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HARMONIC MAPS INTO LIE GROUPS

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Introduction

In these lectures we describe Uhlenbeck's results on harmonic maps $S^2 \to U(N)$. In particular, we give two different proofs of Uhlenbeck's factorization theorem for such maps as finite products of holomorphic objects, called *unitons* (essentially holomorphic subbundles).

Both in the proofs, and in side arguments, we use some elementary algebraic geometry. Moreover, we use the basic fact that it is possible to associate an holomorphic map $S^2 \to \Omega U(N)$ to any harmonic map $S^2 \to U(N)$ (here $\Omega U(N)$ is the based "loop group" of $U(N)$: it is an infinite dimensional Kähler manifold). This comes from a construction which dates back to the theory of solitons and Lax pairs (cf. [Z-M]).

The main references we have used are Uhlenbeck's paper [U], §1, 2, 8, 9 (1st part), 11, 12, 13, 14; then [V1] and [V2] §4,5. Segal's approach ([S]) may be illuminating also, but it's completely different (cf. M. Guest's minicourse in this workshop). Most of this research dates back to 1984-87.

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1. Harmonic maps $M^2 \to U(N)$, and extended solutions

Let (M, g) and (N, h) be two (compact) Riemannian manifolds. For any smooth map $f : M \to N$, the *energy* of f is given by

$$E(f) = 1/2 \int_M |\,\mathrm{d}f|^2 \nu_g \tag{1.1}$$

where ν_g is the volume element on M, and $|\,\mathrm{d}f|^2$ is computed using the metrics g and h.

A smooth map $f : M \to N$ is called *harmonic* if it is a critical point of the energy functional, with respect to compactly supported variations. There is a vast literature concerning harmonic maps (cf. [E-L 1], [E-L 2]). The main problems generally are:

(1) Existence (and regularity) of harmonic maps in homotopy classes of functions.

(2) Explicit description or construction of wide classes of harmonic maps.

Examples.

1) Harmonic maps $S^1 \to N$ are the closed geodesics, parametrized by arc-length.

2) Harmonic maps $M \to \mathbb{R}$ are functions $M \to \mathbb{R}$ which are in the kernel of the Laplace-Beltrani operator on M.

3) If (M, g) and (N, h) are Kähler manifolds, then any holomorphic (or antiholomorphic) map $M \to N$ is harmonic. Moreover, if M is compact, any $\pm$ holomorphic map is an absolute minimum of the energy functional in its homotopy class.

4) $f : \mathbf{C} \to \mathbb{R}$ is harmonic $\Leftrightarrow \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$.

The following (very well known!) theorem is a model for more complicated situations (like the ones in the following sections).

Proposition. *Let $\Omega \subset \mathbf{C}$ be a simply connected domain. Then $f : \Omega \to \mathbb{R}$ is harmonic $\Leftrightarrow \exists\, g : \Omega \to \mathbf{C}$ holomorphic, $Re(g) = f$*

Note: ($\pi_1(\Omega) = (0)$ is only used to prove $\Rightarrow$).

In the following we will take M to be a Riemann surface, mainly S^2; and $N = U(N)$ be the unitary group in N-dimensions. The Lie group $U(N)$ has a left-invariant Riemannian metric, which is the left translation of the inner

product

$$\langle A, B\rangle = -Tr(AB) \tag{1.2}$$

on the Lie algebra $U(N)$ of skew-hermitian $N \times N$ matrices.

On a oriented surface $M = M^2$, the choice of a conformal structure (i.e. the choice of a metric g modulo conformal changes $g \mapsto e^{\varphi}g$, for $\varphi : M \to \mathbb{R}$ smooth) is equivalent to the choice of a complex structure. A Riemann surface is, by definition, a surface M^2, together with the choice of a conformal structure; i.e. M^2 is a complex manifold of complex dimension 1.

The Hodge $*$ (star)-operator maps 1-forms into 1 forms, and is conformally invariant; if ω is a 1-form on M^2, and if

$$\omega = \omega^{1,0} + \omega^{0,1}$$

is the decomposition into components of (1,0), and (0,1) type, then we have

$$*\omega = i\ \omega^{1,0} - i\ \omega^{0,1}$$

and

$$|\omega|^2 = -\int_M \omega \wedge (*\omega)\ . \tag{1.3}$$

Therefore the Riemannian norm on the space of 1-forms is conformally invariant as well. Let $f : M^2 \to U(N)$ be a smooth map and M^2 a Riemann surface. Following Uhlenbeck's notation in [U], we denote:

$$A = \frac{1}{2}(f^{-1}\,\mathrm{d}f) = A_z + A_{\bar{z}} \tag{1.4}$$

where A_z, $A_{\bar{z}}$ are the (1,0) and (0,1) parts of A; $A_{\bar{z}} = (-A_z)^*$.

From the above remarks, it follows that the energy of f is:

$$\begin{aligned} E(f) = \frac{1}{2}\int_M |f^{-1}\,\mathrm{d}f|^2 &= +\frac{1}{2}\int_M Tr(f^{-1}\,\mathrm{d}f \wedge *(f^{-1}\,\mathrm{d}f)) \\ &= -4i\int_M Tr(A_z \wedge A_{\bar{z}}). \end{aligned} \tag{1.5}$$

Therefore the energy of a smooth map $f : M^2 \to U(N)$ is invariant under conformal changes of metrics on $M = M^2$.

Exercise. This is true for general target manifolds, the key point being $\dim(M) = 2$.

We note that A is a 1-form on M^2, with matrix coefficients. We can see A as a connection on the trivial bundle $M^2 \times \mathbf{C}^N$; A is actually a unitary connection. Since the curvature $F(2A)$ of the connection $2A$ (on $M^2 \times \mathbf{C}^N$) is 0, because $2A = f^{-1}\,\mathrm{d}f$, the Maurer-Cartan equation

$$F(2A) = \mathrm{d}(2A) + \frac{1}{2}[2A, 2A] = 0$$

implies

$$\overline{\partial} A_z + \partial A_{\overline{z}} + 2[A_z,\, A_{\overline{z}}] = 0 \tag{1.6}$$

where we have split the exterior differential $\mathrm{d} = \partial + \overline{\partial}$, with standard notation.

Let $f : M^2 \to U(N)$ be a smooth map and let $f_t : M^2 \times (-\varepsilon, \varepsilon) \to U(N)$ be a smooth variation of f. Then

$$\begin{aligned} f \text{ is harmonic} \quad &\Leftrightarrow \tfrac{\mathrm{d}}{\mathrm{d}t} E(f_t)_{|t=0} = 0 \;\forall f_t;\; f_0 = f \\ &\Leftrightarrow \tfrac{\mathrm{d}}{\mathrm{d}t} \textstyle\int_M Tr(f_t^{-1}\,\mathrm{d}f_t \wedge *f_t^{-1}\,\mathrm{d}f_t)_{|t=0} = 0 \\ &\Leftrightarrow 2\textstyle\int_M Tr((\,\mathrm{d}v + [2A, v]) \wedge *A = 0 \quad \forall v : M^2 \to u(N) \end{aligned}$$

where $v = \left(f_t^{-1}\frac{\mathrm{d}f}{\mathrm{d}t}\right)_{|t=0}$

Integrating by parts of M, we get

$$0 = 2\int_M Tr(v \wedge \mathrm{d} * A - 2v[A, *A]) = 2\int_M Tr(v \wedge \mathrm{d} * A) \quad \forall v\,.$$

Therefore f is harmonic if and only if $\left(A = \frac{1}{2} f^{-1}\,\mathrm{d}f\right)$

$$\mathrm{d} * A = 0$$

or

$$\overline{\partial} A_z - \partial A_{\overline{z}} = 0\,. \tag{1.7}$$

Proposition 1.1. *Let $f : M^2 \to U(N)$ be a smooth map and $A = \frac{1}{2}(f^{-1}\,\mathrm{d}f)$. Then f is harmonic if and only if*

$$\begin{cases} \overline{\partial} A_z - \partial A_{\overline{z}} = 0 \\ \overline{\partial} A_z + \partial A_{\overline{z}} + 2[A_z, A_{\overline{z}}] = 0 \end{cases} \tag{1.8}$$

(*where* $A = A_z + A_{\bar z}$). *Equivalently,* f *is harmonic if and only if*

$$\overline{\partial}_A A_z = \overline{\partial} A_z + [A_{\bar z}, A_z] = 0 \ . \tag{1.9}$$

Proof. (1.8) is the union of (1.6) and (1.7). Adding and subtracting these two equations, we get (1.9) and

$$\partial A_{\bar z} + [A_z, A_{\bar z}] = 0 \tag{1.10}$$

which is the conjugate transpose of (1.9).

