

## INSTANTONS, RATIONAL MAPS, AND HARMONIC MAPS

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In these lectures I shall discuss the space  $\text{Hol}(S^2, F_k)$  of holomorphic maps from the Riemann sphere  $S^2$  to the space  $F_k$ , where  $F_k$  is a certain compact (singular) projective variety. Such holomorphic maps are automatically “rational”. The spaces  $F_k$  are subvarieties of the infinite dimensional manifold  $\Omega U_n$ , the loop group of the unitary group. In fact they form a filtration of the space of algebraic loops, the “Mitchell-Segal filtration”. Because of this, it turns out that  $\text{Hol}(S^2, F_k)$  plays a role in problems of gauge theory and topology.

Lecture I is a brief survey of the problems in Yang-Mills theory and the theory of harmonic maps which motivate the discussion of  $\text{Hol}(S^2, F_k)$ . Lecture II gives some basic properties of  $F_k$  and  $\text{Hol}(S^2, F_k)$ . Applications to instantons are presented in Lecture III, and to harmonic maps in Lecture IV.

In Lecture III I shall show that the space  $\text{Hol}(S^2, F_k)$  is a “good approximation” to the space  $\text{Map}(S^2, F_k)$  of continuous maps from  $S^2$  to  $F_k$  (at least in the case  $n = 2$ , and for basepoint preserving maps of a fixed degree). By a correspondence of M. F. Atiyah and S. K. Donaldson, this result may be translated into an approximation theorem for instantons, and in fact re-proves and extends an earlier result of M. F. Atiyah and J. D. S. Jones.

In lecture IV I shall discuss harmonic maps from  $S^2$  to  $U_n$ , which, via a formulation of K. Uhlenbeck and G. Segal, correspond to certain holomorphic maps from  $S^2$  to  $F_k$ . I shall show how the natural action of the (complex) loop group on such harmonic maps agrees with an apparently more complicated action, the “dressing action”, which arose earlier from the theory of integrable systems. Combining this with some ideas from elementary Morse Theory, one

can obtain nontrivial deformations of harmonic maps. I shall illustrate this by giving a simple proof that the space of harmonic maps from  $S^2$  to  $S^4$  of fixed energy is a path connected space.

In describing the spaces  $F_k$  I am reporting on work of S. Mitchell, A. Pressley, W. Richter, and G. Segal. My interest in  $\text{Hol}(S^2, F_k)$  owes much to discussions with A. Pressley. The results on the dressing action in Lecture IV are joint work with Y. Ohnita, and the idea of using the dressing action to obtain results on the connectivity of spaces of harmonic maps came from discussions with N. Ejiri and M. Kotani. This work was supported by the Japan Society for the Promotion of Science and the U.S. National Science Foundation. I am very grateful to the faculty, staff, and students of Tokyo Metropolitan University for their kind assistance during the academic year 1990/1991.

## Lecture I: Instantons, rational maps, and harmonic maps

### §1.1 The moduli space of instantons.

Let  $P \rightarrow S^4$  be a principal  $G$ -bundle, with  $c_2(P) = -d$ , where  $G$  is a compact simple Lie group and  $d$  is a nonnegative integer. Let  $\mathcal{A}_d$  be the space of (smooth) connections  $\nabla$  in  $P \rightarrow S^4$ . The Yang-Mills functional is defined by

$$YM : \mathcal{A}_d \rightarrow \mathbf{R}, \quad YM(\nabla) = \int_{S^4} \|F(\nabla)\|^2$$

where  $F(\nabla)$  is the curvature of  $\nabla$ . The critical points of  $YM$  ("Yang-Mills connections") are the solutions of the equation

$$d^* F(\nabla) = 0. \quad (1)$$

It is not easy to find solutions to this equation. However, the critical points of  $YM$  which are absolute minima ("Yang-Mills instantons") turn out to be the solutions of the simpler equation

$$* F(\nabla) = F(\nabla). \quad (2)$$

Let  $I_d$  be the subset of  $A_d$  consisting of solutions to (2). The “based gauge group”  $\mathcal{G}$  is the group of all automorphisms of  $P \rightarrow S^4$  which are the identity over  $\infty \in S^4$ . This acts freely on  $A_d$  and on  $I_d$ , and the submanifold  $\mathcal{M}_d = I_d/\mathcal{G}$  of  $C_d = A_d/\mathcal{G}$  is called the *moduli space of (framed)  $G$ -instantons of charge  $d$  over  $S^4$* .

