Q)$ satisfies some non-trivial identity. Then G is locally finite.*

Theorem 3.15 ([Sha]). *Let G be a residually finite tosion group with no 2-torsion acted by a finite 2-group Q. Suppose that the centralizer $C_G(Q)$ is solvable or of finite exponent. Then G is locally finite.*

If Q is any finite group let $m(Q)$ denote the maximal prime divisor of the order $|Q|$ of Q.

Then we have the following results of Shumyatsky.

Theorem 3.16 ([Shu]). *Let G be a residually finite group acted on by a finite solvable group Q with $m = m(Q)$. Assume that G has no $|Q|$-torsion and that $C_G(Q)$ is either solvable or of finite exponent. If any of $m - 1$ elements of G generate a finite solvable subgroup then G is locally finite.*

Theorem 3.17 ([Shu]). *Let G be a finitely generated periodic residually solvable group acted on by a finite solvable group Q. Assume that G has no $|Q|$-torsion and $C_G(Q)$ is either solvable or of finite exponent. Then G is finite.*

The above theorem is no longer true if we impose conditions only on $C_G(Q)$. Indeed, for any (not necessarily distinct) odd primes p and q Miller and Obraztsov [MOb] have constructed a finitely generated infinite residually finite periodic group admitting a fixed-point-free automorphism of order q.

4. Theme Four: H-algebras

We start with some known definitions which can be found, for instance, in [M3].

4.1 Some definitions and examples

Definition 4.1.1. *A (unital)* algebra *A over a field F is a vector space over F with a product which is a linear map $m : A\otimes A \to A$ satisfying the associativity:*

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m)$$

and a unit which is a linear map $u : F \to A$ such that we have

$$m \circ (u \otimes id_A) = \mu, \mu' = m \circ (id_A \otimes u).$$

Here id_A is the identity map of A and $\mu : F \otimes A \to A$, $\mu' : A \otimes F \to A$, respectively, the left and the right multiplications by the scalars in A.

Definition 4.1.2. *A* coalgebra *C over a field F is a vector space over F with a coproduct which is a linear map $\Delta : C \to C \otimes C$ satisfying the coassociativity:*

$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta$$

and a counit which is a linear map $\varepsilon : C \to F$ such that we have

$$(\varepsilon \otimes id_C) \circ \Delta = \nu, \nu' = (id_C \otimes \varepsilon) \circ \Delta$$

where $\nu : C \to F\otimes C$, $\nu' : C \to C\otimes F$ are the linear maps given by $\nu(c) = 1\otimes c$, $\nu'(c) = c \otimes 1$.

Definition 4.1.3. *A* bialgebra *B over a field F is an associative algebra over F with 1 (or with the* unit *linear map $u : F \to B$ sending 1 of F into 1 of B) and a coalgebra under some Δ, ε as above such that both Δ and ε are algebra homomorphisms.*

Definition 4.1.4. *If A is an algebra and C is a coalgebra then the space* $\mathrm{Hom}(C, A)$ *of all linear maps from C into A becomes an algebras under the* convolution *product of maps given by*

$$f * g = m \circ (f \otimes g) \circ \Delta.$$

The neutral element of this multiplication is the map $u \circ \varepsilon$. If B is a bialgebra then the inverse S of id_B under the convolution product, if exists, is called the antipode *of B.*

One can verify that S is an antihomomorphism of algebras.

Definition 4.1.5. *A* Hopf algebra *H is a bialgebra with antipode. It is conventional to denote the coproduct in H by so called* Sweedler 's notation*:*

$$\Delta h = \sum h_{(1)} \otimes h_{(2)} \in H \otimes H.$$

The right hand side is simply an element in $H \otimes H$ with the components (!) labeled by $h_{(1)}$ lying in the left tensor factor and those by $h_{(2)}$ in the second one.

Definition 4.1.6. *Let F be a field, A an algebra (not necessarily associative), H a Hopf algebra, both over F. We say that H* acts *on A or that A is an* H-algebra *if the vector space A is made into a left unital H-module and, in addition, we have*

$$h(a_1 a_2) = \sum (h_{(1)} a_1)(h_{(2)} a_2)$$

where $h \in H$ and $a_1, a_2 \in A$. We also have used Sweedler's notation from Definition 4.1.5.

That A is a unital H-module is written through two equations as follows:

$$(h_1 h_2)a = h_1(h_2 a) \text{ and } 1a = a \text{ for any } h_1, h_2 \in H,\ a \in A.$$

Definition 4.1.7. *Let A be an H-algebra, as in Definition 4.1.6. An element $a \in A$ is called an H-*invariant *if for any $h \in H$ one has $ha = \varepsilon(h)a$. The set of all H-invariants of A is denoted by A^H.*

It is a trivial observation that the set of invariants A^H of an H-algebra A is a subalgebra of A.

Our main concern in this part will be determining when for an H-algebra A it follows from A^H being a PI-algebra that also A is a PI-algebra.

Definition 4.1.8. *We say that a Hopf algebra H over a field F belongs to the class $\mathcal{P}$ if for any associative algebra A over F it follows from A^H being a PI-algebra that also A is a PI-algebra.*

Actually, there is a dual situation, and this will be convenient for Part 4.2. We start with so called *comodules.*

Definition 4.1.9. *An H-comodule M is a space endowed with a* comodule map *$\rho : M \to M \otimes H$ which is a linear map such that $(1 \otimes \Delta) \circ \rho = (\rho \otimes 1) \circ \rho$ and $(1 \otimes \varepsilon) \circ \rho = id$. There is notation of the same type as for the coproduct. We write*

$$\rho(x) = \sum_{(x)} x_0 \otimes x_1. \tag{29}$$

If we use this notation then the above conditions take the form of

$$\sum ((x_0)_0 \otimes (x_0)_1) \otimes x_1 = \sum x_0 \otimes ((x_1)_1 \otimes (x_1)_2)$$

and

$$\sum x_0 \varepsilon(x_1) = x.$$

Definition 4.1.10. *A right H-comodule algebra A is an algebra endowed with the structure of a right H-comodule such that the structure map ρ is a homomorphism of algebras A and $A \otimes H$. An element $a \in A$ is called* coinvariant *if $\rho(a) = a \otimes 1$. The set A^{co-H} of all coinvariants is a subalgebra.*

It is important to remark that, in the case where H is finite-dimensional, the H-module algebra is an H^*-comodule algebra where H^* is the dual Hopf algebra, that is, the space of linear functions on H with values in the base field, $\Delta(f)(h \otimes k) = f(hk)$ and $(fg)(h) = (f \otimes g)(\Delta h)$. Here $f, g \in H^*$ and $h, k \in H$ (see [M3]).

4.1.1 Examples

There are very simple examples of Hopf algebras which are *not* in $\mathcal{P}$. As it follows from our Main Theorem 4.1 any finite-dimensional Hopf algebra which has a non-zero Jacobson radical is such. So over any F one can take as such an example the 4-dimensional Hopf algebra H_4 which has a basis $1, g, x, gx$ where 1 is the identity element, g a group-like, i.e. $\Delta g = g \otimes g$, and $g^2 = 1$, $x^2 = 0$, $\Delta x = x \otimes 1 + g \otimes x$, $Sg = g$, $Sx = -x$, $xg = -gx$. The counit is given by $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$. If we do not insist on zero characteristic of F then one can take any group algebra $F[G]$ of a finite group G such that char$F$ divides $|G|$. We recall that any group algebra $F[G]$ becomes a Hopf algebra if one sets $\Delta g = g \otimes g$ for any $g \in G$. For the antipode one has to set $Sg = g^{-1}$, $g \in G$. For the counit we set $\varepsilon(g) = 1$ for any $g \in G$. If $H = F[G]$ acts on an algebra A then $g \cdot (ab) = (g \cdot a)(g \cdot b)$ which means actually that G acts on A by automorphisms. The converse is obviously true as well. The question of when $F[G] \in \mathcal{P}$, for G finite and char$F = 0$, was resolved by Kharchenko [Kh] who showed that this is always the case. Kharchenko also gave examples in the modular case (i.e. char$F$ divides $|G|$) showing that then it is not necessary that $F[G] \in \mathcal{P}$.

A good example of semisimple finite-dimensional Hopf algebras is provided by the dual algebras $(F[G])^*$ to the group algebras $F[G]$ of finite groups, with arbitrary characteristic of F. If we set H to be the space of linear functions on $F[G]$ with values in F, endowed with ordinary multiplication of functions then it is obvious that this is a commutative algebra, the direct sum of n copies of F, $n = |G|$, each generated by a function p_g, $g \in G$, a characteristic function on G: $p_g(g) = 1, p_g(h) = 0$ if $h \neq g$. Canonically, $\Delta p_g = \sum_{hk=g} p_h \otimes p_k$, and $\epsilon(p_g) = 0$ if $g \neq e$ and $\epsilon(p_e) = 1, Sp_g = p_{g^{-1}}$. It is well-known that an algebra A with an action of such an H is actually a G-graded algebra, that is, $A = \sum_{g \in G} A_g$ where $A_g A_h \subset A_{gh}$ for any $g, h \in G$. Actually, $A_g = \{a \in A | p_g \cdot a = a\}$, for any $g \in G$. So the elements p_g act as projections on "homogeneous" components A_g. It is immediate that the subalgebra of invariants of this action is just the identity component A_e of the grading. It has been shown in [BC] that $(F[G])^* \in \mathcal{P}$.