Note: The equations above are written in the notation of gauge theory. Indeed, they are (locally) obtained by reducing Yang-Mills equations on $\mathbb{R}^4$, with signature $(++--)$, imposing independence from 2 variables. In §2 we shall interpret (1.9) as a holomorphicity condition.

Exercise. Let M be a Riemann surface, $p \in M$. Prove that the map $f \mapsto f^{-1}\partial f = \partial(\log f)$ establishes a $1-1$ correspondence between

$$\left\{ \begin{array}{ll} f : M \to S^1 & \text{harmonic} \\ f(p) = 1 & \end{array} \right\} \quad \text{and}$$

$$\{\text{holomorphic 1-forms on } M, \quad \text{with integral periods.}\ \}$$

We have seen that the harmonicity equations for maps $M^2 \to U(N)$ are equivalent to a system of first order, non linear equations.

The Zakharov-Shabat technique for solving a wide class of "integrable" non linear systems in 2 variables (originally t time variable and x space variable) consists in expressing the system as compatibility condition of an associated linear system: or of a family of linear systems, indexed by a "spectral parameter" $\lambda \in \mathbf{C}$. The Zakharov-Shabat method is a standard tool in soliton theory.

In our case, let $f : M^2 \to U(N)$, $A = \frac{1}{2} f^{-1}\,\mathrm{d}f = A_z + A_{\bar z}$. For $\lambda \in \mathbf{C}^*$ we define:

$$A_\lambda = (1 - \lambda^{-1}) A_z + (1 - \lambda) A_{\bar z} \ . \tag{1.11}$$

We see A_λ as a family of connections on the trivial bundle $M^2 \times \mathbf{C}^N$, unitary for $|\lambda| = 1$.

Lemma 1.2. *$f : M^2 \to U(N)$ is harmonic $\Leftrightarrow \forall \lambda \in \mathbf{C}^*$, the connection A_λ has curvature $F(A_\lambda) = 0$.*

Proof.

$$\begin{aligned} F(A_\lambda) &= \mathrm{d}A_\lambda + \tfrac{1}{2}[A_\lambda, A_\lambda] = \\ &= (1-\lambda^{-1})\overline{\partial} A_z + (1-\lambda)\partial A_{\overline{z}} + (1-\lambda)(1-\lambda^{-1})[A_z, A_{\overline{z}}] = \\ &= (1-\lambda^{-1})(\overline{\partial} A_z + [A_{\overline{z}}, A_z]) + (1-\lambda)(\partial A_{\overline{z}} + [A_z, A_{\overline{z}}]) \ . \end{aligned}$$

Remark. The equation $F(A_\lambda) = 0$ is the integrability condition for the non linear system:

$$\mathrm{d}_{A_\lambda} E_\lambda = \mathrm{d}E_\lambda - E_\lambda A_\lambda = 0 \tag{1.12}$$

with $E_\lambda : \mathbf{C}^* \times M \to GL(N, \mathbf{C})$. Indeed, the system (1.12) is locally solvable on M if and only if $F(A_\lambda) = 0$, because

$$\mathrm{d}_{A_\lambda} \mathrm{d}_{A_\lambda} E_\lambda = -F(A_\lambda) E_\lambda \ .$$

Therefore we get the following key proposition.

Proposition 1.3.
1) Let $f : M^2 \to U(N)$ be an harmonic map; suppose $\pi_1(M^2) = (0)$. Then there exists

$$E_\lambda : \mathbf{C}^* \times M \to GL(N, \mathbf{C}) \qquad \text{such that}$$

i) $E_\lambda^{-1} \mathrm{d}E_\lambda = (1-\lambda^{-1}) A_z + (1-\lambda) A_{\overline{z}}$.

ii) $E_{-1} = Q \cdot f \quad Q \in U(N)$.

iii) $E_1 = I$.

iv) $E_\lambda \in U(N)$ *for* $|\lambda| = 1$.

v) *E_λ is holomorphic in λ, real analytic on M.*

2) *Let* $E_\lambda : \mathbf{C}^* \times M \to GL(N, \mathbf{C})$ *satisfying* (iii) (iv) (v) *above. Suppose that*

$$\frac{E_\lambda^{-1}\overline{\partial}E_\lambda}{(1-\lambda)}$$

is independent of λ. *Then* $f = E_{-1} : M^2 \to U(N)$ *is harmonic.*

Proof. The existence of E_λ comes from understanding the previous Lemma 1.2, and the meaning of the 0-curvature property as locally integrability condition. When $\pi_1(M) = (0)$, the holonomy of A_λ vanishes and A_λ flat implies A_λ trivial. The regularity of E_λ comes from the analiticity of f. Moreover i) implies $\mathrm{d}E_{+1} = 0$, therefore E_1 is constant on M, and by left multiplication, we may suppose $E_{+1} = I$.

For $|\lambda| = 1$ we have

$$\overline{\partial}(E_\lambda E_{-\lambda}^*) = (1-\lambda)E_\lambda A_{\bar{z}} E_\lambda^* + E_\lambda((1-\lambda^{-1})E_\lambda A_z)^* = 0 \ .$$

Similarly, we have

$$\partial(E_\lambda E_\lambda^*) = 0 \ .$$

Therefore $E_\lambda E_\lambda^* = K(\lambda)$. Choose $p \in M$ and take $\tilde{E}_\lambda = E_\lambda^{-1}(p)E_\lambda$. Then $\tilde{E}_\lambda \tilde{E}_\lambda^* = I$ for $|\lambda| = 1$, and $\tilde{E}_\lambda$ satisfies i) iii) v) as well.

It's easy to show $\mathrm{d}(\hat{E}_{-1}f^{-1}) = 0$. Therefore we have

$$\tilde{E}_{-1} = Qf \qquad\qquad Q \in U(N) \ .$$

We leave 2) as an exercise for the reader. One should first conjugate transpose, using iv), to show that

$$\frac{E_\lambda^{-1}\partial E_\lambda}{(1-\lambda^{-1})}$$

is also independent of λ and then use lemma 1.2.

A map $E_\lambda : \mathbf{C}^* \times M^2 \to GL(N, \mathbf{C})$, satisfying the properties listed above, is called an *extended solution* for the harmonic map f.

We remark that the extended solution is not unique. Two such solutions differ by left multiplication with a holomorphic map

$$R : \mathbf{C}^* \to GL(N, \mathbf{C}) \ R(1) = I \quad R(S^1) \subseteq U(N) \ .$$

At the same time, we tend to identify two harmonic maps $M^2 \to U(N)$, which differ by left multiplication with a $Q \in U(N)$. In general, extended solution only exist locally. Since S^2 is simply-connected, we have:

Corollary 1.4. *Any harmonic map $S^2 \to U(N)$ has an extended solution.*

Corollary 1.5. *Let M^2 be a compact Riemann surface, $f : M^2 \to U(N)$ an harmonic map. Suppose f has an extended solution. Then f is unstable.*

Proof. Let f_t be the variation $f_t = E_{e^{i(t+s)}}$ $t \in (-\varepsilon, \varepsilon)$. Then $\frac{d^2}{dt^2} E(f_t)|_{t=0} < 0$. This follows easily by computation or by looking at the following picture in the space of connections on $M^2 \times \mathbf{C}^N$, in the affine plane containing 0 (the trivial connection), A, $*A$. Here the circle represents the loop of connections A_λ, for $|\lambda| = 1$ and the energy of each map E_λ, for $|\lambda| = 1$, is given by a constant multiple of the distance between A_λ and 0.

