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ELLIPTIC BOUNDARY VALUE PROBLEMS FOR CONNECTIONS A NON-LINEAR HODGE THEORY

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0. Introduction and statement of the result

We define some elliptic boundary value problems for connection one-forms. The equations in the interior are the Yang-Mills equations. These problems arise in a natural way, via a variational principle, as explained in §2. They do not arise immediately as elliptic problems. In fact, the solutions of the systems considered are geometrical objects. They are acted upon by a symmetry group, that does not leave ellipticity invariant. This constitutes a source of complexity in the study of boundary value problems for connections. The problems that we define should be viewed as the most natural generalization of linear Hodge theory for forms, to a non-linear Hodge theory.

The main results could be exemplified as follows.

Theorem 1. *Given a_τ, a smooth connection at ∂M, there exists a smooth solution of the Dirichlet problem, defined in §2.*

When studying the Neumann problem, in order to avoid ending up with a trivial solution, we need to put a topological obstruction on the bundles where the minimizing procedure takes place. This obstruction is defined via Cěch cohomology and is the second Stieffel Whittney class if the structure group in the theory is S0(3). This obstruction had been already introduced for finding Yang-Mills connections on a compact Riemannian four-dimensional manifold, [7]. The result for the Neumann problem is stated as follows.

Theorem 2. *There exists a smooth solution A for the Neumann problem, defined in §2, with given obstruction $\eta \in H^2(M, \pi_1 G)$.*

In the next section, we give a simplified description of the the geometrical objects involved in the theory.

1. Brief geometrical description

We consider a principal bundle P, over a compact Riemannian manifold M, of dimension four with boundary ∂M. We take the structure group of the bundle to be $G \equiv SO(l)$. (Most of the interesting analysis can be carried out on $M \equiv B^4$, the four-dimensional disk, and on the trivial bundle, $P = B^4 \times SO(l)$). We cover M with neighborhoods of type

$$U^{(1)} \equiv \{x \equiv (x^1, ..., x^n) \text{ such that } |x| < 1\}$$

in the interior, and

$$U^{(2)} \equiv \{x \equiv (x^1, ..., x^n) \equiv (\underline{x}', x^n) \text{ s.t. } |x| < 1, x^n \geq 0\}$$

at the boundary.

We consider connection one-forms on P, i.e. differential one-forms valued in the Lie algebra of $SO(l)$, described in coordi nates by $A(x) = \sum_i A_i(x)\, dx^i$, (the $A_i^{\prime}s$ are matrices in $so(l)$), that can be gauge-transformed in the following way,

$$A_i \to g^{-1} A_i g + g^{-1} \frac{\partial g}{\partial x_i}, \qquad \forall i = 1, ...n.$$

The $g's$ above are maps from the manifold into the Lie group $SO(l)$. The curvature of A is

$$F = \sum_{ij} F_{ij}\, dx^i \wedge dx^j, \text{ where } F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j],$$

and, under gauge transformations, $F_{ij} \to g^{-1} F_{ij} g$. Various types of action could be considered. We restrict our attention to the pure Yang-Mills functional,

$$\mathcal{Y}.\mathcal{M}.(A) \equiv \int_{B^4} trace\, F \wedge *F \equiv \int_{B^4} |F|^2.$$

2. Definition of the Dirichlet and Neumann boundary value problems

Two boundary value problems arise from the Yang-Mills functional, via the calculus of variations. One of them requires fixing the tangential part of the connection A on the bundle restricted to ∂M. If ∂M is locally described as $\{x^4 = 0\}$, this amounts to prescribing $A_1(x)\,dx^1 + A_2(x)\,dx^2 + A_3(x)\,dx^3$ on ∂M. The connection is fixed in the sense that a particular choice is fixed in the gauge-equivalence class of A on the boundary. The equations in the interior are the Yang-Mills equations. This corresponds to the Dirichlet problem, i.e. to the following system of equations,

$$\mathcal{D}: \quad \begin{cases} D * F \equiv d * F + [A, *F] = 0 & \text{on} \quad M \\ A_i = a_i & \text{on} \quad \partial M \quad i = 1, .., n-1. \end{cases}$$