Thus we have seen that both algebras with actions of automorphism groups and graded algebras fall into the framework of algebras with actions of Hopf algebras. It should be emphasized, however, that these examples and appropriate crossed products (see [MW]) of such algebras are very important examples of semisimple finite-dimensional Hopf algebras.

Our last remark here: every Hopf algebra H is an H-comodule algebra, if we set $\rho = \Delta$.

4.2 Main theorem about H-algebras

In our discussion in this section we denote the set of invariants of an H-module M by $I(M)$. Moreover the invariants of the left regular action of H on itself will be denoted sometimes simply as I. If $T(H)$ is the tensor algebra of a vector space H without the component of degree 0 then there is a unique extension of this latter action of H on H to $T(H)$ that makes $T(H)$ into an H-algebra. On a homogeneous tensor $v = h_1 \otimes \ldots \otimes h_n$ an element $h \in H$ acts by

$$h * v = \sum (h_{(1)}h_1) \otimes \ldots \otimes (h_{(n)}h_n).$$

To simplify the notation we often write T in place of $T(H)$. We write T_n for the homogeneous component of T of degree n. We write $I_n = I(T_n)$.

The following is the Main Theorem about H-algebras from [BL].

Theorem 4.1. *Let H be a finite-dimensional Hopf algebra over a field F. The following conditions are equivalent.*

1. *For any associative algebra A over F with an action of H it follows from A^H being a PI-algebra that also A is a PI-algebra (i.e. $H \in \mathcal{P}$)*
2. *There exists a function $f : N \to N$ such that for any associative algebra A with an action of H if A^H satisfies a non-trivial identity of degree t then A satisfies a non-trivial identity of degree $f(t)$*

3. *There exists a function* $g(t)$ *such that for any natural* t *and any* H*-algebra* A *with* $I(A)^t = \{0\}$ *one has* $A^{g(t)} = \{0\}$

4. *There exists a number* N *such that any* H*-algebra* A*, where* $I(A)$ *has zero multiplication, satisfies* $A^N = \{0\}$

5. $\dim T/(\mathrm{Ideal}_T I(T)) < \infty$

Any of Conditions (1) to (5) implies H *being semisimple.*

A technical condition which is equivalent to any of Conditions (1) to (5) is called

Condition (*)

For any $t \in N$ there is $h(t)$ such that for any $n \geq h(t)$ one has

$$T_n = \sum_{\substack{k_0+k_1+\dots k_t+k_{t+1}=n \\ k_1,\dots,k_t\geq 1}} T_{k_0} \otimes I_{k_1} \otimes \dots \otimes I_{k_t} \otimes T_{k_{t+1}}$$

The proof requires some techniques with *free* H*-algebras* as the central notion.

Let H be a Hopf algebra over a field F, X a non-empty set of variables, $T = T(H)$ the tensor algebra of the vector space H .

Let further $T\langle X\rangle$ be the free algebra over T with free generators X. Every element u of $T\langle X\rangle$ can be written in the form

$$u = \sum_\alpha t_\alpha X^\alpha \tag{30}$$

where X^α is a word (or non-commutative monomial) in X, α is a label uniquely defining X^α. If γ is a label for $X^\alpha X^\beta$ then we set $\gamma = \alpha\beta$. The ordinary degree of X^α will be denoted by $|\alpha|$. Obviously, $|\alpha\beta| = |\alpha| + |\beta|$.

We have a natural action of H on $T\langle X\rangle$ given by

$$h * u = \sum_\alpha (h * t_\alpha) X^\alpha. \tag{31}$$

It is immediate that, in this way, $T\langle X\rangle$ becomes an H-algebra), that is

$$h * (u_1u_2) = \sum(h_{(1)} * u_1)(h_{(2)} * u_2).$$

The set $\mathcal{H} = \mathcal{H}(X)$ of all elements of the form (30) with each t_α homogeneous of degree $|\alpha|$ is an H-subalgebra of $T\langle X\rangle$. We call $\mathcal{H}$ the *free H-algebra with the free generating set X*.

Some basic properties of $\mathcal{H}$ are as follows.

Proposition 4.2.1. *Let A be an arbitrary H-algebra. Then any map $\varphi : X \to A$ extends to a unique homomorphism of H-algebras $\bar{\varphi} : \mathcal{H} \to A$ such that $\bar{\varphi}|_X = \varphi$. Thus any H-algebra A can be written as a quotient H-algebra $A \cong \mathcal{H}(X)/J$ for a suitable set X and a suitable H-ideal J of $\mathcal{H}(X)$.*

Proof. It is sufficient to define $\bar{\varphi}$ on the elements $t_\alpha X^\alpha$. If $X^\alpha = x_{i_1} \ldots x_{i_n}$ then $t_\alpha = \sum h_1 \otimes \ldots \otimes h_n$ for some $h_1, \ldots, h_n \in H$ and we set

$$\bar{\varphi}(t_\alpha X^\alpha) = \sum(h_1 * \varphi(x_{i_1})) \ldots (h_n * \varphi(x_{i_n})). \tag{32}$$

The verification of $\bar{\varphi}$ being a homomorphism of H-algebras is immediate and left to the reader.

□

If we look at the invariants of $\mathcal{H}$ and their behavior under H-homomorphisms then we observe the following. It is well-known that if $\dim H < \infty$ then $\dim H^H = 1$ and if H is semisimple we have $H^H = F.\int$ with $\varepsilon(\int) = 1$ (see [M3]). Let $I = \oplus I_k$ be the set of left invariants of $T = T(H)$. Then the following is true.

Proposition 4.2.2. *The H-invariants of $\mathcal{H}$ form a homogeneous subalgebra $\mathcal{I}$ of $\mathcal{H}$ whose elements have the form (30) with each $t_\alpha \in I$. If H is semisimple then $\mathcal{I} = \int * \mathcal{H}$. If φ is an H-homomorphism of $\mathcal{H}$ and H is semisimple then*

$$\varphi(I(\mathcal{H})) = I(\varphi(\mathcal{H})). \tag{33}$$

Proof. The form of an element in $\mathcal{I}$ is immediate from (31). Also if $u \in \mathcal{I}$ then $\int * u = \varepsilon(\int)u = u$ and if $v \in \mathcal{H}$ then

$$h * (\textstyle\int * v) = (h\textstyle\int) * v = (\varepsilon(h)\textstyle\int) * v = \varepsilon(h)(\textstyle\int * v),$$

i.e. $\int * v \in \mathcal{I}$. To prove the final statement we write

$$\varphi(I(\mathcal{H})) = \varphi(\textstyle\int * \mathcal{H}) = \textstyle\int * \varphi(\mathcal{H}) = I(\varphi(\mathcal{H})).$$

□

It is natural to define now the notion of H-identities.

We take $X = \{x_1, x_2, \ldots\}$ countable and define two kinds of identities of H-algebras. Given such an algebra A any element $u \in \mathcal{H}$ (of the form (30)) is called an *H-identity* if under any H-homomorphism φ of $\mathcal{H}$ into A we have $\varphi(u) = 0$. The set of all such identities will be denoted by $\mathcal{T}_H(A)$. A particular case of H-identities are those where, in (30), each coefficient t_α is a scalar multiple of $\underbrace{1 \otimes \ldots \otimes 1}_{k}$, $k = \mid \alpha \mid$. The set of these elements is denoted by $\mathcal{T}(A)$. These are *ordinary polynomial identities* of A. Now we introduce two kinds of subspaces of $\mathcal{H}$, $V_{n,H}$ and V_n, for each natural $n = 1, 2, \ldots$. In the case of $V_{n,H}$ we have to take all (30) where $\alpha = \sigma \in S_n$, i.e. $X^\alpha = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$, with arbitrary t_α of degree n and σ a permutation of $\{1, \ldots, n\}$. If t_α is a scalar multiple of $\underbrace{1 \otimes \ldots \otimes 1}_{n}$ then we obtain V_n. It is natural to call $V_{n,H}$ (resp. V_n) *H-multilinear* (resp. *multilinear*) components of $\mathcal{H}$. Now we introduce the numbers (the n-th codimensions)

$$c_{n,H}(A) = \dim V_{n,H} / \mathcal{T}_H(A) \cap V_{n,H}$$

and

$$c_n(A) = \dim V_n / \mathcal{T}(A) \cap V_n.$$

As in Part 2.2, we have

Proposition 4.2.3. *For any H-algebra A and any natural n one has*

$$c_n(A) \leq c_{n,H}(A). \tag{34}$$

Remark 4.2.1. The inequality (34) is important for the proof of the Main Theorem because showing that A a PI-algebra is the same as finding n such that $c_n(A) < n!$. From (34) it follows that it is sufficient to prove $c_{n,H}(A) < n!$ for some n. In fact, this condition is also necessary.