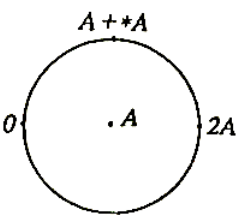

Figure 1: $A_{e^{is}} = A + (\sin(s)) * A - (\cos(s))A$.

2. Loop groups; unitons

We introduce now the loop group of $U(N)$

$$\Omega U(N) = \{\gamma : S^1 \to U(N) \quad \text{smooth } |\gamma(1) = I\} \tag{2.1}$$

$\Omega u(N)$ is a Frechet Lie group. Its Lie algebra is:

$$\Omega u(N) = \{\eta : S^1 \to U(N) \quad \text{smooth } |\eta(1) = 0\} \ . \tag{2.2}$$

We may define a complex structure J on $\Omega U(N)$, as the left translation of the operator

$$\tilde{J} : \Omega u(N) \to \Omega u(N)$$

defined by:

$$\text{if } \eta = \sum_{\alpha \in \mathbb{Z}^*} (1 - \lambda^\alpha)\eta_\alpha \tag{2.3}$$

$$\tilde{J}\eta = i \sum_{\alpha > 0} (1 - \lambda^\alpha)\eta_\alpha - i \sum_{\alpha < 0} (1 - \lambda^\alpha)\eta_\alpha . \tag{2.4}$$

We can define a hermitian metric on $\Omega U(N)$, by left translation of the Hilbert norm on $\Omega U(N)$

$$|\eta|^2 = \sum_{\alpha \in \mathbb{Z}^*} |\alpha||\eta_\alpha|^2 . \tag{2.5}$$

The basic fact is that the resulting metric is Kähler with respect to the complex structure J (cf. [P-S], [E-L 2]).

The associated symplectic 2-form, normalized so as to be the positive generator of $H^2(\Omega U(N), \mathbb{Z}) \cong \mathbb{Z}$, is given by the left translation S of the alternating form on $\Omega U(N)$

$$\tilde{S}(\eta, \xi) = -1/4\pi^2 \int_{S^1} Tr(\eta \xi') . \tag{2.6}$$

S is an integral closed 2-form of type (1,1). If M is a complex manifold, then

$$F : M \to \Omega U(N)$$

is holomorphic if and only if the 1-form $F_\lambda^{-1}\overline{\partial}F_\lambda$ extends from $\lambda \in S^1$ to $\lambda \in D = \{\lambda \in \mathbf{C} | \lambda| < 1\}$, holomorphically in λ.

Let $e_{-1} : \Omega U(N) \to U(N)$ be the evaluation map at $\lambda = -1$. We can rephrase Proposition 1.3 1).

Theorem 2.1 *Let $f : M^2 \to U(N)$, $\pi_1(M^2) = (0)$, be an harmonic map. Then there exists a holomorphic map $F : M \to \Omega U(N)$, such that the following diagram commutes*

$$\begin{array}{ccc} & & \Omega U(N) \\ & F \nearrow & \downarrow e_{-1} \\ M & \xrightarrow{f} & U(N) \end{array} \tag{2.7}$$

Note: Here we use the convention of considering equivalence classes of both harmonic maps and extended solutions, modulo left multiplication by constants. If we want (2.7) to be true in a strong sense, we should take $p \in M$, $f(p) = I$, then it would be possible to choose $E_\lambda(p) = I$ as well.

Let $F : M^2 \to \Omega U(N)$, and M a compact Riemann surface. We can define the degree of F as the degree of the induced map on second dimensional integral cohomology. We have

$$\deg F = \int_{M^2} F^* S = -\frac{1}{4\pi^2} \int_{M^2 \times S^1} Tr(F^{-1}\partial F \wedge (F^{-1}\overline{\partial} F)') =$$

$$= -\frac{1}{4\pi^2} \int_{M^2 \times S^1} Tr(F^{-1}\, \mathrm{d}F \wedge F^{-1}\, \mathrm{d}F \wedge F^{-1}\dot{F}) \tag{2.8}$$

Theorem 2.2 *Let $f : M^2 \to U(N)$ be harmonic, and let M^2 be a compact Riemann surface. Suppose f has an extended solution E_λ. Then we have:*

$$E(f) = 8\pi \deg\,(E_\lambda)\ . \tag{2.9}$$

In particular, the energy of harmonic maps $S^2 \to U(N)$ is always an integral multiple of 8π.

Proof. We have:

$$\begin{aligned}
\deg\,(F_\lambda) &= -\frac{1}{4\pi^2} \int_{M^2 \times S^1} Tr(E^{-1}\partial E \wedge (E^{-1}\overline{\partial} E)' = \\
&= -\frac{1}{4\pi^2} \int_{M^2 \times S^1} Tr((1-\lambda^{-1})A_z \wedge (-i\lambda)A_{\overline{z}}) = \\
&= \frac{i}{4\pi} \int_{M^2 \times S^1} Tr(-A_z A_{\overline{z}}) = -\frac{2i\pi}{4\pi^2} \int_{M^2} Tr(A_z A_{\overline{z}}) = \\
&= -\frac{2i\pi}{4\pi^2} \frac{1}{-4i} E(f) = \frac{1}{8\pi} E(f).
\end{aligned}$$

Theorem 2.2 (better called observation 2.2) has been generalized to general compact Lie groups by Eells and Freed.

We want now to find the "simplest" harmonic maps $M^2 \to U(N)$, corresponding to extended solutions which are monomial in the loop variable. Let $G_k(\mathbf{C}^N)$ be the Grassmannian manifold of complex k-subspaces of $\mathbf{C}^N$. It is

well known that $G_k(\mathbf{C}^N)$ is a complex Kähler manifold. For each $V \in G_k(\mathbf{C}^N)$ let $p = p_v : \mathbf{C}^N \to V$ be the hermitian projection operator onto V. Let $p_v^\perp = I - p_v$ be the projection onto $(V^\perp) \in G_{N-k}(\mathbf{C}^N)$.

The following is well known: it generalizes to embeddings of hermitian symmetric spaces into their (compact) Lie groups of isometries (cf. [B-R]).

Lemma 2.3 *The map $G_k(\mathbf{C}^N) \xrightarrow{i} U(N)$ defined by $V \mapsto P_v - P_v^\perp$ is a totally geodesic embedding; i induces a multiple of the standard Kähler metric on $G_k(\mathbf{C}^N)$, and is called "Cartan embedding".*

We remark that i identifies $GR(N) = \cup G_k(\mathbf{C}^N)$ as the fixed set of the involution $s \mapsto s^{-1}$ on $U(N)$.

Corollary 2.4. *Any harmonic map $M^2 \to G_k(\mathbf{C}^N)$ induces a harmonic map $M^2 \to U(N)$.*

In particular, since M^2 and $G_k(\mathbf{C}^N)$ are Kähler, any holomorphic map $M^2 \to G_k(\mathbf{C}^N)$ is harmonic.