All these concepts extend to principal  $G$ -bundles  $P \rightarrow M^4$  where  $M^4$  is a compact oriented Riemannian manifold of dimension 4. Since the work of Donaldson in 1984 we know that the corresponding moduli space  $\mathcal{M}_d(M^4, G)$  is a fundamental object which reflects deep properties of  $M^4$  and  $G$ . However, in these lectures, we shall be concerned mainly with the simplest case  $M^4 = S^4, G = SU_2$ ; even in this case the moduli space  $\mathcal{M}_d = \mathcal{M}_d(S^4, SU_2)$  is quite nontrivial!

### §1.2 Basic properties of the moduli space of instantons.

The first interesting examples of  $SU_2$ -instantons on  $S^4$  were obtained by the “t Hooft construction” (around 1977): given a collection  $\{q_1, \dots, q_d\}$  of distinct points in  $\mathbb{R}^4$ , it is possible to write down an explicit solution of (2). Shortly afterwards, Atiyah, Hitchin and Singer showed that  $\mathcal{M}_d(S^4, G)$  is a finite dimensional manifold, and computed its dimension. For example,  $\dim \mathcal{M}_d(S^4, SU_n) = 4nd$ . Then Atiyah, Drinfeld, Hitchin and Manin gave a linear algebraic description of all the solutions to (2) (the “ADHM construction”), for the compact simple classical groups.

What can be said about the space  $\mathcal{M}_d$ ? For  $d = 1$ , it is known that there is a diffeomorphism

$$\mathcal{M}_1 \cong SO_3 \times \{x \in \mathbb{R}^5 \mid \|x\| < 1\}.$$

For  $d = 2$  (see [Ha],[Hh],[Au]) there is a homotopy equivalence

$$\mathcal{M}_2/SU_2 \simeq Gr_2(\mathbb{R}^5).$$

(In general,  $G$  acts on  $\mathcal{M}_d(M^4, G)$ , although the action is not usually free.) For any  $d$ , it is known that  $\mathcal{M}_d$  is connected (see [Ta1]), and it is known that

$\pi_1 M_d \cong \mathbb{Z}/2\mathbb{Z}$  (see [Hu1]). However, it is not very easy to get this kind of explicit topological (or geometrical) information about  $M_d$  from the ADHM construction.

### §1.3 Morse theoretic principles.

A different way of obtaining topological information about  $M_d$  is suggested by Morse theory. We have a functional  $YM : C_d \rightarrow \mathbb{R}$ , whose critical points are the Yang-Mills connections, and for which  $M_d$  is the set of absolute minima. This can be compared with another, simpler, situation. If  $P, Q$  are two points of a Riemannian manifold  $M$ , we have the energy functional  $E : \Omega M \rightarrow \mathbb{R}$  on (a component of) the set of smooth paths from  $P$  to  $Q$ . The critical points of  $E$  are the geodesics connecting  $P$  and  $Q$ , and of course the shortest geodesics constitute the absolute minima of  $E$ . It is a classical fact that Morse theory applies to  $E$ . A simple consequence of Morse theory is that if  $M_{min}$  denotes the set of shortest geodesics (in a fixed component), then the induced maps  $H_i M_{min} \rightarrow H_i \Omega M$  in homology, and  $\pi_i M_{min} \rightarrow \pi_i \Omega M$  in homotopy, are isomorphisms for  $i < n$ , where  $n + 1$  is a lower bound for the index of any nonminimal critical point.