For the proof of the Main Theorem the following construction is essential. Let us fix a natural number t. If A is an H-algebra then we define the subspaces $J^{(i)}(A)$ and H-ideals $M^{(i)}(A)$ by induction on natural $i \in N$ starting with $J^{(0)}(A) = M^{(0)}(A) = \{0\}$. If $i > 0$ then we set

$$J^{(i+1)}(A) = \{a \in A | \forall h \in H \Longrightarrow h * a = \varepsilon(h)a + m_h,\ m_h \in M^{(i)}(A)\},$$

and define $M^{(i+1)}(A)$ as an H-ideal of A generated by $(J^{(i+1)}(A))^t$ (the t-th power of $J^{(i+1)}(A)$).

If $A = T(H)$ we abbreviate the notation as just above and simply write $J^{(i)}$ or $M^{(i)}$.

The following is true.

Proposition 4.2.4. *1. For any A as above and any $i \in N$ one has*

$$J^{(i)}(A) \subset J^{(i+1)}(A),\ M^{(i)}(A) \subset M^{(i+1)}(A),\ M^{(i)}(A) \subset J^{(i+1)}(A)$$

and $M^{(i)}(A)$ is the ordinary two-sided ideal of A generated by $(J^{(i+1)}(A))^t$.

2. For any H-algebra homomorphism $\varphi : A \to B$ we have

$$\varphi(J^{(i)}(A)) \subset J^{(i)}(B),\ \varphi(M^{(i)}(A)) \subset M^{(i)}(B).$$

For any H-algebra A we set

$$J(A) = \bigcup_{i=1}^{\infty} J^{(i)}(A),\ M(A) = \bigcup_{i=1}^{\infty} M^{(i)}(A).$$

It follows that $M(A) \subset J(A)$.

Remark 4.2.2. If H is a finite-dimensional semisimple Hopf algebra then for any H-algebra one has

$$J(A) = I(A) + M^{(1)}(A),\ M(A) = M^{(1)}(A) = \mathrm{Ideal}_A(I(A)).$$

Proof. By what we have shown above, $J^{(k+1)}(A)/M^{(k)}(A) = I(A/M^{(k)}(A))$. If we take $k = 1$ then $J^{(2)}(A) \subset I(A)+M^{(1)}(A)$, actually $J^{(2)}(A) = I(A)+M^{(1)}(A)$ by Part (1) in Proposition 4.2.4. It follows that $M^{(2)}(A) = M^{(1)}(A)$ and then, of course, for any $k \geq 2$ we have $J^{(k)}(A) = I(A)+M^{(1)}(A)$, and $M^{(k-1)} = M^{(1)}$, proving the required.

□

To proceed further we use the notion and the properties of good and bad permutations from section 2.2.4. As before, we have

Proposition 4.2.5. *An H-algebra is a PI-algebra if and only if for some natural n we have $c_{n,H} < n!$ (actually, there exists $\tilde{N}$ such that for any $n > \tilde{N}$ we have the above inequality).*

Now the proposition proving the necessity of Condition (*):

Proposition 4.2.6.
1. *If $H \in \mathcal{P}$ then H satisfies Condition (*) and H is semisimple.*
2. *If H is not semisimple (but still finite-dimensional) then*
   (a) *If* char $F = 0$ *then there exists an H-algebra A with $A^H = \{0\}$ but A is not a PI-algebra.*
   (b) *If* char $F \neq 0$ *then there exists an H-algebra A with $(A^H)^2 = \{0\}$ but A is not a PI-algebra.*

For the proof the following technical result is of importance.

Proposition 4.2.7. *The following equations are true*

1. $J^{(n)}(\mathcal{H}(X)) = \bigoplus_{i=1}^{\infty} J^{(n)}(\mathcal{H}(X)) \bigcap \mathcal{H}_i$.
2. $(J^{(n)}(\mathcal{H}(X)))_i = \{\sum_\alpha f_\alpha X^\alpha | f_\alpha \in J_i^{(n)},\ |\alpha| = i\}$

Similar equations hold if J is replaced by M.

The next step is proving the implication Condition (*) $\Longrightarrow$ Condition (2) of the Main Theorem.

Proposition 4.2.8. *Let A be an H-algebra, H satisfying Condition (*), and A^H has a non-trivial identity of degree s. Fix any integers $n \geq m \geq h(s)$ where $h(s)$ is given by Condition (*). Then*

$$V_{n,H}/V_{n,H} \cap \mathcal{T}_H(A) = G + \mathcal{T}_H(A) \cap V_{n,H}/V_{n,H} \cap \mathcal{T}_H(A)$$

where G is the linear span of the monomials $t_\sigma X^\sigma$, σ an m-good permutation.

Then the following is true.

Corollary 4.2.1. *Let H be a finite-dimensional Hopf algebra, $\dim H = d$, satisfying Condition (*), A as in Proposition 4.2.8. Then for any $n \geq m \geq h(s)$ we have*

$$c_{n,H}(A) \leq \frac{d^n(m-1)^{2n}}{(m-1)!}.$$

Proof. This is immediate from the observation about the number of m-good permutations at the beginning of the section.

□

Theorem 4.2 ([BL]). *Let A be an H-algebra over a Hopf algebra H, $\dim H = d$, satisfying (*) with function $h(t)$. Suppose A^H satisfies a non-trivial polynomial identity of degree s. Then A satisfies a non-trivial identity of degree n where n is any integer satisfying the inequality*

$$\frac{d^n(h(t)-1)^{2n}}{(h(t)-1)!} < n!.$$

In particular, if n is the least integer with $ed(h(t)-1)^2 \leq n$ then A satisfies a non-trivial identity of degree n.

Proof. Very similar to the one in Part 2.2.

□

After this it remains to prove that Condition (*) is equivalent to the remaining Conditions in Main Theorem, but we skip this quite technical material.

Theorem 4.2 also leads to the following quantitative version of [BC, Theorem 7, part (4)].

Corollary 4.2.2. *Let H be an s-dimensional semisimple commutative Hopf-algebra over F and let R be an H-module algebra. Suppose that $s < \infty$ and that*

$$R^H = \{a \in R |\, h \cdot a = \epsilon(h)a, \text{ for all } h \in H\}$$

satisfies a polynomial identity of degree d. Then R satisfies a polynomial identity of degree n, where n is any integer satisfying the inequality

$$\frac{s^n(sd-1)^{2n}}{(sd-1)!} < n!$$

We complete this Part by formulating a very strong result of Vitaly Linchenko [LV] based on the techniques just described.

Theorem 4.3 ([LV]). *Let G be a finite subgroup in the automorphism group of a Lie algebra L over a field F where* char$F$ *is not a divisor of $|G|$ and let L^G satisfies a non-trivial identity. Then L satisfies a non-trivial identity.*

5. Theme Five: Bicharacters and Discoloration

As we learned earlier, given any abelian group G, and any "commutation factor" β on G, any (G, β)-Lie color algebra can be twisted into an ordinary Lie algebra, or an ordinary Lie superalgebra. In order to do this twisting, we showed the purely group-theoretic fact that there exists a 2-cocycle σ on G such that β is a "skew-symmetrization" of σ.

The object of this part is to extend the results of Part 3.1 to (H, β)-Lie algebras, where H is a cocommutative Hopf algebra and β is a skew-symmetric Hopf bicharacter on H. We obtain a complete answer when the base field k is algebraically closed of characteristic $p > 2$.

The results are divided into three parts. In subsection 5.1, we try to extend the known fact that, using β, the group G can be decomposed as $G = G_+ \cup G_-$. In Scheunert's work, such a decomposition of G determined whether the twisted object was an ordinary Lie algebra (the case when $G = G_+$) or an ordinary Lie superalgebra (the case when G_- is non-trivial). When H is cocommutative, we prove that such decomposition holds: one can define suitable H_+ and H_-

such that $H = H_+ \oplus H_-$. Some of our results here apply to more general Hopf algebras, called *weakly coquasitriangular.*

In subsection 5.2, we prove the analog of Scheunert's first theorem; that is, when H is cocommutative and β is a skew-symmetric bicharacter, then we can find a Hopf 2-cocycle σ on H so that β can be suitably factored using σ. Here we must assume, as mentioned above, that k has characteristic $p > 2$.

Finally in subsection 5.3, we consider (H, β)-Lie algebras and their twistings, and extend Scheunert's second theorem. We prove that if H is both commutative and cocommutative, with the same assumptions on k as above, and β has a suitable factorization involving some σ, then any such Lie algebra can be twisted to an ordinary Lie algebra or an ordinary Lie superalgebra.

We now fix some notation. Unless otherwise stated, k will denote throughout a field of characteristic not 2, and $\otimes := \otimes_k$. H will denote a Hopf algebra over k with comultiplication $\Delta : H \to H \otimes H$ and bijective antipode S. We use the Sweedler notation $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, although we usually omit the parentheses on subscripts, and when H is cocommutative, we sometimes omit the subscripts altogether. We also frequently omit the summation sign. R will denote a commutative k-algebra.

Given two maps $\alpha, \beta \colon H \to A$, where A is an algebra, we denote the *convolution product* of α and β by $\alpha * \beta \colon H \to A$; that is $(f * g)(h) = \sum f(h_1)g(h_2)$.

Given any function f, we will denote its convolution inverse by f^{-1}, and its composition inverse by $\overline{f}$.