Definition. (Uhlenbeck) *A 1-uniton* is a holomorphic map:

$$f = (p - p^\perp) : M^2 \to G_k(\mathbf{C}^N) \xrightarrow{i} U(N) .$$

Proposition 2.5. *Any 1-uniton $f : M^2 \to G_k(\mathbf{C}^N)$, M^2 Riemann surface, defines a harmonic map $M^2 \to U(N)$, with extended solution $E_\lambda = (p + \lambda p^\perp) : M^2 \to \Omega U(N)$.*

Proof. f is holomorphic $\Leftrightarrow p^\perp \overline{\partial} p = 0$. Moreover,

$$E_\lambda^{-1} \overline{\partial} E_{-\lambda} = (p + \lambda^{-1} p^\perp)(1 - \lambda)\overline{\partial} p = (1 - \lambda) p \overline{\partial} p$$

and we can now use Proposition 1.3 2).

Remark. It is easy to see that $f = (p - p^\perp) : M^2 \to G_k(\mathbf{C}^N)$ is harmonic if

and only if the equation:

$$[\partial\overline{\partial}p, p] = 0 \tag{2.10}$$

is satisfied. Eq. (2.10) is an easy algebraic consequence of $p^{\perp}\overline{\partial}p = 0$.

The constructions above may be put in the following picture:

$$\begin{array}{ccc} & & \Omega U(N) \\ & j \nearrow & \downarrow e_{-1} \\ G_k(\mathbf{C}^N) & \xrightarrow{i} & U(N) \end{array} \tag{2.11}$$

where i is totally geodesic, and j, defined by:

$$j((p - p^{\perp})) = p + \lambda p^{\perp}$$

is holomorphic. Moreover, j induces an isomorphism

$$\mathbb{Z} \cong H^2(G_k(\mathbf{C}^N), \mathbb{Z}) \cong H^2(\Omega U(N), \mathbb{Z}) \ .$$

Remark. The map j constructed above is not the only possible lift of i.

Indeed, for any choice of $\alpha \in \mathbf{C}$, $|\alpha| < 1$, we can define

$$\xi_\alpha(\lambda) = \frac{\lambda - \alpha}{\overline{\alpha}\lambda - 1} \, \frac{\overline{\alpha} - 1}{1 - \alpha} \ . \tag{2.12}$$

Then a completely equivalent choice for an holomorphic j is:

$$j(f) = (p + \xi_\alpha p^{\perp}) \tag{2.13}$$

if $f = p - p^{\perp}$.

We have $\xi_0 = \lambda$. The only reason for choosing $\alpha = 0$ in (2.11) is that our connection A_λ has poles for $\lambda = 0, \infty$, while $(p + \xi_\alpha p^{\perp}) \in GL(N, \mathbf{C})$ for $\lambda \neq \alpha, 1/\overline{\alpha}$,

The choice of a general α is anyway important when classifying the whole set of holomorphic maps

$$M \to \Omega U(N)$$

with M compact Kähler manifold (cf. [V2]).

The following theorem is an adaptation of the naive idea of producing extended solutions, by taking products of simplest ones. Unfortunately, the pointwise product

$$\Omega U(N) \times \Omega U(N) \to \Omega U(N)$$

is not holomorphic.

Theorem 2.6. *Let $f : M^2 \to U(N)$ be an harmonic map, with extended solution $E_\lambda : M^2 \to \Omega U(N)$; and let $(p - p^\perp) : M^2 \to U(N)$. Then $\tilde{f} = f(p - p^\perp)$ is harmonic, with extended solution $\tilde{E}_\lambda = E_\lambda(p + \lambda p^\perp)$ if and only if we have:*

$$\begin{cases} p^\perp \overline{\partial} p + p^\perp A_{\overline{z}} p = p^\perp \overline{\partial}_A p = 0 \\ p^\perp A_z p = 0 \ . \end{cases} \tag{2.14}$$

Proof.

$$\tilde{E}_\lambda^{-1} \overline{\partial} \tilde{E}_\lambda = (p + \lambda^{-1} p^\perp)(1 - \lambda) A_{\overline{z}} (p + \lambda p^\perp) + (1 - \lambda)(p + \lambda^{-1} p^\perp) \overline{\partial} p$$

$\frac{E_\lambda^{-1} \overline{\partial} E_\lambda}{1-\lambda}$ is independent of λ if and only if

$$(p + \lambda^{-1} p^\perp) A_{\overline{z}} (p + \lambda p^\perp) + (p + \lambda^{-1} p^\perp) \overline{\partial} p$$

is independent of λ. Computing the coefficients of λ^{-1}, λ, we must have

$$\begin{cases} p^\perp \overline{\partial} p + p^\perp A_{\overline{z}} p = 0 \\ p A_{\overline{z}} p^\perp = 0 \ . \end{cases}$$

By transposing the second equation, this system is equivalent to (2.14).

We also observe

$$\tilde{E}_\lambda^{-1} \overline{\partial} \tilde{E}_\lambda = (1 - \lambda) \tilde{A}_{\overline{z}} = (1 - \lambda)(p A_{\overline{z}} p + p^\perp A_{\overline{z}} p^\perp + p \overline{\partial} p) \ . \tag{2.15}$$

We call equations (2.14) the uniton equations. We will give their geometrical interpretation in §3. If $\tilde{f} = f(p - p^\perp)$ with p satisfying the uniton equations, we say that $\tilde{f}$ has been obtained by addition of the uniton $\underline{p} = Im(p)$ to f.

Theorem 2.6 gives a recursive procedure to generate new harmonic maps $M^2 \to U(N)$ from given ones. It is an analogue of Bäcklund transformations in the theory of solitons.

The aim of the next two lectures will be to prove *Uhlenbeck's theorem*: any harmonic map $S^2 \to U(N)$ can be obtained by $K \leq N-1$ additions of unitons, starting from a constant map.

3. Stable bundles and the first proof of Uhlenbeck's factorization theorem

We will give two different proofs of Uhlenbeck's theorem, in slightly different versions. In this section we will give a proof which is conceptually very simple, using some elementary algebraic geometry on the structure of holomorphic vector bundle over S^2. This proof is in [V1]. In section 4 we will give a proof more in the spirit of Uhlenbeck's original proof. Examining in detail the extended solution, this will yield also a unicity result for the factorization into unitons.

First, we give a geometrical interpretation of the "uniton equations" (2.14). For any (0,1) form with matrix coefficients $A_{\bar{z}}$ on a Riemann surface M^2, the differential operator

$$\overline{\partial}_A = \overline{\partial} + A_{\bar{z}} \tag{3.1}$$

defines a complex structure on the topologically trivial bundle $M^2 \times \mathbf{C}^N$: this is a special case of a theorem by Koszul & Malgrange (cf. [A-B] for an elementary proof).

Such a complex structure is characterized by having, as local holomorphic sections, all the maps $v : M^2 \to \mathbf{C}^N$ such that

$$\overline{\partial}_A v = \overline{\partial} v + A_{\bar{z}} v = 0 \ .$$

We denote $M^2 \times \mathbf{C}^N$, with the Koszul-Malgrange complex structure induced by $\overline{\partial}_A$, as

$$(\underline{\mathbf{C}}^N \, , \overline{\partial}_A) \ .$$

It is, in general, a non-trivial holomorphic vector bundle. Given a subbundle of constant rank $F \subseteq (\underline{\mathbf{C}}^N, \overline{\partial}_A)$, then F is holomorphic if and only if we have

$$p^{\perp} \overline{\partial}_A p = 0 \tag{3.2}$$

where $p = p(z)$ is the hermitian projection $(\underline{\mathbb{C}}^N, \overline{\partial}_A) \to F$.

Defnition. (cf. [H1]). Let $E \to M^2$ be a holomorphic vector bundle over the Riemann surface M^2; and let Φ be a holomorphic section of

$$\text{End}\,(E) \otimes T^*_{(1,0)}(M^2)\ .$$

Then the pair (E, Φ) is called a *Higgs field* (over M^2). A holomorphic subbundle $F \subseteq E$ is called a *Higgs subfield* if

$$\Phi(F) \subseteq F \otimes T^*_{1,0}(M^2)\ .$$

It is easy to see that in this case, $(F, \Phi_{|F})$ is also a Higgs field.