For example, if  $M = S^n$  and  $P, Q$  are the north and south poles of the  $n$ -sphere, then  $(S^n)_{min} \cong S^{n-1}$  and we obtain an inclusion  $S^{n-1} \rightarrow \Omega S^n$  which induces isomorphisms in dimensions less than  $2n - 3$ . This is the "Freudenthal Suspension Theorem". A direct generalization of this example can be made when  $M$  is a compact symmetric space:  $M_{min}$  is also a compact symmetric space (for suitable  $P, Q$ ), and in 1958 Bott used the equivalence  $\pi_i M_{min} \rightarrow \pi_i \Omega M$  to obtain his famous "Periodicity Theorem".

Now, the classical Morse theory does not apply to  $YM$  in the same way that it applies to  $E$ , but there are some formal analogies between the two situations. In the version of the Yang-Mills equations for a principal  $G$ -bundle  $P \rightarrow M^2$ , where  $M^2$  is a compact oriented Riemann surface, Atiyah and Bott made this analogy very precise. This suggests the first basic principle:

APPROXIMATION PRINCIPLE:  $\mathcal{M}_d(M^4, G)$  should approximate  $\mathcal{C}_d(M^4, G)$  in homology and homotopy, up to some dimension which increases with  $d$ .

In the case of  $E : \Omega M \rightarrow \mathbf{R}$ , this approximation principle is valid and it allows one to study  $\Omega M$  by studying the simpler space  $M_{\min}$ . In the case of  $YM : \mathcal{C}_d(M^4, G) \rightarrow \mathbf{R}$  the emphasis is switched, as it is  $\mathcal{C}_d(M^4, G)$  which is simpler than  $\mathcal{M}_d(S^4, G)$ . In fact:

**Proposition [AJ].** *The space  $\mathcal{C}_d(M^4, G)$  has the homotopy type of  $\Omega_d^3 G = \text{Map}_d^+(S^3, G)$ .*

Here,  $\text{Map}_d^+(S^3, G)$  denotes the set of smooth (or continuous) maps  $f : S^3 \rightarrow G$  such that  $f(\infty) = e$  (the identity element of  $G$ ), and such that in  $\pi_3 G \cong \mathbf{Z}$  the class  $[f]$  corresponds to  $d$ . Note that  $\Omega_d^3 G$  is connected, and that  $\pi_1 \Omega_d^3 SU_2 \cong \pi_4 SU_2 \cong \pi_4 S^3 \cong \mathbf{Z}/2\mathbf{Z}$ , so the approximation principle is at least consistent with the information given earlier on  $\mathcal{M}_d$ .

The second basic principle (again from the analogy with geodesics) is:

LOWEST INDEX PRINCIPLE: The space  $\mathcal{M}_d(M^4, G)$  of minimal (index zero) Yang-Mills connections should “generate” all Yang-Mills connections in  $\mathcal{C}_d(M^4, G)$ .

This was shown to hold for the Yang-Mills problem over a Riemann surface, in the work of Atiyah and Bott just mentioned. For the Yang-Mills problem over  $M^4$ , very little is known about nonminimal Yang-Mills connections; in fact, existence of such connections for  $M^4 = S^4, G = SU_2$  was proved only recently, by Sibner, Sibner, and Uhlenbeck and by Sadun and Segert, as well as by Parker in the case of a perturbed metric.