If $g \in H$ satisfies $\Delta g = g \otimes g$ and $\varepsilon(g) = 1$, then g is called *grouplike*; the set of all such elements is denoted $G(H)$. An element $x \in H$ is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$, and we denote by $P(H)$ the k-space of all primitive elements. If $x \in H$ satisfies $\Delta x = x \otimes g + h \otimes x$ then x is called $(g, h) - primitive$.

The flip map is denoted by $\tau : H \otimes H \to H \otimes H$, that is, $\tau(h \otimes k) = k \otimes h$ for any $h, k \in H$.

Recall that a tensor product of Hopf algebras $H = K \otimes L$ is also a Hopf algebra, with $\Delta = \Delta_H := \Delta_K \otimes \Delta_L$.

Consider a convolution invertible map $\gamma \colon H \otimes H \to R$ for which $\gamma(1, h) =$

$\gamma(h,1)=\varepsilon(h)1$. If γ satisfies

$$\sum \gamma(x_1y_1,z)\gamma(x_2,y_2)=\sum \gamma(x,y_1z_1)\gamma(y_2,z_2) \tag{35}$$

then γ is a *right cocycle.*

If γ satisfies

$$\sum \gamma(x_1,y_1)\gamma(x_2y_2,z)=\sum \gamma(y_1,z_1)\gamma(x,y_2z_2) \tag{36}$$

then γ is a *left cocycle.*

Note that if γ is a left cocycle then γ^{-1} is a right cocycle. Moreover, in the case where H is cocommutative, conditions (35) and (36) are equivalent, and the set of all cocycles on H with convolution multiplication forms a group which we denote by $\mathcal{Z}^2(H,R)$ [Sw1].

5.1 Decomposition of H into Positive and Negative Parts

This section is motivated by Scheunert's work on $H=kG$ [Sch]: when G has a commutation factor (that is, a skew-symmetric bicharacter) $\beta\colon G\times G\to k^*$, then $G=G_+\cup G_-$, where

$$G_+=\{g\in G\mid \beta(g,g)=1\},\quad \text{and}\quad G_-=\{g\in G\mid \beta(g,g)=-1\}.$$

Here G_+ is always a subgroup of G with index ≤ 2.

Our goal is to generalize this decomposition to Hopf algebras and their comodules.

Definition 5.1.1. *Let H be a Hopf algebra.*

(a) A function $\beta\colon H\otimes H\to R$ is called a bicharacter on H *if β is bilinear and $\forall h,k,l\in H$,*

(i) $\beta(hk,l)=\sum\beta(h,l_1)\beta(k,l_2)$

(ii) $\beta(h,kl)=\sum\beta(h_2,k)\beta(h_1,l)$

(iii) β is normal, i.e. $\beta(h,1)=\beta(1,h)=\varepsilon(h),\quad \forall h\in H,$

(iv) β is convolution invertible.

(b) β is called skew-symmetric *if $\beta^{-1} = \beta \circ \tau$.*

We denote by $\mathcal{B}(H, R)$ the set of skew-symmetric bicharacters on H; this is a group under convolution multiplication whenever H is cocommutative.

When H is the enveloping algebra of a Lie algebra $\mathcal{L}$, then this definition of a skew-symmetric bicharacter generalizes the usual skew-symmetric bilinear form on Lie algebras. For that reason, and since many of our results specialize to Lie algebras, we have chosen this terminology rather than the usual Hopf algebra term "symmetric bicharacter".

For any bicharacter β it is easy to see that

$$\beta^{-1} = \beta \circ (id \otimes \overline{S}) = \beta \circ (S \otimes id), \text{ and} \tag{37}$$

$$\beta = \beta \circ (S \otimes S). \tag{38}$$

Definition 5.1.2. *Define*

$$u(h) := \sum \beta(h_2, Sh_1) \ \ \textit{for all} \ \ h \in H.$$

Examples 5.1.1. *(a) If $H = kG$ is a group algebra of a group then*

$$u(g) = \beta(g, g^{-1}) = \beta(g, g)^{-1} = \pm 1,$$

depending on g belonging to $G_{\pm}$. Here $g \in G$.

(b) If $H = u(G)$, the restricted enveloping algebra then for any $x \in L$ we have

$$u(x) = \beta(x, S1) + \beta(1, Sx) = \varepsilon(x) - \varepsilon(x) = 0.$$

(c) Consider any H and define $\beta(h, k) := \varepsilon(hk)$. Then β is a bicharacter.

Lemma 5.1.1. *Let H be a cocommutative Hopf algebra, and let β be a bicharacter on H. Then β is a left and a right 2-cocycle.*

Definition 5.1.3. If $f \in H^*$ satisfies $f * id = id * f$ we say that f is *cocentral.*

Definition 5.1.4. Suppose H has a bicharacter β and let V be a (right) H-comodule. Define $\Phi_V\colon V \to V$ by

$$\Phi_V(x) := \sum u(x_1)x_0,$$

and set

$$V_+ := \{x \in V \mid \Phi_V(x) = x\} \quad \text{and} \quad V_- := \{x \in V \mid \Phi_V(x) = -x\}.$$

In particular we may consider H itself as a right H-comodule via Δ; then Φ_H is given by $\Phi_H = \sum u(h_2)h_1$. Note that if $h \in H_+$, then $h = \sum u(h_2)h_1$. Applying ε to both sides we see that $u(h) = \varepsilon(h)$. Similarly if $h \in H_-$, then $u(h) = -\varepsilon(h)$. However, these conditions are not sufficient to determine when h is in H_+ or H_-. We will get such a condition below, in Corollary 5.1.1.

The situation in which $u * u = \varepsilon$ is particularly interesting; in the following lemma we see that it allows us to decompose H-comodules as a sum of positive and negative parts. In particular this happens when $u = \varepsilon$. In this case all H-comodules have only a positive part; this situation was studied in [CWZ] and [CW].

Recall that the *coefficient space* $C(V)$ of a (right) H-comodule V is the span in H of all $\{x_1\}$ such that $\rho(x) = \sum x_0 \otimes x_1$, for $x \in V$. For $x \in V$, the coefficient space $C(x)$ of x is the coefficient space of the smallest H-subcomodule of V containing x. $C(V)$ is always a subcoalgebra of H. See [L], [G].

Proposition 5.1.1. *Suppose H has a bicharacter β such that u is cocentral. Let V be any right H-comodule with Φ_V as above. Then*

(a) Φ_V is H-colinear, and V_+ and V_- are H-subcomodules.

(b) $V = V_+ \oplus V_-$ if and only if $\Phi_V^2 = id$.

(c) $x \in V_+$ (respectively V_-) if and only if $C(x) \subseteq H_+$ (resp. H_-).

(d) $V = V_+ \oplus V_-$ if and only if $u * u = \varepsilon$ on the coefficient space $C(V)$.

Remark 5.1.1. A similar version holds for left H-comodules W. That is, define $\Phi'\colon W \to W$ by $\Phi'(x) = \sum u(x_{-1})x_0$, and define W_+ and W_- as before. Then Φ' is H-colinear and W_+ and W_- are left H-subcomodules, assuming $u * id = id * u$.

Corollary 5.1.1. *Assume that H is a Hopf algebra with bicharacter β such that u is cocentral. Then*

(a) H_+ and H_- are subcoalgebras.

(b) Let $\langle h\rangle$ denote the subcoalgebra generated by $h \in H$. Then

$$h \in H_+ \quad \text{if and only if} \quad u(c) = \varepsilon(c)\ \forall c \in \langle h\rangle$$
$$h \in H_- \quad \text{if and only if} \quad u(c) = -\varepsilon(c)\ \forall c \in \langle h\rangle$$

(c) $H = H_+ \oplus H_-$ if and only if $u * u = \varepsilon$.

Proof. (a) By Proposition 5.1.1, considering H as a right H-comodule via Δ, H_+ and H_- are right coideals. Similarly, by Remark 5.1.1, H_+ and H_- are both left coideals. Thus both H_+ and H_- are subcoalgebras of H.
(b) ($\Rightarrow$) For $h \in H_+$, we know that $u(h) = \varepsilon(h)$. Now H_+ is a subcoalgebra by part (a), so the result follows.
($\Leftarrow$). Assume that $u(c) = \varepsilon(c)$ for all $c \in \langle h\rangle$. Then $\Phi(h) = \sum u(h_2)h_1 = \sum \varepsilon(h_2)h_1 = h$, so $h \in H_+$.

A similar argument works for H_-.
(c) This is just Proposition 5.1.1(d) with $V = H$.

□

We give examples of cases when the condition $u * u = \varepsilon$ holds.

Example 5.1.1. If H is cocommutative and β is skew-symmetric, then $u*u = \varepsilon$. For $u(h) = \beta(h_2, Sh_1) = \beta^{-1}(Sh_1, h_2) = \beta^{-1}(Sh_2, h_1) = v(h)$. The second equality follows from the skew-symmetry of β, the third from cocommutativity of H.

Notice that in this case $S^2 = id$ so u also satisfies:

$$u(h) = \sum \beta(h_1, h_2) \quad \text{and} \quad v(h) = \sum \beta^{-1}(h_2, h_1).$$

Lemma 5.1.2. *Let H be cocommutative. Then*

(a) $u : H \longrightarrow k$ is an algebra homomorphism.

(b) Φ is an algebra automorphism of H.