Let $E \to M^2$ be a holomorphic vector bundle over a compact Riemann surface M^2; suppose $c_1(E) = 0$. A Higgs field (E, Φ) is called *semistable* (resp. stable) if, for any Higgs subfield $F \subset E$, we have $c_1(F) \leq 0$ (resp. $c_1(F) < 0$}.

Proposition 3.1

1) *Let $f : M^2 \to U(N)$ be a smooth map, and let*

$$A = A_z + A_{\bar{z}} = \frac{1}{2}(f^{-1}\,\mathrm{d}f)\ .$$

Then f is harmonic $\Leftrightarrow$ $((\underline{\mathbb{C}}^N, \overline{\partial}_A), A_z)$ is a Higgs field.

2) *Suppose f is harmonic. Then*

$$\{\text{unitons for } f\} = \{\text{Higgs subfields of } ((\underline{\mathbb{C}}^N, \overline{\partial}_A), A_z)\}\ .$$

3) *Suppose f is harmonic and M^2 is compact. Then the Higgs field $((\underline{\mathbb{C}}^N, \overline{\partial}_A), A_z)$ is not semistable if and only if there exists a uniton $\underline{p}$ for f which added to f produces an harmonic map of strictly smaller energy.*

Proof.

(1) is a rephrasing of Eq. (1.9), Proposition 1.1.

(2) is almost obvious, from the remarks above: the uniton equations say precisely that a uniton $\underline{p} = Im(p)$ is a holomorphic subbundle of $(\underline{\mathbf{C}}^N, \overline{\partial}_A)$, and that

$$A_z(\underline{p}) \subseteq \underline{p} \otimes T^*_{(1,0)}M \ .$$

(3) is just a rephrasing, once we prove the following lemma.

Lemma 3.2. *(Energy formula) Let M^2 be a compact Riemann surface, $f : M^2 \to U(N)$ a harmonic map, $\underline{p}$ a uniton for f, $\tilde{f} = f(p - p^\perp)$ the harmonic map obtained by adding the uniton $\underline{p}$ to f. Then we have*

$$\Delta E = E(\tilde{f}) - E(f) = -8\pi c_1(\underline{p}) \ . \tag{3.3}$$

Proof. (Sketch) The proof consists of 2 steps. First one shows, using Eq. (2.15), that we have

$$\Delta E = 8 \int_M -|p^\perp \partial p|^2 + |p^\perp \overline{\partial} p|^2 \ . \tag{3.4}$$

This is a quite straightforward computation (cf. [V1]). Then one identifies the right-hand side in (3.4) as a multiple of the 1st Chern class of $\underline{p}$. The simplest way to do it is the following: hermitian projection produces a connection B on $\underline{p}$, starting from the trivial one on $M^2 \times \mathbf{C}^N$. We have $F(B) = p\,dp \wedge dp$. We conclude the proof by using the Chern-Weil formulas for characteristic classes (cf. [O-V] §5).

Remark. Compare Lemma 3.2 with Theorem 2.2.

Lemma 3.3 *Let (E, Φ) be a Higgs field over S^2, $c_1(E) = 0$. Suppose (E, Φ) is semistable. Then E is holomorphically trivial, and $\Phi = 0$.*

Proof. We recall the following well-known theorems on holomorphic vector bundles on S^2.

Theorem 3.4 (cf. [A-B]).

(1) *For any $k \in \mathbb{Z}$, there exists a unique holomorphic line bundle $L^k \to S^2$ of 1st Chern class k, up to isomorphism;*

(2) *(Birkhoff-Grothendieck) any holomorphic vector bundle $E \to S^2$ is a direct sum of holomorphic line subbundles:*

$$E = L^{k_1} \oplus \quad \oplus L^{k_m} \qquad k_1 \leq k_2 \leq \quad \leq k_n \ .$$

The integers $\{k_i\}$ are uniquely determined by E.

(3) *The Harder-Narasimhan filtration*

$$E_i = \bigoplus_{k_j \geq i} L^{kj}$$

is uniquely determined by E. Indeed, each E_i is spanned by the neromorphic sections of E, of divisor order $\geq i$.

Proof of Lemma 3.3. Let (E, Φ), $c_1(E) = 0$, be a semistable Higgs field over S^2. Let $\{E_i\}$ be the Harder-Narasinhan filtration of $E \to S^2$. We claim that each E_i is a Higgs subfield. Indeed, Φ is a holomorphic section of $\mathrm{End}(E) \otimes T^*_{(1,0)}(S^2)$ and $\Phi(E_i)$ is spanned by $\Phi(\varphi)$ [v], where φ is a meromorphic section of E_i, and v meromorphic vector field. Therefore, $\Phi(\varphi)$ [v] has divisor order $\geq$ divisor order (φ)+divisor order $(v) \geq i + 2$ because $c_1(TS^2) = 2$; and therefore v has divisor order ≥ 2. Therefore we have $\Phi(E_i) \subseteq E_{i+2} \otimes T^*_{1,0}(S^2)$; in particular, each E_i is a Higgs subfield of positive degree, unless $E_i = E$ or (0).

If $E_i = E \ \forall i$, we have, by the above discussion, $\Phi = 0$, and E trivial. In particular, E_1 is a Higgs subfield of maximal degree.

Corollary 3.5 *(1st version of Uhlenbeck's theorem). Any harmonic map $f : S^2 \to U(N)$ is obtained canonically from a constant map, by a finite number of additions of unitons.*

Proof. By Lemma 3.3 we see that it is possible to decrease the energy of f, by adding a uniton, unless $A_z = 0$, i.e. f is constant. Therefore, using the

"quantization" of the energy, we see that, if we repeat the procedure, we must arrive to a constant map, after a finite number of steps.

Going backwards, we have produced a factorization of f. Moreover, at each step, we can choose the most energy – decreasing uniton in a canonical way, as in the proof of Lemma 3.3.

The above version of Uhlenbeck's theorem is not the strongest possible. For example, the factorization is canonical, but not unique.

Corollary 3.6 *Let $f : M^2 \to U(N)$ be a harmonic map which is a product of unitons. Let $A_z = \frac{1}{2} f^{-1} \partial f$. Then f satisfies the "isotropicity" condition*

$$Tr(A_z)^k = 0 \qquad \forall k \ . \tag{3.5}$$

In other words, A_z is nilpotent.

Remark. For $k = 2$ this is equivalent to conformality of f.

Proof. From Eq. (2.15) we see that the quantities $Tr(A_z)^k$ are invariant under addition of unitons. Therefore, if f is a product of unitons, they must equal the quantities computed for constant maps.

In §2, we saw that the Cartan embedding:

$$GR(N) = \bigcup_k G_k(\mathbf{C}^N) \rightarrow U(N)$$

identifies $GR(N)$ with the set $\{s \in U(N) | s^2 = I\}$. Indeed, $s^2 = I \Rightarrow s = p - p^\perp$. Let now $f : S^2 \to GR(N)$ be a harmonic map. We would like to have a factorization

$$f = Q(p_1 - p_1^\perp) \qquad\qquad (p_R - p_R^\perp) \ .$$

So that each particular product $f_i = Q(p_1 - p_1^\perp) \cdots (p_i - p_i^\perp)$ has values in $GR(N)$ as well.