#### §1.4 The Atiyah-Jones theorem on instantons.

The first work on the approximation principle for  $SU_2$ -instantons on  $S^4$  was done by Atiyah and Jones [AJ]. They established a homotopy commutative

diagram

$$\begin{array}{ccc} \mathcal{M}_d & \xrightarrow{I} & C_d \simeq \Omega_d^3 S^3 \\ T \uparrow & \cdot & H \uparrow \\ C_d(\mathbf{R}^4) & \xrightarrow{E} & \Omega_d^4 S^4 \end{array}$$

where  $I$  is the natural inclusion and  $T$  is the inclusion of the 't Hooft instantons. The map  $E$  is the "electric field map", well known to topologists. The map  $H$  is the composition  $\Omega_d^4 S^4 \cong \Omega_d^4 \mathbf{H}P^1 \rightarrow \Omega_d^4 \mathbf{H}P^\infty \cong \Omega_d^4 BS^3 \cong \Omega_d^3 S^3$ . It is elementary that  $H$  induces a surjection in homology groups  $H_i$ , and it is a well known theorem in topology that  $E$  induces an isomorphism in  $H_i$  for  $i < d$ . Hence (from the diagram):

**Theorem [AJ].** *The map  $I : \mathcal{M}_d \rightarrow C_d$  induces a surjection in homology groups  $H_i$  for  $i < d$ .*

Thus, the philosophy of the proof is to identify a subset  $C_d(\mathbf{R}^4)$  of  $\mathcal{M}_d$  whose topological behaviour is better understood.

Atiyah and Jones conjectured that in fact the map  $I$  induces isomorphisms in  $H_i$  and  $\pi_i$  (for  $i$  in some computable range), in particular that the approximation principle holds for  $SU_2$ -instantons on  $S^4$ . (It should be noted, however, that the above diagram gives no information about homotopy groups, as  $\pi_1 C_d \cong \mathbf{Z}/2\mathbf{Z}$  whereas  $\pi_1 C_d(\mathbf{R}^4)$  is the symmetric group on  $d$  letters.) At the time of writing, this conjecture has still not been proved. However, it has almost acquired the status of a "folk theorem", as substantial progress has been made both via analysis and via topology. The analytical approach is due to Taubes, who has established the "stable" approximation principle for any  $M^4$  and  $G$  (see [Ta2] for a precise statement of this). The topological approach depends on a reformulation of the problem, which will be described in the next section. Using this approach, Graveson [Gr] has also proved the stable approximation principle, in the case  $M^4 = S^4$ . In the case  $M^4 = S^4, G = SU_2$ , the "expected" range of isomorphisms is  $i < d$ . The main evidence for this was provided by Boyer and Mann [BM], who showed that  $H_* \mathcal{M}_d$  has a product structure, which permits the construction of various nonzero homology

classes. As a consequence, the approximation principle cannot be valid in general beyond  $i = d$ .

### §1.5 Reformulation of instantons in terms of rational maps.

Atiyah [At1] and Donaldson [Do1] obtained the following remarkable description of the map  $I : \mathcal{M}_d(S^4, G) \rightarrow \mathcal{C}_d(S^4, G)$  (at least for  $G$  a compact simple classical Lie group):

**Theorem [At1],[Do1].** *The map  $I : \mathcal{M}_d(S^4, G) \rightarrow \mathcal{C}_d(S^4, G)$  is homotopy equivalent to the natural inclusion  $J : \text{Hol}_d^*(S^2, \Omega G) \rightarrow \text{Map}_d^*(S^2, \Omega G)$ .*

Here we use the notation  $\text{Map}_d(S^2, \Omega G)$  for the space of smooth maps  $f : S^2 \rightarrow \Omega G$  such that  $[f] \in \pi_2 \Omega G \cong \pi_3 G \cong \mathbb{Z}$  corresponds to  $d$ , and  $\text{Map}_d^*(S^2, \Omega G)$  for the subspace consisting of maps which satisfy in addition the basepoint condition  $f(\infty) = \delta$ , where  $\delta$  is a fixed basepoint in  $\Omega G$ . Similar definitions apply to  $\text{Hol}_d(S^2, \Omega G)$  and  $\text{Hol}_d^*(S^2, \Omega G)$ , using holomorphic maps instead of smooth maps; the complex structure of  $\Omega G$  being used here will be explained later. The identification of  $\mathcal{C}_d(S^4, G) \simeq \Omega_d^3 G$  with  $\text{Map}_d^*(S^2, \Omega G) = \Omega_d^2(\Omega G)$  is elementary, but the identification of  $\mathcal{M}_d(S^4, G)$  with  $\text{Hol}_d^*(S^2, \Omega G)$  uses the twistor description of instantons as holomorphic bundles.