We can now give our result on the decomposition of H and its comodules. Recall that a coalgebra C is *irreducible* if the intersection of any two non-zero subcoalgebras is non-zero.

Theorem 5.1 ([BFM]). *Assume that H is cocommutative and $u * u = \varepsilon$. Then*

(a) $V = V_+ \oplus V_-$ for any right H-comodule V; in particular $H = H_+ \oplus H_-$.

(b) If H is irreducible as a coalgebra, then $H = H_+$.

(c) H_+ is a normal Hopf subalgebra of H, and $(H_-)^2 \subseteq H_+$.

However, when H is pointed cocommutative, more can be said. First we need some other known facts:

An *irreducible component* of any coalgebra C is a maximal irreducible subcoalgebra. A coalgebra C is *pointed* if any minimal subcoalgebra D is one-dimensional (equivalently, D is spanned by a group-like element). For each $g \in G(C)$, let C_g be the irreducible component containing g. When H is pointed cocommutative, it is known that any irreducible component of H must be H_g for some $g \in G(H)$, and that $H = \oplus_{g \in G} H_g$. Moreover, $H_g = H_1 g = g H_1$ (see [M3, 5.6.4] or [Sw2]), and so $H = H_1 \# kG$.

Theorem 5.2 ([BFM]). *Let H be pointed cocommutative with a skew-symmetric bicharacter β. Then the conclusions of Theorem 5.1 hold. In addition*

(a) $H_+ = \oplus_{g \in G_+} H_g = H_1 \# kG_+$ and $H_- = \oplus_{g \in G_-} H_g$.

(b) If $H_- \neq \{0\}$ then $[H : H_+] = 2$; that is, the quotient Hopf algebra $\overline{H} := H/H(H_+)^+$ has dimension 2.

(c) If $x \in H$ is a (g,g)-primitive element, then $x \in H_g$ and

$$x \in H_+ \iff g \in G_+ \quad \text{and} \quad x \in H_- \iff g \in G_-.$$

Remark 5.1.2. (a) Theorem 5.1(b) applies to any Hopf algebra generated by its primitive elements $P(H)$, as well as to any divided power algebra, as these are all irreducible Hopf algebras.
(b) Theorem 5.2(c) actually holds for a (g,g)-primitive element x in any H, as the Hopf subalgebra generated by $\{1,g,x\}$ is pointed cocommutative. More generally, if x is a (g,h)-primitive element, where also $h \in G(H)$, then one can show that $\Phi(x) = (\beta(x,h) + \beta(g,x))g + \beta(h,h)x$. An element x is called (g,h)-primitive if $\Delta x = x \otimes g + h \otimes x$.

5.2 Constructing the Cocycle

The main aim of this subsection is to investigate properties of bicharacters of cocommutative Hopf algebras, leading up to a generalization of Theorem 3.8 about the bicharacters of abelian groups. This theorem states, in particular, that for any bicharacter $\beta : G \times G \to k^*$ such that $G = G_+$, there is a 2-cocycle $\sigma : G \times G \to k^*$ such that for any $g,h \in G$ one can write β as a "skew-symmetrization of σ; that is,

$$\beta(g,h) = \sigma(h,g)\sigma^{-1}(g,h). \tag{39}$$

In the case of finitely generated abelian groups and algebraically closed fields of charcteristic 0 this result is due to [Sch]. For finite abelian groups over $\mathbb{C}$ this result was also independently established in [Mos]. The relation (39) also holds for arbitrary abelian groups, as was shown independently in [Po], [BM], [Pa1]; in [BM] k may be replaced by any commutative ring R.

An important application of this result was to give a correspondence between Lie coloralgebras and certain ordinary Lie superalgebras; in the next section we generalize this correspondence to (H,β)-Lie algebras for commutative and cocommutative H with a skew-symmetric bicharacter β.

In this section we will show that an analog of (39) is true for most cocommutative Hopf algebras.

Theorem 5.3 ([BFM, BKM]). *Let H be a pointed cocommutative Hopf algebra over a field k of characteristic not equal to 2. Let α be a bicharacter on H with values in a commutative algebra R. Then α can be written in the form*

$$\alpha(h,k) = \sigma(h,k) * \sigma^{-1}(k,h),$$

for some 2-cocycle σ if and only if $\alpha(g,g) = 1$ for any group-like element $g \in H$.

That is, given a skew-symmetric bicharacter β on H such that $H = H_+$, there exists a Hopf 2-cocycle σ such that $\beta = (\sigma \circ \tau) * \sigma^{-1}$; equivalently,

$$\beta(h,k) = \sum \sigma(k_1,h_1)\sigma^{-1}(h_2,k_2)\ \forall h,k \in H. \tag{40}$$

Conversely if H is cocommutative and β can be expressed in the form (40) then necessarily $H = H_+$. For then $\Phi(h) = \sum h_1\beta(h_2,h_3)$ (using Example 5.1.1(a)). Now apply (40) and cocommutativity to get $\Phi(h) = h$.

First, we see that, given a cocycle, we can construct a bicharacter.

Lemma 5.2.1. *Let H be a Hopf algebra with a bilinear map $\sigma\colon H \otimes H \to k$. Define $\beta\colon H \otimes H \to k$ via $\beta := (\sigma \circ \tau) * \sigma^{-1}$. Then*

(a) If H is commutative and σ is a left 2-cocycle, then β is a skew-symmetric bicharacter on H.

(b) If H is cocommutative and σ is a bicharacter, then so is β. Moreover $\beta^{-1} := \sigma * (\sigma^{-1} \circ \tau)$ is also a bicharacter.

We begin our "factorization of bicharacters by showing that when H is cocommutative, it suffices to also assume that H is commutative.

First, recall that for any bilinear form γ on a vector space V, the *kernel* of γ is defined to be $\mathrm{Ker}(\gamma) := \{v \in V \mid \gamma(v,V) = \gamma(V,v) = 0\}$.

Remark 5.2.1. Let β be a bicharacter on H. Using Definition 2.1(i) and (ii), it is easy to verify the following facts:
(a) If $a \in \mathrm{Ker}(\beta)$, then also $HaH \subseteq \mathrm{Ker}(\beta)$, where HaH is the ideal of H generated by a.
(b) Assume H is cocommutative. Then for any $a_1, \dots, a_n \in H$ and any permutation $\sigma \in \mathcal{S}_n$, $a_1 \cdots a_n - a_{\sigma(1)} \cdots a_{\sigma(n)} \in \mathrm{Ker}(\beta)$.

Lemma 5.2.2. *Let $\pi : H \longrightarrow \overline{H}$ be a surjective morphism of Hopf algebras.*
(a) If $\overline{\beta}$ is a bicharacter on $\overline{H}$, then $\overline{\beta}$ lifts to a bicharacter β of H. Conversely, if β is a skew-symmetric bicharacter on H such that $\mathrm{Ker}(\pi) \subseteq \mathrm{Ker}(\beta)$, then β induces a well-defined bicharacter $\overline{\beta}$ on $\overline{H}$. Moreover β is skew-symmetric if and only if $\overline{\beta}$ is skew-symmetric.
(b)If there exists a 2-cocycle $\overline{\sigma}$ on $\overline{H}$ such that $\overline{\beta} = (\overline{\sigma} \circ \tau) * \overline{\sigma}^{-1}$, then there exists a 2-cocycle σ on H such that the same equation holds for β and σ on H; in fact we may define $\sigma(x, y) = \overline{\sigma}(\pi(x), \pi(y))$.

Conversely, if β can be written in the form $\beta = (\sigma \circ \tau) * \sigma^{-1}$, for σ a 2-cocycle on H, then $\overline{\beta} = (\overline{\sigma} \circ \tau) * \overline{\sigma}^{-1}$.

Recall that for any H, and $h, l \in H$, the *commutator* of h and l is given by $(h, l) := \sum h_1 l_1 (Sh_2)(Sl_2)$. The *commutator subalgebra* H' of H is the k-span of all products of commutators. It is straightforward to check that when H is cocommutative, H' is actually a normal Hopf subalgebra of H. Clearly the Hopf quotient $H/H(H')^+$ is commutative.

Corollary 5.2.1. *Assume that, given a bicharacter β on H such that $H = H_+$, we may solve for σ as in (40) whenever H is both commutative and cocommutative. Then we may solve (40) for σ whenever H is only cocommutative.*

We next establish a connection between cocycles of a tensor product and those of its components.

Proposition 5.2.1. *Let $H = K \otimes L$ be as above, with K and L (and so H) commutative and cocommutative, and let β be a bicharacter on H. Consider the restrictions $\beta_K := \beta|_{K \times K}$ and $\beta_L := \beta|_{L \times L}$. If there exist cocycles σ on K*

and γ on L such that $\beta_K = (\sigma \circ \tau) * \sigma^{-1}$ and $\beta_L = (\gamma \circ \tau) * \gamma^{-1}$, then there exists a cocycle ρ on H such that

$$\beta = (\rho \circ \tau) * \rho^{-1}.$$

Proof. Let $x = a \otimes l$, $y = b \otimes m$, and $z = c \otimes s$, for $a, b, c \in K$, $l, m, s \in L$. We define

$$\rho(x, y) := \sum \sigma(a, b_1)\beta(b_2, l_1)\gamma(l_2, m).$$

The verification of the cocycle condition and the other properties required is of technical nature, so we omit it.