If $g, \tilde{g} \in U(N)$, $g^2 = I$, $\tilde{g} = g(p - p^\perp)$, then, by elementary undergraduate algebra, we have

$$\tilde{g}^2 = I \Leftrightarrow gp = pg \ .$$

If $f : S^2 \to GR(N)$ is harmonic, $A = \frac{1}{2}g^{-1}\,\mathrm{d}g$, and $\underline{p}$ is the maximal energy decreasing uniton for f, we want to prove $fp = pf$. From $f^2 = I$ one gets, by differentiating,

$$fA + Af = 0\,. \tag{3.7}$$

Eq. (3.7) implies that f defines a holomorphic endomorphism of the Higgs field associated to f. Therefore f preserves the Harder-Narasimhan filtration and, being unitary, it commutes with the associated hermitian projections; so that each term of the filtration is a uniton which, added to f, produces a new map into $GR(N)$.

We have proved the following

Corollary 3.7. *Let $f : S^2 \to GR(N) = \bigcup_k G_k(\mathbf{C}^N)$ be harmonic. Then f is canonically a product of unitons:*

$$f = Q(p_1 - p_1^\perp) \qquad\qquad (p_R - p_R^\perp)$$

where each partial product $f_i = Q(p_1 - p_1^\perp)\ \ (p_i - p_i^\perp)$ is a harmonic map $S^2 \to GR(N)$.

With the methods in this section, we have not been able to prove Uhlenbeck's estimate: number of unitons $\leq N - 1$. We will do it in §4. Meanwhile, we prove the following.

Corollary 3.8 *Let $f : S^2 \to S^3 = SU(2)$ be a harmonic map. Then f is a conformal (= holomorphic) map into some equator.*

Proof. In this case the Cartan embedding restricts to

$$G_1(\mathbf{C}^2) = \mathbf{CP}^1 = S^2 \to S^3 = SU(2)\,.$$

We have to prove that $f = Q \cdot (p - p^\perp)$. Suppose we want to add a uniton to any such f.

If $A = \frac{1}{2}f^{-1}\,\mathrm{d}f$, we have $(\underline{\mathbf{C}}^2, \overline{\partial}_A) = L^k \oplus L^{-k}$, with $k > 0$. Since $\underline{p}^\perp$ is a uniton for f, we must have $\underline{p}^\perp = L^k$ (first term in the Harder-Narasimhan

term in the Harder-Narasimhan filtration), $k = -c_1(\underline{p})$. With respect to this decomposition, we have $A_z = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, by the proof of Lemma 3.3). Therefore the only possible uniton for f is $\underline{p}^\perp$. But, by adding $\underline{p}^\perp$ to f, one gets back $Q \in U(N)$.

4. Uniton number of harmonic maps $M^2 \to U(N)$; a second proof of Uhlenbeck's Theorem

The proof of Uhlenbeck's factorization theorem presented in §3 uses the energy of harmonic maps $S^2 \to U(N)$ as a measure of complexity: the lesser the energy, the simpler the map. Uhlenbeck's original proof uses the degree, as a polynomial in λ, of the extended solution, corresponding to the number of unitons needed for the factorization, as a measure.

Since extended solutions are defined only modulo left multiplication by constant loops, some "canonical" normalization must be chosen. Moreover, we need to prove this normalized extended solution is a polynomial in the loop variable λ.

The following is a key lemma: it is taken from [V2]; Uhlenbeck's approach in [U] is slightly different.

Lemma 4.1 *Let M^2 be a compact Riemann surface, and let $F : M^2 \to \Omega U(N)$ be a holomorphic map (into the real-analytic loops). Then there exists a unique*

$$G : M^2 \to \Omega U(N) \qquad \text{such that}$$

(1) *$G^{-1}\,\mathrm{d}G = F^{-1}\,\mathrm{d}F$ for $\lambda \in S^1$ (in particular, G is holomorphic);*

(2) *G extends holomorphically in λ to a map*

$$M^2 \times D \to gl(N, \mathbf{C})$$

(3) *For any $H : M^2 \to \Omega U(N)$ satisfying 1) and 2), there exists $q \in \Omega U(N)$, q extendible to a holomorphic map $D \to gl(N, \mathbf{C})$, such that*

$$H = q \cdot G\,.$$

Note: In [V2] G (or rather its inverse) is called the "maximal unitary expansion" of F. In [U], in the case F is an extended solution, G is called the "canonical normalization of the extended solution".

Proof. *Uniqueness*: (This is the easy part.) Let G_1, G_2 satisfy 1) 2) 3). Then there exist loops q, r, extendible holomorphically to maps $D \to gl(N, \mathbf{C})$, such that

$$\begin{aligned} G_1 &= qG_2 \\ G_2 &= rG_1 \end{aligned} \qquad \text{as functions on } M^2 \times D\ .$$

Therefore $G_1 = qrG_1 \Rightarrow qr = I$ on the open dense subset of D where G_1 is invertible. Hence $q = r^{-1}$ so that $q \in \Omega U(N)$, and q is extendible to a map $D \to GL(N, \mathbf{C})$. Therefore, q is constant $\equiv I$, by analytic reflection through S^1.

Existence: Let $A_{e^{i\theta}} = F_{e^{i\theta}}^{-1}\overline{\partial}F_{e^{i\theta}}$. Then A extends holomorphically in λ to $A_\lambda(z) = A(\lambda, z)$, for $\lambda \in \overline{D} = \{\lambda \in \mathbf{C}\ \ |\lambda| \leq 1\}$, because F is holomorphic (cf. §2).

On the topologically trivial vector bundle:

$$D \times M^2 \times \mathbf{C}^2 \to D \times M^2$$

we consider the $\overline{\partial}$ operator

$$\overline{\nabla} = (\overline{\partial}_\lambda, \overline{\partial}_{A_\lambda}) \qquad \text{where } \overline{\partial}_\lambda = \frac{\partial}{\partial_{\overline{\lambda}}}(\,)\,\mathrm{d}\overline{\lambda}\ .$$

Extending $\overline{\nabla}$ to an exterior differential of $\overline{\partial}$-type on forms, we have $\overline{\nabla}^2 = 0$. By Koszul-Malgrange theorem in higher dimension, $\overline{\nabla}$ defines a holomorphic structure on $D \times M^2 \times \mathbf{C}^N \to D \times M^2$. Call E the resulting holomorphic vector bundle on $D \times M^2$.

Note: (Here we prefer to make $\overline{\partial}_{A_\lambda}$ act on functions

$$V : D \times M^2 \to \mathbf{C}^N \quad \text{as } \overline{\partial}_A V = \overline{\partial}V - VA)\ .$$

Let $\tau : D \times M^2 \to D$ be the projection. It is a proper map, because M^2 is compact. By Grauert's direct image theorem, $\tau_* E$ is a torsion-free coherent sheaf

of $O(D)$-modules over D. Because D is 1-dimensional, and any holomorphic vector bundle over D is trivial, $\tau_* E$ is globally free.

Let $v_1, \quad , v_m$ be global generating sections of $\tau_* E$, as an $O(D)$-module. They satisfy

$$\frac{\partial v_i}{\partial \overline{\lambda}} = 0 \qquad \overline{\partial} v_i - v_i A_\lambda = 0 \ . \tag{4.1}$$

Put the v_i together as column vectors to form K

$$K : D \times M^2 \to \{M \times N \qquad \text{complex matrices.}\}$$

We have $M = N$, and $\det K \neq 0$ for $|\lambda| = 1$, because on a neighbourhood of S^1, $\overline{\partial}_{A_\lambda}$ is trivialized by F.