In the Neumann problem, A is free at the boundary, but the normal part of the curvature is set equal to zero. The Neumann problem is obtained by imposing that the variation with respect to A of the Yang-mills functional be equal to zero with no constraints, and corresponds to the following system

$$\mathcal{N}: \quad \begin{cases} D * F \equiv d * F + [A, *F] = 0 & \text{on} \quad M \\ F_{in} = 0 & \text{on} \quad \partial M \quad i = 1, .., n-1. \end{cases}$$

The Neumann problem is meant in a weak sense, or for smooth solutions. For the Dirichlet problem, we need some admissibility conditions, when prescribing a_r [8].

Comments: The Neumann problem presents itself as a natural problem from all points of view (analytical, geometrical, physical). It arises naturally from the energy functional, as a consequence, it is gauge invariant. It could be done completely by reflecting the manifold across the boundary, hence it could be transformed into an interior problem.

To make the Dirichlet problem gauge invariant, we need to allow gauge transformations at the boundary that extend to the interior. The Dirichlet problem cannot be done completely by doubling the manifold, because the doubling that would be required does not work geometrically to define a new bundle on a compact manifold. The Dirichlet problem is non-homogeneous, to allow non-trivial solutions.

We should think of the above problems as problems for A. In fact the differential operators that we are considering depend upon the connection. The above systems are non-linear and second order for A. They are not elliptic, also (but not only) because we need to prescribe more boundary conditions (only "half" boundary condition has been prescribed) [2]. We use the gauge invariance of connections to find extra interior equations and boundary conditions to add to the systems above. Under a suitable gauge transformation we gain ellipticity.

Sections 3. and 5. illustrate these remarks and relate these problems to the linear theory for forms.

3. Boundary value problems for differential forms

Let us consider a differential operator which is formally self-adjoint and elliptic of order o, say Q. At first, let us define $\mathcal{N}$ for functions. We start with $Q \equiv \Delta^2$, the square of the Laplace operator and o=4. Following the variational approach, by imposing

$$\delta_\varphi \int_M |\Delta f|^2 = 0$$

with no constraints on φ, we obtain the classical Neumann problem

$$\begin{cases} \Delta^2 f = 0 & \text{on } M \\ \frac{\partial}{\partial n} \Delta f = 0 & \text{at } \partial M \\ \Delta f = 0 & \text{at } \partial M. \end{cases}$$

Notice that there are two boundary conditions, that is "half" the order of Δ^2.

Next, let us consider the Clifford algebra of differential forms of all degrees on a smooth n-dimensional manifold, $\Lambda(M) = \sum_{k=0}^{n} \Lambda^k T^* M$ and let us take $Q \equiv d + d^*$. Then Q has order o= 1. The following systems of equations represent well posed boundary value problems for $\theta \in \Lambda(M)$:

$$\mathcal{N}: \quad \begin{cases} (d + d^*)\theta = 0 & \text{on } M \\ i^*(*\theta) = 0 & \text{at } \partial M, \end{cases} \qquad \mathcal{D}: \quad \begin{cases} (d + d^*)\theta = 0 & \text{on } M \\ i^*(\theta) = 0 & \text{at } \partial M, \end{cases}$$

where $i^*: \quad \Lambda^k T^* M \to \Lambda^k T^*(\partial M)$ is the pull back of the inclusion map, $*$ is the Hodge operator.

In coordinates, close to ∂M we can write

$$\Lambda(M) = \Lambda(\partial M) \times \Lambda(\partial M) \otimes d\nu \qquad \text{and} \qquad \theta = \theta_1 + \theta_2 \wedge d\nu.$$

Using this notation, the boundary condition for $\mathcal{D}$ is $\theta_1 = 0$, and for $\mathcal{N}$ is $\theta_2 = 0$. These are also called relative and absolute boundary conditions, respectively. The dimension of the fiber is 2^n for the vector bundle $\Lambda(M)$ and 2^{n-1} for $\Lambda(\partial M)$, thus also in this case we are prescribing "half" boundary conditions. We refer to [2] for a more formal exposition.