□

5.2.1 r-characters on connected Hopf algebras

Let k be a field, H a cocommutative Hopf algebra over k, R a commutative algebra over k.

Definition 5.2.1. *An r-multilinear function $\alpha : H \times \ldots \times H \to R$ is called an r-character if it is convolution invertible and, for any $i = 1, \ldots, r$, the following two conditions are satisfied:*

$$\alpha(h^1, \ldots, h^{i-1}, 1, h^{i+1}, \ldots, h^r) = \varepsilon(h^1 \ldots h^{i-1}h^{i+1} \ldots h^r), \tag{41}$$

$$\begin{aligned} & \alpha(h^1, \ldots, h^{i-1}, lm, h^{i+1}, \ldots, h^r) \\ = & \sum \alpha(h^1_1, \ldots, h^{i-1}_1, l, h^{i+1}_1, \ldots, h^r_1)\alpha(h^1_2, \ldots, h^{i-1}_2, m, h^{i+1}_2, \ldots, h^r_2), \end{aligned} \tag{42}$$

for all $h^1, \ldots, h^{i-1}, l, m, h^{i+1}, \ldots, h^r \in H$.

It is easy to verify that all r-characters form a group under the convolution product (since H is cocommutative). We will denote this group by $\mathrm{Ch}^r(H, R)$.

Let S_r be the symmetric group on r elements. Then for any $\pi \in S_r$ we can consider an r-character

$$(\pi \circ \alpha)(h^1, \ldots, h^r) = \alpha(h^{\pi(1)}, \ldots, h^{\pi(r)}),$$

for any $h^1, \ldots, h^r \in H$.

Definition 5.2.2. *An r-character α is called* symmetric *if, for any $\pi \in S_r$, we have $\pi \circ \alpha = \alpha$. All symmetric r-characters form a group, which will be denoted by* $\mathrm{Sym}^r(H, R)$.

Definition 5.2.3. *An r-character α is called* skew-symmetric *if for any $\pi \in S_r$, we have $\pi \circ \alpha = \alpha^{\mathrm{sgn}\,\pi}$. All skew-symmetric r-characters form a group, which we denote* $\mathrm{Alt}^r(H, R)$.

In the case of $r = 1$ we simply obtain the algebra maps from H to R, which we will simply call characters. In the case of $r = 2$ our definitions agree with those of bicharacters and skew-symmetric bicharacters given in section 5.1.

Now we will explore the structure of groups of r-characters with the special interest in bicharacters. At first we assume that H is connected. Our aim is to prove the following result.

Theorem 5.4 ([BKM]). *Let H be a cocommutative connected Hopf algebra over k, m be a positive integer such that $\mathrm{char}\, k \nmid m$. Let α be an r-character on H with values in a commutative algebra R over k. Then there exists a unique r-character β such that $\alpha = \beta^m$ (under the convolution product). Moreover, if α is symmetric (or skew-symmetric), then so is β.*

We first consider a slightly more general situation. Suppose we have a multilinear map of r variables $f : H \times \ldots \times H \to R$, which is normalized with respect to the i-th variable in the sense that

$$f(h^1, \ldots, h^{i-1}, 1, h^{i+1}, \ldots, h^r) = \varepsilon(h^1 \ldots h^{i-1} h^{i+1} \ldots h^r),$$

for any $h^1, \ldots, h^{i-1}, h^{i+1}, \ldots, h^r \in H$.

We define a multilinear map by setting

$$f_0(h^1, \ldots, h^r) = f(h^1, \ldots, h^r) - \varepsilon(h^1 \ldots h^r), \ \ \forall h^1, \ldots, h^r \in H. \qquad (43)$$

In the following important proposition H is not necessarily cocommutative (but it is still assumed connected). The proof is similar to an argument of Takeuchi [T].

Proposition 5.2.2. *Let f_0 be defined as in (43). Then f_0 is locally nilpotent.*

After this it is sufficient to prove the following.

Proposition 5.2.3. *For any multilinear map $f : H \times \ldots \times H \to R$, normalized with respect to the i-th variable (or, more generally, such that $f - \varepsilon$ is locally nilpotent), there exists a unique multilinear map $g : H \times \ldots \times H \to R$, normalized with respect to the i-th variable, such that $f = g^m$ under the convolution product, provided* $\operatorname{char} k \nmid m$.

Proof. Since $\operatorname{char} k \nmid m$, we can consider the formal power series

$$A(t) = (1+t)^{\frac{1}{m}} = 1 + \frac{1}{m}t + \ldots$$

Set $g = A(f_0)$. Obviously, g is normalized with respect to the i-th variable, and $f = g^m$ by construction. We omit the proof of the uniqueness of such g.

□

Let us define two operators:

$$sym : \operatorname{Ch}^r(H,R) \to \operatorname{Sym}^r(H,R) : \alpha \mapsto \prod_{\pi \in S_r} (\pi \circ \alpha),$$

and

$$alt : \operatorname{Ch}^r(H,R) \to \operatorname{Alt}^r(H,R) : \alpha \mapsto \prod_{\pi \in S_r} (\pi \circ \alpha)^{\operatorname{sgn} \pi},$$

for any cocommutative Hopf algebra H.

Corollary 5.2.2. *If H is connected and $r < \operatorname{char} k$ or $\operatorname{char} k = 0$, then sym and alt are projections of $\operatorname{Ch}^r(H,R)$ on $\operatorname{Sym}^r(H,R)$ and $\operatorname{Alt}^r(H,R)$, respectively.*

In particular, we obtain the following corollary.

Corollary 5.2.3. *Let H be a connected cocommutative Hopf algebra over a field k of characteristic not equal to 2. Let α be a skew-symmetric bicharacter on H with values in a commutative algebra R. Then there exists a (unique) skew-symmetric bicharacter β such that*

$$\alpha(h,k) = \beta^2(h,k) = \beta(h,k) * \beta^{-1}(k,h).$$

We next allow the possibility that $G = G(H)$ is not trivial and also $G \neq G_+$, or equivalently when $H \neq H_+$. By Theorem 5.2 we know that $H = H_+ \oplus H_-$. Thus we may define the *sign bicharacter* β_0 on H as follows: for any homogeneous $h, k \in H$

$$\beta_0(h,k) = \begin{cases} -\varepsilon(h)\varepsilon(k) & \text{if } h,k \in H_- \\ \varepsilon(h)\varepsilon(k) & \text{otherwise} \end{cases} \tag{44}$$

and extend β_0 linearly to H. Note that $\beta_0^{-1} = \beta_0$. Since both H_+ and H_- are subcomodules of H by Proposition 5.1.1, one can easily verify that β_0 is a bicharacter on H.

We can now obtain our best result on factoring bicharacters with cocycles; it extends [Sch, Lemma 2].

Theorem 5.5 ([BFM, BKM]). *Let H be cocommutative and k be algebraically closed. Assume* char $k = p > 2$. *Let β be a skew-symmetric bicharacter on H. Then:*

(a) If $H = H_+$ then there exists a cocycle σ such that $\beta = (\sigma \circ \tau) * \sigma^{-1}$.

(b) If $H_- \neq 0$ then there exists a cocycle σ such that $\beta = \beta_0 * (\sigma \circ \tau) * \sigma^{-1}$, where β_0 is the sign bicharacter.

Proof. Since k is algebraically closed, H is pointed, and so $H \cong H_1 \# kG$, where H_1 is the irreducible component of 1 (as noted before Theorem 5.2).

Thus (a) follows from Corollary 5.2.3 and Proposition 5.2.1.

For (b), we apply (a) to the bicharacter $\beta * \beta_0$, which satisfies $H = H_+$. The result now follows since $\beta_0^{-1} = \beta_0$.

□

Remark 5.2.2. Recently Etingof and Gelaki have shown, in the case of characteristic zero, that if (H, β) is cotriangular and $tr(S^2|_C) = \dim C$ for each finite-dimensional subcoalgebra C of H, then there exists a cocycle σ such that $\beta = \beta_c * (\sigma \circ \tau) * \sigma^{-1}$, where β_c is a certain bicharacter with $\beta_c^2 = \varepsilon$. Their methods are similar to those in [EG].

5.2.2 Groups of r-characters in particular cases

The description of polycharacters on two types of connected Hopf algebras is particularly transparent. We do not give proofs but just formulate the final result and all necessary preliminary results, whose proofs are not very hard.

We will later need the following general results about the r-characters, which are straightforward generalizations of the bicharacter versions given in [BFM].

Proposition 5.2.4. *Let H be a cocommutative Hopf algebra, R a commutative algebra over k. Let $\alpha \in \mathrm{Ch}^r(H,R)$. Then α vanishes on the ideal $I \triangleleft H$, generated by commutators $kl - lk$, for all $k, l \in H$, i.e., $\alpha(H,\dots,I,\dots,H) = 0$, for any position of I among the arguments of α.*

Proposition 5.2.5. *Let $I \triangleleft H$ be any Hopf ideal of H, $\bar{H} = H/I$. Then the r-characters $\bar{\alpha}$ of $\bar{H}$ are in one-to-one correspondence with the r-characters α of H such that $\alpha(H,\dots,I,\dots,H) = 0$, for any position of I. This correspondence is given by*

$$\alpha(h^1,\dots,h^r) = \bar{\alpha}(h^1 + I,\dots,h^r + I),$$

for any $h^1,\dots,h^r \in H$.