We can normalize K so that $K(1, z) \equiv I$, by left multiplication. It is easy to prove that $\det K = 0$ only on a subset of $D \times M^2$ of the form $\{\alpha_1, \ldots, \alpha_k\} \times M^2$, by topological reasons (cf. [V2] §4). If K is not unitary for $|\lambda| = S^1$, we consider $p \in M$ and $K^{-1}(p) : S^1 \to GL(N, \mathbf{C})$. By Theorem 8.1.1 in the book of Pressley and Segal [P-S], we have:

$$K^{-1}(p) = \gamma \cdot \sigma$$

where $\gamma \in \Omega U(N)$, and σ extends to a holomorphic map $D \to GL(N, \mathbf{C})$. We have:

$$K^{-1}(p) \cdot K = \gamma \cdot \sigma \cdot K \ .$$

Let $G = \sigma \cdot K$. Then G satisfies property 2) and

$$\overline{\partial} G - G A_\lambda = 0 \ .$$

For $|\lambda| = 1$ we have:

$$\begin{aligned} \overline{\partial} K - KA &= 0 \ \Leftrightarrow \overline{\partial} K - K F^{-1} \overline{\partial} F = 0 \Leftrightarrow \overline{\partial}(K F^{-1}) = 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow \\ \overline{\partial} K^* + A K^* &= 0 \ \Leftrightarrow \partial K - K F^{-1} \partial F = 0 \ \Leftrightarrow \partial(K F^{-1}) = 0 \ . \end{aligned}$$

Therefore:

$$\overline{\partial}(K K^*) = \overline{\partial} K K^* + K \overline{\partial} K^* = K A K^* - K A K^* = 0$$

and KK^* is independent of z, for λ on S^1. But

$$GG^* = \gamma^{-1}(K^{-1}(p)KK^*K^{-1}(p)^*)\gamma^{*\,-1} .$$

The term inside the bracket is independent of z, but it is $\equiv I$ for $z = p \Rightarrow$ it is $\equiv I$. Therefore, for $|\lambda| = 1$

$$GG^* = \gamma^{-1}I\gamma^{*\,-1} = I$$

because γ is unitary. Moreover, $G^{-1}\partial G = F^{-1}\partial F$ for $|\lambda| = 1$, by hermitian transposing of $G^{-1}\overline{\partial}G = F^{-1}\overline{\partial}F$.

Property 3) is an easy consequence of the fact that the column vectors of K (and hence of G, since $\det\sigma \neq 0$) are generating sections of $\tau_* E$.

Remark. The key point is Grauert's theorem, and not the loop decomposition.

We may apply Lemma 4.1 to the case when F is an extended solution.

Corollary 4.2 *Let M^2 be a compact Riemann surface, and let $f : M^2 \to U(N)$ be an harmonic map admitting an extended solution. Then there exists a unique extended solution $E_\lambda : M^2 \to \Omega U(N)$ for f with the following property:*

$$\forall q \text{ constant projection operator } qE_0(z) \equiv 0 \implies q = 0.$$

Proof. The property above is equivalent to condition 3) in lemma 4.1. It is also equivalent to Uhlenbeck's condition

$$\{\mathrm{Im}\ E_0(z)\}_{z\in M^2} \quad \mathrm{span}\ \mathbf{C}^N .$$

We call the canonical extended solution of Corollary 4.2 normalized extended solution (or n.e.s.). At this point, the following exercise may be useful.

Exercise. Let $\gamma \in \Omega U(N)$; suppose γ extends to a holomorphic map $D \to gl(N, \mathbf{C})$. Then γ is rational in the loop variable λ. Moreover, if $\det\gamma \neq 0$ for $\lambda \neq 0$, then γ is polynomial in λ.

This sort of results goes back to Potapov and Masani (cf. [M]) on infinite Blaschke products, and factorization of matrix-valued functions on D, of Hardy classes: unitons "ante litteram".

Theorem 4.3. *Let M^2 be a compact Riemann surface, and let $E_\lambda : M^2 \to \Omega U(N)$ be a normalized extended solution. Then there exists a unique factorization of E_λ into unitons*

$$E_\lambda = (p_1 + \lambda p_1^\perp) \cdot \qquad \cdot (p_k + \lambda p_k^\perp)$$

such that

(1) $\forall i$ $\underline{p}_{i+1} \cap \underline{p}_i^\perp = (0)$ *as holomorphic vector bundles*

(2) $\forall q$, $q^2 = q^* = q$ *constant projector, we have*

$$q(p_1 \cdot \quad \cdot p_i) \equiv 0 \quad \Rightarrow \quad q = 0 \, .$$

Moreover we have:

(3) $k \leq N-1$ $rk(\underline{p}_{i+1}) < rk(\underline{p}_i)$ $c_i(\underline{p}_i) < 0$

(4) *Any partial product*

$$E_\lambda^i = (p_1 + \lambda p_1^\perp) \dots (p_i + \lambda p_i^\perp)$$

is a normalized extended solution.

Proof. Theorem 4.3 is Uhlenbeck's theorem in full version. We will need some lemmas as intermediary steps.

Lemma 4.4. *Let $E_\lambda : M^2 \to \Omega U(N)$ be a n.e.s. Let $E_\lambda = \sum_{\lambda \geq 0} \lambda^i T_i(z)$. Then we have:*

(1) $\underline{p} = \ker(T_0)$ *defines a uniton for E_λ, with* $c_1(\underline{p}) > 0$

(2) $\tilde{E}_\lambda = \lambda^{-1} E_\lambda (p + \lambda p^\perp)$ *is a n.e.s.*, $\deg \tilde{E}_\lambda < \deg E_\lambda$

(3) *if* $\tilde{E}_\lambda = \Sigma \lambda^i \tilde{T}_i$, *we have* $rk(\tilde{T}_0) > rk(T_0)$.

Proof of Lemma 4.4.

1) The equation $\overline{\partial} E_\lambda - E_\lambda A_\lambda = 0$ implies, by evaluating at $\lambda = 0$, that $\overline{\partial} T_0 - T_0 A_{\overline{z}} = 0$. Therefore

$$T_0 : (\underline{\mathbb{C}}^N, \overline{\partial}_A) \to (\underline{\mathbb{C}}^N, \overline{\partial}) \tag{4.2}$$

is a holomorphic morphism. In particular, there exist holomorphic subbundles $F_1 \subseteq (\underline{\mathbb{C}}^N, \overline{\partial}_A)$, $F_2 \subseteq (\underline{\mathbb{C}}^N, \overline{\partial})$ which coincides with ker T_0, Im T_0, where rank (T_0) is maximal, on an open dense subset. We denote $F_1 = \ker T_0$ $F_2 = \operatorname{Im} T_0$, with a slight abuse of notation. We have $c_1(\ker T_0) > 0$, unless $c_1(\operatorname{Im} T_0) = 0$. But, if this is the case, $\operatorname{Im}(T_0)$ is a trivial bundle of $(\underline{\mathbb{C}}^N, \overline{\partial})$, therefore constant: but this is forbidden by the normalization of E_λ.

From the equation $\partial E = (1 - \lambda^{-1}) E_0 A_z$ we get, expanding in power series of λ,

$$T_0 A_z = 0 \ . \tag{4.3}$$

Therefore $A_z(\operatorname{Ker} T_0) \subseteq A_z \subseteq \operatorname{Ker} T_0$; and $\underline{p} = \operatorname{Ker}(T_0)$ is a uniton for E_λ.

2) Suppose $\exists\, q$, $q\tilde{T}_0 \equiv 0$. Since $\tilde{T}_0 = T_0 p^\perp + T_1 p$

$$q(T_0 p^\perp + T_1 p) \equiv 0 \ \Rightarrow \ q T_0 p^\perp = q T_0 \equiv 0 \qquad \Rightarrow q = 0 \ .$$

3) $\tilde{T}_0 = T_0 + T_1 p = T_0 p^\perp + T_1 p$.

This implies $rk\tilde{T}_0 \geq rkT_0$. But we have

$$\partial T_0 = T_0 A_z - T_1 A_z \qquad \text{and} \tag{4.4}$$

$$\operatorname{Im}\,(T_1 A_z) = T_1(\operatorname{Im} A_z) = T_1(\underline{p}) \ . \tag{4.5}$$

Because $\underline{p} = \ker T_0 = \operatorname{Im}(A_z)$.