The Hodge operator takes one problem into the other and makes them "isomorphic". More specifically, let us consider the following de Rham complexes with added relative and absolute boundary conditions,

$$\mathcal{D}: \qquad \cdots \xrightarrow{d} C^\infty(\Lambda^j_{rel}) \xrightarrow{d} C^\infty(\Lambda^{j+1}_{rel}) \xrightarrow{d} C^\infty(\Lambda^{j+2}_{rel}) \cdots,$$

$$\mathcal{N}: \qquad \cdots \xrightarrow{d^*} C^\infty(\Lambda^j_{abs}) \xrightarrow{d^*} C^\infty(\Lambda^{j-1}_{abs}) \xrightarrow{d^*} C^\infty(\Lambda^{j-2}_{abs}) \cdots,$$

and corresponding cohomology groups,

$$H^j_{rel}(M;\mathbb{C}) \equiv \left(\frac{ker\, d}{im\, d}\right)_{rel} \quad \text{and} \quad H^j_{abs}(M;\mathbb{C}) \equiv \left(\frac{ker\, d^*}{im\, d^*}\right)_{abs}.$$

The de Rham theory for manifolds with boundary gives the following isomorphisms:

$$H^j_{rel}(M;\mathbb{C}) \sim H^j(M, \partial M;\mathbb{C}) \qquad \text{and} \qquad H^j_{abs}(M;\mathbb{C}) \sim H^j(M;\mathbb{C}).$$

Clearly Hodge duality, that transforms $\mathcal{D}$ into $\mathcal{N}$, translates into Lefschetz duality. On the de Rham theory on manifolds with boundary see [1]. The Hodge decomposition theorem for manifolds with boundary gives a formal motivation to the definition of $\mathcal{D}$ and $\mathcal{N}$ above. In fact, this theorem implies existence and uniqueness of a solution to $\mathcal{D}$ and $\mathcal{N}$ in $H^j_{rel}(M;\mathbb{C}) \equiv (\frac{ker\, d}{im\, d})_{rel}$ and $H^j_{abs}(M;\mathbb{C}) \equiv (\frac{ker\, d^*}{im\, d^*})_{abs}$, respectively. For a precise statement of Hodge decomposition theorem for manifolds with boundary see [6]. Notice that both problems are defined in the homogeneous case.

Unfortunately, the non-linear theory does not have such a good geometrical description. If there were a more general theory of boundary value problems for non-linear equations, the problems for connections defined in §2 would fit into it very naturally. Non-linear Hodge theory is more natural for one-forms, neverthelesss the Yang-Mills equation and the Bianchi identity for the curvature form $F(A)$ seem to be a natural extension. In the case the structure group in the gauge-theory is abelian, A transforms like a differential form $(A \to A + du)$, $D \equiv d$, and our results are contained in the Hodge theory for manifolds with boundary.

4. Existence theory: the minimizing procedure

Existence of a solution for the Dirichlet and Neumann boundary value problems in §2 is found via a minimizing procedure, analogous to the one used in [7] on a closed manifold. Here we examine the Dirichlet problem. The Neumann problem is done in similar fashion.

Let us define

$$m(a_\tau) \equiv \min_{\mathcal{A}} \quad \mathcal{Y}.\mathcal{M}(A),$$

where $\mathcal{A} \equiv \{$ connections A on G-bundles on $M \quad s.t.\ A_\tau|_{\partial M} = a_\tau\}$. Let $\{A_i\}$ be a smooth minimizing sequence for $\mathcal{Y}.\mathcal{M}.$, $A_i \in \mathcal{A}$. Then (cfr [4]),

(1) there exists a subsequence weakly convergent in L^2_1 to a connection A on M, except at most for a finite number of points.