The proofs are straightforward.

These two propositions allow us to reduce the study of r-characters of any cocommutative Hopf algebra to those of a Hopf algebra which is both commutative and cocommutative.

At first we consider the polynomial algebra $H = k[X]$, where X is any set of variables (not necessarily finite), and $\Delta x = x \otimes 1 + 1 \otimes x$, for any $x \in X$.

Proposition 5.2.6. *Let $w^1 = x_1^1 \dots x_{m^1}^1, \dots, w^r = x_1^r \dots x_{m^r}^r$ be monomials, α an r-character on $k[X]$. Then $\alpha(w^1,\dots,w^r) = 0$ unless $m^1 = \dots = m^r = m$, and in the latter case we have*

$$\begin{aligned}&\alpha(w^1,\dots,w^r) \\ = &\sum_{\pi^2,\dots,\pi^r \in S_m} \alpha(x_1^1, x_{\pi^2(1)}^2,\dots,x_{\pi^r(1)}^r)\cdot\ldots\cdot\alpha(x_m^1, x_{\pi^2(m)}^2,\dots,x_{\pi^r(m)}^r).\end{aligned} \tag{45}$$

In particular, α is completely determined by its values on X.

Proof. Induction on $m^1, \ldots, m^r$ using the definition of the r-character.

□

Proposition 5.2.7. *Let α be an r-character of $k[X]$,* $\operatorname{char} k = p > 0$, $r \geq 2$. *Then α vanishes on the ideal generated by x^p, for all $x \in X$.*

Proof. Let $w^1 = x^p$, $w^2, \ldots, w^r$ any other monomials. Then by Proposition 5.2.6 $\alpha(w^1, \ldots, w^r) = 0$, unless all $w^2, \ldots, w^r$ have degree p. In the latter case by (45) we obtain

$$\begin{aligned}
& \alpha(w^1, \ldots, w^r) \\
= & \sum_{\pi^2, \ldots, \pi^r \in S_p} \alpha(x, x^2_{\pi^2(1)}, \ldots, x^r_{\pi^r(1)}) \cdot \ldots \cdot \alpha(x, x^2_{\pi^2(p)}, \ldots, x^r_{\pi^r(p)}) \\
= & \; p! \sum_{\pi^3, \ldots, \pi^r \in S_p} \alpha(x, x^2_1, x^3_{\pi^3(1)} \ldots, x^r_{\pi^r(1)}) \cdot \ldots \cdot \alpha(x, x^2_p, x^3_{\pi^3(p)}, \ldots, x^r_{\pi^r(p)}) \\
= & \; 0.
\end{aligned}$$

The proposition follows.

□

Now let L be a Lie algebra over k, $H = U(L)$ the universal enveloping algebra of L. By Propositions 5.2.4 and 5.2.5, the r-characters of H are in one-to-one correspondence with the r-characters of $\bar{H} = U(L/[L, L]) \cong k[X]$, where X is any basis of $L/[L, L]$. Let α be any r-character of H, $\bar{\alpha}$ the corresponding r-character of $\bar{H}$ and $\mathcal{A}$ the restriction of $\bar{\alpha}$ on the subspace $L/[L, L]$. By Proposition 5.2.6, $\bar{\alpha}$ is completely determined by $\mathcal{A}$. On the other hand, it can be easily verified that (45) defines an r-character on $k[X]$ for any given values on the elements of X. It follows, that the correspondence $\alpha \mapsto \mathcal{A}$ is one-to-one, and it is actually an isomorphism of the Abelian groups $\mathrm{Ch}^r(U(L), R)$ and $\mathrm{Hom}((L/[L, L])^{\otimes r}, R)$, since $\overline{\alpha * \beta}(x^1, \ldots, x^r) = (\bar{\alpha} * \bar{\beta})(x^1, \ldots, x^r) = \bar{\alpha}(x^1, \ldots, x^r) + \bar{\beta}(x^1, \ldots, x^r)$, for any $x^1, \ldots, x^r \in X$. Obviously, α is (skew-)symmetric if and only if $\mathcal{A}$ is (skew-)symmetric. So we have proved the following.

Theorem 5.6 ([BKM]). *For any Lie algebra L, commutative algebra R and $r \geq 1$*

$$\begin{aligned} \mathrm{Ch}^r(U(L),R) &\cong \mathrm{Hom}(T^r(L/[L,L]),R), \\ \mathrm{Sym}^r(U(L),R) &\cong \mathrm{Hom}(S^r(L/[L,L]),R), \\ \mathrm{Alt}^r(U(L),R) &\cong \mathrm{Hom}(\Lambda^r(L/[L,L]),R), \end{aligned}$$

as Abelian groups.

Here $T^r(V)$, $S^r(V)$ or $\Lambda^r(V)$ stand for the tensor, symmetric or skew-symmetric power of a vector space V, respectively.

Let L be a restricted Lie algebra over k, $\operatorname{char} k = p > 0$, and $H = u(L)$ the restricted enveloping algebra of L. We want to describe the r-characters of H, $r \geq 2$. By Propositions 5.2.4 and 5.2.5 we may assume L Abelian. Fix a basis X in L. Then $u(L) \cong k[X]/Ideal(x^p - x^{[p]} \mid \forall x \in X)$. By Proposition 5.2.5, the r-characters of $u(L)$ are in one-to-one correspondence with the r-characters of $k[X]$ that vanish on $Ideal(x^p - x^{[p]} \mid \forall x \in X)$. But by Proposition 5.2.7, any r-character of $k[X]$ vanishes on $Ideal(x^p \mid \forall x \in X)$, so the r-characters of $u(L)$ are in one-to-one correspondence with the r-characters of $k[X]$ that vanish on $Ideal(x^{[p]} \mid \forall x \in X)$, and the latter are in one-to-one correspondence with the r-characters of $k[X]/Ideal(x^{[p]} \mid \forall x \in X) \cong u(L/L^{[p]})$. So we have proved the following.

Theorem 5.7 ([BKM]). *For any restricted Lie algebra L, commutative algebra R and $r \geq 2$*

$$\begin{aligned} \mathrm{Ch}^r(u(L),R) &\cong \mathrm{Hom}(T^r(L/([L,L]+L^{[p]})),R) \\ \mathrm{Sym}^r(u(L),R) &\cong \mathrm{Hom}(S^r(L/([L,L]+L^{[p]})),R) \\ \mathrm{Alt}^r(u(L),R) &\cong \mathrm{Hom}(\Lambda^r(L/([L,L]+L^{[p]})),R) \end{aligned}$$

as Abelian groups.

5.3 Bicharacters and Generalized Lie Structures

In this section we apply our previous results to give a bijection between certain types of generalized Lie algebras and classical Lie algebras or Lie superalgebras, under appropriate conditions. In order to do this, we will "twist" the Lie algebra structure using cocycles and bicharacters.

Throughout this section, H will be a cocommutative Hopf algebra with a skew-symmetric bicharacter $\beta\colon H \otimes H \to R$. In this situation, given right H-comodules X, Y, we may define a "twist" map $\theta\colon X \otimes Y \to Y \otimes X$:

$$\theta(x \otimes y) = \sum \beta(x_1, y_1) y_0 \otimes x_0 \quad \forall x \in X, y \in Y.$$

In fact, since β is skew-symmetric we have $\theta^2 = \mathrm{id}$, and we may use this map to form a generalized Lie algebra:

Definition 5.3.1. A (right) (H, β)-*Lie algebra* is a (right) H-comodule $\mathcal{L}$ together with a β-*Lie bracket* $[\ ,\ ]\colon \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ which is an H-comodule morphism satisfying, for all $a, b, c \in \mathcal{L}$:

(a) anticommutativity: $[\ ,\ ] = -[\ ,\ ] \circ \theta$, that is,

$$[a, b] = -\beta(a_1, b_1)[b_0, a_0].$$

(b) Jacobi identity:

$$[\ ,\ ] \circ ([\ ,\ ] \otimes id) + [\ ,\ ] \circ ([\ ,\ ] \otimes id) \circ \theta_{12,3} + [\ ,\ ] \circ ([\ ,\ ] \otimes id) \circ \theta_{1,23} = 0, \quad \text{i.e.}$$

$$[[a, b], c] + \beta(a_1 b_1, c_1)[[c_0, a_0], b_0] + \beta(a_1, b_1 c_1)[[b_0, c_0], a_0] = 0$$

A more symmetric form of this identity is

$$\beta(c_1, a_1)[[a_0, b], c_0] + \beta(b_1, c_1)[[c_0, a], b_0] + \beta(a_1, b_1)[[b_0, c], a_0] = 0.$$

(H, β)-Lie algebras are a special case of Lie algebras in symmetric monoidal categories, as described in [Gu]. Some basic properties of (H, β)-Lie algebras are considered in [FM].

Examples 5.3.1. (a) Let A be a right H-comodule algebra, and define $[\ ,\ ]_\beta$ to be

$$[a,b]_\beta := ab - \sum \beta(a_1, b_1) b_0 a_0.$$

Then $[\ ,\ ]_\beta$ is a β-Lie bracket.