Therefore, if $rk(T_0) = rk(\tilde{T}_0)$ we have that, by (4.4), (4.5), Im (T_0) is a antiholomorphic subbundle of $\underline{\mathbb{C}}^N$, since

$$\partial(\operatorname{Im} T_0) \subseteq \operatorname{Im} T_0 \ .$$

But $\operatorname{Im}(T_0)$ is also a holomorphic subbundle of $(\underline{\mathbb{C}}^N)$, by (4.2). Therefore, it is constant. Arguing as in 1), this is impossible, because E_λ is normalized.

We see at once that Lemma 4.4 implies that, if E_λ is a n.e.s., then E_λ is a canonical product of at most $(N-1)$ unitons, by arguing on induction on $rk(T_0)$. Indeed, $rk(T_0) = 0$ is impossible, because E_λ is normalized; and $rk(T_0) = N \Rightarrow E_\lambda : D \times M^2 \to GL(N, \mathbf{C})$: and, since E_λ is unitary for $|\lambda| = 1$, we see that E_λ must be constant $\equiv I$.

Definition. (Uhlenbeck) The *uniton number* of an extended solution

$$\begin{array}{ll} E_\lambda & : M^2 \to \Omega U(N) \\ E_\lambda & : \sum_{\lambda=0}^{n} \lambda^i T_i \qquad \text{is} \qquad n \in N \cup \{\infty\} \ . \end{array}$$

The *minimal uniton number* of an harmonic map $f : M^2 \to \Omega U(N)$ is the minimum of the uniton numbers of the extended solutions of f, if there exists any.

Proposition 4.5. *Let $f : M^2 \to U(N)$ be harmonic, admitting an extended solution, and let M^2 be compact. Then the minimal uniton number of f is $\leq N-1$, and it is equal to the uniton number of its normalized extended solution.*

Proof. By the remarks after the Proof of Lemma 4.4, we see that the uniton number must be finite, and $\leq N-1$. Let F_λ be an extended solution with minimal uniton number and let E_λ be the n.e.s. We have:

$$F_\lambda = Q(\lambda) E_\lambda$$

with $Q(\lambda) : \overline{D} \to gl(N, \mathbf{C})$, holomorphic, $Q \in \Omega U(N)$. If $\det Q(0) \neq 0$, then $Q = I$; suppose then $Q(0)$ is not invertible.

We have

$$E_\lambda^{-1} = F_\lambda^{-1} Q(\lambda) \ . \tag{4.6}$$

By unitarity on S^1, the uniton number of E (or F) is equal to the order of the pole at $\lambda = 0$ of E^{-1} (or F^{-1}). Expanding (4.6) in (finite) Laurent series around $\lambda = 0$, we easily get that

$$\text{uniton number of } E_\lambda \leq \text{ uniton number of } F_\lambda \ .$$

Problem. Describe the set of extended solutions with minimal uniton number. This problem is more interesting than it looks like, being related to the choice of a normalization of the extended solution, and to the choice of a "best" possible uniton to subtract. Is there something related to quantum groups and R-matrix theory?

So far in this section, we have produced a canonical factorization of an harmonic map f by choosing a canonical extended solution E_λ, and by choosing $\mathrm{Ker}(E_0)$ as uniton to subtract. By Proposition 4.5 and Lemma 4.4, this procedure decreases the uniton number by 1, (producing a new canonical extended solution, so that we can iterate). To get a *unique* factorization, we have to show how to increase the uniton number.

Lemma 4.6. *Let $E_\lambda : M^2 \to \Omega U(N)$ be a normalized extended solution, of uniton $n^\circ = k$. Let $\underline{p}$ be a uniton for E_λ. Then $\tilde{E}_\lambda = E_\lambda(p + \lambda p^\perp)$ is a n.e.s. of uniton number $k+1$, such that adding Ker $(\tilde{E}_0)$ to $\tilde{E}_\lambda$ we get back E_λ, if and only if:*

(1) Ker $E_0 \cap \underline{p} = (0)$ *as holomorphic vector bundles*

(2) $\forall q$, $q^* = q^2 = q$ *constant*, $qE_0p \equiv 0 \;\Rightarrow\; q = 0$.

Proof.

2) Is precisely the condition for $\tilde{E}_\lambda$ to be normalized, because $\tilde{E}_0 = E_0p$. The condition Ker $(\tilde{E}_0) = \underline{p}^\perp$ is satisfied

$$\Leftrightarrow \;\; \mathrm{Ker}\,(E_0p) = \underline{p}^\perp \; . \tag{4.6}$$

We have $\underline{p}^\perp \subseteq$Ker (E_0p) anyway. Let $V \in$ Ker (E_0p). Then $V = V_1 + V_2$, $pV_1 = V_1$, $pV_2 = 0$.

$$E_0pV = E_0pV_1 + E_0pV_2 = E_0V_1 = 0 \; .$$

Therefore $V_1 \in \underline{p}\cap$Ker (E_0), and $V_1 \neq 0 \Leftrightarrow \; V \in$Ker $(E_0p) - \underline{p}^\perp$. So, (4.6) is equivalent to condition 1), on a open dense subset of M^2.

Note: We ask the reader to forgive us for having changed the notation again from T_0 to E_0.

We have now finished the proof of Theorem 4.3. Indeed, condition 4.3 1) is equivalent to 4.6 1), by use of induction; and the iteration of 4.6 2) produces 4.3 2). The other statements have also been proved. Here the key point is that when we have the factorization (4.1), then Ker $(E_0) = \underline{p}_k^{\perp}$: and this for each k.

We have shown two proofs of Uhlenbeck's theorem. There are other approaches on the market.

1) In [U], Uhlenbeck first proved that there exists a Laplace-type operator L on S^2 so that $LE_\lambda = 0\ \forall\lambda$. From dim Ker $L < \infty$, she deduced that E_λ is polynomial in λ. The unique factorization theorem then followed, as in this section.

2) Segal's proof ([S]) used the "Grassmannian model" of $\Omega U(N)$ as a space of certain Hilbert subspaces of $L^2(S^1, \mathbf{C}^N)$. Using the fact that any holomorphic map $\varphi : M \to \mathbf{P}(\mathcal{H})$ M compact, $\mathcal{H}$ Hilbert space, has a finite dimensional image, he proved that any holomorphic map $\varphi : M \to \Omega U(N)$ can be normalized, to have image in the rational loops. In particular, any extended solution is polynomial in λ, modulo normalization. Then a decomposition of polynomial loops as product of loops of the form $p + \lambda p^{\perp}$, *automatically* induces the factorization into unitons for maps. The space of harmonic maps $S^2 \to U(N)$ decomposes as union of spaces of holomorphic maps $S^2 \to F_m$, where F_m is a flag manifold. We refer to M. Guest's lectures in this workshop for more details.

3) Wood refined the methods in [U] and [V1] to give an *explicit* construction of harmonic maps $S^2 \to G_k(\mathbf{C}^N)$, $S^2 \to U(N)$, using $\mathrm{Im}(A_z)$ and $\mathrm{Ker}(A_z)$ as simplifying unitons.

4) Burstall & Rawnsley have generalized Uhlenbeck's theorem to compact simple Lie groups, admitting hermitian symmetric spaces, using the method

in §3 (cf. [B-R]).

Concerning other generalizations of the material of these lectures, we note:

Hitchin [H2] has given a detailed analysis of harmonic maps $T^2 \to SU(2)$, using a "spectral curve" constructed through the holonomy of the loop of flat connections.

The results do not generalize to harmonic maps $M \to U(N)$ when M has higher dimension. They rather generalize to "pluriharmonic" maps $M \to U(N)$, M simply connected, compact Kähler manifold (a map $M \to N$, M complex manifold, N Riemannian, is called pluriharmonic if its restrictions to germs of complex curves in M are harmonic): cf. [O-V].

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