(2) the limiting A satisfies $\mathcal{D}$, (cfr §2)

(3) A is smooth up to ∂M

(4) point singularities can be removed (also at the boundary).

By a counting argument, if M is 4-dimensional, it can be covered by neighborhoods of type one and two with eventually small energy for the sequence.

Applying the good gauge theorems (cfr. §5 estimate $(*)$), (1) is obtained by means of weak compactness. The dimension $n = 4$ is crucial in this argument. In dimension $n > 4$ singularities of Hausdorff codimension four arise. The dimension four is a border line case, in which isolated singularities might occur apriori. (In [3] there is an example of what can happen if the structure group is simply connected, (*i.e.,* with zero obstruction η). Here M is a compact manifold, $M = S^4$, and a sequence of connections $\{A_i\}$ with fixed energy is given. The energy is prescribed to be equal to the topological minimum for the bundle where all the A_i's live. By means of weak-compactness of L_1^2, there is always a limiting connection, but the limit could be flat and an instanton come out at a point of the manifold.)

At this point, two type of questions need to be analysed. One of them is of local nature and very important from the analytical point of view. It is discussed in the next section. The other is of global nature and is related to the geometrical picture. It is the problem of patching the local solutions to form a global solution on a reasonably smooth bundle. This can be done if the base manifold considered is of dimension $n \leq 4$. If the dimension is 5 or more, Sobolev embeddings and multiplication theorems used in the 4−dimensional case do not work any more to give a global solution.

5. Ellipticity: choice of gauge

What are the right boundary conditions to add to systems $\mathcal{D}$ and $\mathcal{N}$ in §2, in order to make them elliptic?

The problem of making the equations elliptic is already encountered in the interior case, it is more involved in the case a boundary is present. The interior problem is done by [9], by finding a gauge transformation g such that $d * (g^{-1}dg + g^{-1}Ag) \equiv d * \hat{A} = 0$. This is called the Hodge gauge, or Lorentz gauge, by the physicists. This problem is similar to finding harmonic maps from the manifold M to the Lie group $SO(l)$. The newly obtained connection satisfies the estimate

$$(*) \qquad \|A\|_{p;1} \leq \text{const} \cdot \|F\|_p.$$

This estimate is essential to the existence and regularity theory. Observe that in the Abelian case A satisfies $\Delta A = d^*F + dd^*A = 0$.

The boundary case is done in [4], by proving new good gauge theorems. In these good gauges, the new potential A satisfies

$$\mathcal{D}: \quad \begin{array}{ll} \text{(a)} & d_\tau^* A_\tau = 0 \text{ on } \partial_1(U) \equiv \{x^n = 0\} \\ \text{(b)} & A_\nu = 0 \text{ on } \partial_2(U) \equiv \{|x| = 1\} \\ \text{(c)} & d^* A = 0 \text{ on } U, \end{array}$$

in the Dirichlet case;

$$\mathcal{N}: \quad \begin{array}{ll} \text{(a)}' & A_n = 0 \text{ on } \partial_1(U) \equiv \{x^n = 0\} \\ \text{(b)}' & A_\nu = 0 \text{ on } \partial_2(U) \equiv \{|x| = 1\} \\ \text{(c)}' & d^* A = 0 \text{ on } U, \end{array}$$

in the Neumann case.

Notice that in $\mathcal{D}$ the boundary condition on $\partial(U) = \partial_1(U) \cup \partial_2(U)$ is non-homogeneous in rank. In some sense, conditions (a) and (c) introduce a Neumann condition $\frac{\partial}{\partial x^n} A_n = 0$, on $\partial_2(U)$, as a hidden boundary condition, that will be essential later to perform reflections; and prove regularity.

The proofs of the new good gauge theorems do not come as straight-forward consequence of the old ones. These theorems make estimate $(*)$ available, also for boundary value problems.

The Neumann case looks like the interior case, eccept for the corners in the domain.

6. Boundary regularity

The problem of interior regularity has been solved by K. Uhlenbeck in [9] and [10].