Denote the β-Lie algebra created in this manner by $[A]_\beta$.

(b) Let $\beta = \varepsilon \otimes \varepsilon$, the trivial bicharacter. Then an $(H, \varepsilon \otimes \varepsilon)$-Lie algebra is an ordinary Lie algebra $\mathcal{L}$ which is an H-comodule and such that $[\ ,\ ]$ is an H-comodule morphism.

(c) Similarly, let $\beta = \beta_0$ be the sign bicharacter. Then an (H, β_0)-Lie algebra is an ordinary Lie superalgebra $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$, such that $\mathcal{L}$ is an H-comodule and $[\ ,\ ]$ is an H-comodule morphism. It is easy to see that $\mathcal{L}_{\bar{0}} = \mathcal{L}_+$ and $\mathcal{L}_{\bar{1}} = \mathcal{L}_-$ and so these subspaces are also H-subcomodules.

We remark that if char$k = 2$ then condition 5.3.1(a) is not appropriate. For an ordinary Lie algebra $\mathcal{L}$ in characteristic 2, 5.3.1(a) is replaced by the condition that $[a,a] = 0$ for all $a \in \mathcal{L}$.

In the presence of a cocycle in addition to the bicharacter, we may twist the operations of H and its comodules. The following collects definitions and results from [KS, S10.2.3] :

Definition 5.3.2. *Let H be a Hopf algebra with a skew-symmetric bicharacter $\beta\colon H \otimes H \to R$, and suppose $\sigma\colon H \otimes H \to k$ is a left cocycle.*

(a) Define H_σ to be H as a coalgebra, with multiplication defined to be

$$h \cdot_\sigma k := \sigma(h_1, k_1) h_2 g_2 \sigma^{-1}(h_3, k_3)$$

Then H_σ (with a suitable antipode) is a Hopf algebra.

(b) Define the map $\beta_\sigma\colon H_\sigma \otimes H_\sigma \to R$ by $\beta_\sigma := (\sigma \circ \tau) * \beta * \sigma^{-1}$; that is, for all $k, h \in H$,

$$\beta_\sigma(h,k) := \sigma(k_1, h_1) \beta(h_2, k_2) \sigma^{-1}(h_3, k_3).$$

If (H, β) is CQT then (H_σ, β_σ) is also CQT (so in particular β is a bicharacter on H).

c) If A is a right H-comodule algebra, define A^σ to be A as a vector space and H_σ-comodule, with multiplication given by:

$$a \cdot^\sigma b := \sigma(a_1, b_1)a_0 b_0 \quad \forall a, b \in A$$

Then A^σ is an H_σ-comodule algebra.

For the rest of this section, we assume that H is cocommutative; in this case σ is a two-sided cocycle and $H = H_\sigma$ as a Hopf algebra.

When H is cocommutative and β is a bicharacter on H, then H is commutative if and only if (H, β) is CQT. Thus the results in 5.3.2 apply. However in this case, a direct computation also shows that β_σ is a bicharacter on H.

Analogous to the definition of A^σ, we define a cocycle twist $\mathcal{L}^\sigma$ of an (H, β)-Lie algebra $\mathcal{L}$:

Definition 5.3.3. *Define $\mathcal{L}^\sigma$ to be $\mathcal{L}$ as a right H-comodule, with the map $[\ ,\ ]^\sigma : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ given by*

$$[a, b]^\sigma := \sum \sigma(a_1, b_1)[a_0, b_0] \quad \forall a, b \in \mathcal{L}.$$

Our goal in this section is to find a cocycle σ on H so that if $\mathcal{L}$ is an (H, β)-Lie algebra, then the new Lie algebra $\mathcal{L}^\sigma$ will be either a classical Lie algebra or a Lie superalgebra. The next result extends [Sch, Prop 3].

Proposition 5.3.1. *Let H be a commutative and cocommutative Hopf algebra with a bicharacter β, let $\mathcal{L}$ be an (H, β)-Lie algebra and assume σ is a cocycle on H. Then $\mathcal{L}^\sigma$ is an $(H, \beta_{\sigma^{-1}})$-Lie algebra.*

In the case of Lie algebras that come from associative algebras, all these concepts mesh nicely, and we have the following analog of [BM, 2.4](see Part 2.3):

Lemma 5.3.1. *Let H be a commutative and cocommutative Hopf algebra with a skew-symmetric bicharacter β and a cocycle σ. Then for any right H-comodule algebra A,*

$$[A^\sigma]_{\beta_{\sigma^{-1}}} \cong ([A]_\beta)^\sigma .$$

Let V be a finite-dimensional (right) H-comodule, and $\mathcal{L}$ an (H,β)- Lie algebra. We say that V is an *(H,β)-representation* of $\mathcal{L}$ if there is a morphism of (H,β)-Lie algebras

$$\Psi : \mathcal{L} \to [\mathrm{End(V)}]_\beta.$$

Since V is finite-dimensional, $\mathrm{End}(V)$ becomes an H-comodule algebra [FM, Lemma 2.10] and so can be made into an (H,β)-Lie algebra as in 5.3.1(a).

If V is not finite-dimensional, then we do not know at this point how to define a representation of $\mathcal{L}$ on V; one needs to have a suitable subset of $\mathrm{End}(V)$ which is an H-comodule algebra. This can always be done when $H = kG$, as in [Sch].

We can now extend [Sch, Prop. 4]

Proposition 5.3.2. *Assume that H is commutative and cocommmutative with a bicharacter β and cocycle σ. Suppose that $\mathcal{L}$ is an (H,β)-Lie algebra and let V be a finite-dimensional H-comodule. If $\Psi : \mathcal{L} \to [\mathrm{End(V)}]_\beta$ is an (H,β)-representation of $\mathcal{L}$, then*

$$\Psi^\sigma : \mathcal{L}^\sigma \to [\mathrm{End(V)}]_{\beta_{\sigma^{-1}}}$$

defined by

$$\Psi^\sigma(a)(x) := \sum \sigma(a_1, x_1)\Psi(a_0)(x_0),\ \forall a \in \mathcal{L}, x \in V$$

is an $(H,\beta_{\sigma^{-1}})$-representation of $\mathcal{L}^\sigma$.

We now formulate the main theorem of this Part; it extends Scheunert's theorem [Sch, Th.2].

Theorem 5.8 ([BFM, BKM]). *Let H be a commutative and cocommutative Hopf algebra over k with skew-symmetric bicharacter β. Assume that k is algebraically closed such that either k has characteristic 0, or* $\mathrm{char} k = p > 2$. *Then:*

(a) *If $H = H_+$ then there exists a cocycle σ on H such that $\mathcal{L} \mapsto \mathcal{L}^\sigma$ is a bijection between the set of (H,β)-Lie algebras and the set of ordinary Lie algebras which are $(H,\varepsilon\otimes\varepsilon)$-Lie algebras as in 5.3.1(b).*

(b) *If $H_- \neq 0$ then there exists a cocycle σ on H such that $\mathcal{L} \to \mathcal{L}^\sigma$ is a bijection between the set of (H,β)-Lie algebras and the set of ordinary Lie superalgebras which are (H,β_0)-Lie algebras as in 5.3.1 (c).*

Furthermore, in each of the cases (a) or (b), the transformation $\Psi \to \Psi^\sigma$ is a bijection between the finite-dimensional (H,β)-representations of $\mathcal{L}$ and the finite-dimensional H-comodule representations of the Lie algebra (respectively, Lie superalgebra) $\mathcal{L}^\sigma$.

Example 5.3.1. In any characteristic $p > 2$: let $H = k[x_1, x_2, \ldots, x_n | x_i^p = 0, \forall i]$ with all x_i primitive as before, and let $A = k[t_1, t_2, \ldots, t_n]$ be the polynomial ring. A becomes an H-comodule algebra by defining $\rho(t_i) = t_i \otimes 1 + 1 \otimes x_i$. Extending multiplicatively to all of A, we see that for $f = f(t_1, \ldots, t_n) \in A$,

$$\rho(f) = \sum_{k_1,\ldots,k_n=0}^{p-1} f^{(k_1,\ldots,k_n)} \otimes x_1^{k_1} \ldots x_n^{k_n}.$$

where

$$f^{(k_1,\ldots,k_n)} = \frac{1}{k_1! \ldots k_n!} \frac{\partial^{k_1+\cdots+k_n} f}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}}.$$

Any bicharacter β on H is determined by its restriction to the space $V = \operatorname{span}_k\{x_1, \ldots, x_n\}$, and the restriction is a skew-symmetric bilinear form. Thus β is completely determined by

$$\beta(x_i, x_j) = \lambda_{ij} \in k \text{ for all } i < j.$$

For any such β, we may form the (H,β)-Lie algebra $[A]_\beta$ by setting

$$[f, g] = \sum_{\substack{k_1,\ldots,k_n=0 \\ l_1,\ldots,l_n=0}}^{p-1} f^{(k_1,\ldots,k_n)} g^{(l_1,\ldots,l_n)} \beta(x_1^{k_1} \ldots x_n^{k_n}, x_1^{l_1} \ldots x_n^{l_n}).$$

This is a rather nontrivial bracket but by Theorem 5.8, $[A]_\beta$ can be twisted to an ordinary Lie algebra, since $H = H_+$.

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