For the proof of boundary regularity, i.e. on neighborhoods of type two, see [4]. An outline of the method used for the Dirichlet problem is the following.

We extend a_τ to a smooth one-form a on U in a suitable way. We define a one-form $H \equiv A - a$. After some easy computation, using the field equations and the "good gauge", we show that this form satisfies the following boundary value problem:

$$\begin{cases} L(H) \equiv \Delta_F(H) + *_F d(* - *_F)d(H) + Q(H) = \alpha(a, A) & \text{in} \quad U \\ H_\tau = 0 \qquad \text{“}\frac{\partial}{\partial x^n} H_n = o\text{”} & \text{at} \quad \partial_1 U, \end{cases}$$

$(H \in L_1^2(U))$, where $Q(H) \equiv *_F(1/2d^*[H, A] + [H, *F(A)])$, Δ_F is the flat Laplace operator on forms, and $\alpha(a, A)$ is a one-form in L^2. The operator Q has been introduced to take care of the non-linearity of the problem. To study this system, we double the neighborhood U in the g_{ij} metric. We call $\tilde{U}$ the doubled neighborhood (this will have a metric that is only Lipschitz bounded) and define an operator $\tilde{L} : \Lambda^1(\tilde{U}) \otimes g \to \Lambda^1(\tilde{U}) \otimes g$ whose coefficients extend the coefficients of L, in such a way that $\tilde{L}r^* = r^*\tilde{L}$. The coefficients of $\tilde{L}$ present a jump discontinuity at $\partial_1 U$, because of characteristic functions appearing in the definition. We will overcome this obstacle by showing that the double of the flat Laplace operator on U is the flat Laplace operator on $\tilde{U}$, and that the difference $\tilde{L} - \Delta_F$ is small in some sense. The first thing to show is an easy computation. We also extend the one-form α, to the one-form $\tilde{\alpha}$ defined on $\tilde{U}$ that satisfies $\tilde{\alpha}(x', x^n) = -r^*\tilde{\alpha}(x', x^n)$. Let now $\tilde{H}$ be the odd extension of H; i.e. such that $r^*\tilde{H} = -\tilde{H}$. Given the boundary conditions that we are considering, the extended form $\tilde{H}$, also belongs to $L_1^2(\tilde{U})$. The one-form $\tilde{H} \in L_1^2(\tilde{U})$ satisfies $\tilde{L}(\tilde{H}) = \tilde{\alpha}$ in $\tilde{U}$. After carrying out all the estimates, we reduce the problem to studying regularity of the following system of equations,

$$S : \quad \begin{cases} \tilde{L}\omega = \gamma & \text{on} \quad \tilde{U} \\ \omega = 0 & \text{at} \quad \partial\tilde{U}; \end{cases}$$

where $\quad \gamma \in L^2 \quad$ and $\quad \omega \in L_1^2(\tilde{U})$.

To show regularity of the solution, we make a dilation from the ball of size σ, $\tilde{U}_\sigma$, to the ball of size one $\tilde{U}_1$. In the new coordinates, we need to show that the operator $\tilde{L} - \Delta_F$ is small.

Very schematically, we proceed as follows. We start from system S. The right hand side is in L^p_{-1} for $p < 4$. The flat Laplace operator has a bounded inverse that goes from L^p_{-1} to L^p_1 and $\tilde{L} - \Delta_F$ is small. Therefore also $\tilde{L}$ has a bounded inverse. This gives $H \in L^p_1$ for $p < 4$ in a smaller neighborhood. Now we can take the non-linear terms to the right hand side and get something in L^p for $p < 4$. Iterating the procedure used before, we prove that $H \in L^p_2$. We can now start taking derivatives. Regularity is proven first for tangential derivatives, then the equations relate them to normal derivatives and standard techniques show smoothness of the solution.

As stated in §4, isolated singularities might occur on a 4-dimensional manifold as a result of the minimizing procedure, A removable singularities theorem for boundary points is in [4].

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