This reformulation is useful only if one has a good understanding of  $\Omega G$ , of course. Fortunately, the theory of loop groups (see [PS]) provides such an understanding. Surprisingly,  $\Omega G$  behaves very much like a compact complex manifold; it is closely analogous to the familiar finite dimensional “generalized flag manifolds” such as  $\mathbb{C}P^n$ ,  $Gr_k(\mathbb{C}^n)$  or a complex flag manifold. In particular, holomorphic maps  $S^2 \rightarrow \Omega G$  are, in a certain sense (see lecture IV), given by rational functions.

### §1.6 A related example: harmonic maps.

The two Morse theoretic principles of 1.3 are supported by evidence both from physics and mathematics, not merely by comparison with the energy function  $E : \Omega M \rightarrow \mathbb{R}$ . A significant piece of mathematical evidence comes

from the theory of *harmonic maps* of a Riemann surface  $M^2$  into a compact Kähler homogeneous space  $G/H$  (see [EL1],[EL2]). One has an energy functional  $E : \text{Map}(M^2, G/H) \rightarrow \mathbb{R}$ ,  $f \mapsto \int_{M^2} \|df\|^2$ , whose critical points are by definition the harmonic maps  $M^2 \rightarrow G/H$ , and for which the absolute minima (in a suitable connected component of  $\text{Map}(M^2, G/H)$ ) are the holomorphic maps. A comparison of the harmonic maps problem with the Yang-Mills problem is given in [Bo], which illustrates why the former may be considered as a simple “model” of the latter. From our point of view, the most compelling evidence is this: for many spaces  $G/H$ , both Morse theoretic principles are valid. In the case  $G/H = \mathbb{C}P^n$ , the approximation principle is justified by the following theorem of Segal:

**Theorem [Se1].** *The inclusion  $\text{Hol}_d(S^2, \mathbb{C}P^n) \rightarrow \text{Map}_d(S^2, \mathbb{C}P^n)$  induces isomorphisms in homology groups  $H_i$  and homotopy groups  $\pi_i$  for  $i < (2n-1)d$ , and a surjection for  $i = (2n-1)d$ .*

The lowest index principle in the case  $G/H = \mathbb{C}P^n$  is justified by the well known “classification theorem” for harmonic maps, which describes how  $\text{Hol}(S^2, \mathbb{C}P^n)$  generates all harmonic maps by a sequence of simple operations (differentiation and orthogonalization). For the history and precise statement of this theorem we refer to [EL2].

Regarding compact Kähler manifolds  $G/H$  other than  $\mathbb{C}P^n$ , progress has been made essentially on a case by case basis. See [CM],[CS],[Gu1],[Ki],[MM1],[MM2] for the approximation principle, and [BR],[EL2] (and the references therein) for the lowest index principle.

### §1.7 Reformulation of harmonic maps in terms of rational maps.

Since complex Grassmannians and projective spaces can be embedded totally geodesically in  $U_n$ , harmonic maps  $S^2 \rightarrow Gr_k(\mathbb{C}^n)$  can be regarded as examples of harmonic maps  $S^2 \rightarrow U_n$ . So the study of harmonic maps  $S^2 \rightarrow U_n$ , may be regarded as a generalization of the problem considered in 1.6. As in the case of instantons, there is an (equally remarkable) reformulation in terms

of holomorphic maps  $S^2 \rightarrow \Omega U_n$ , due to Uhlenbeck [Uh] (see also [ZM], [ZS] for earlier work in this direction).

**Definition.** An *extended solution* is a map  $f : S^2 \rightarrow \Omega U_n$  which satisfies the conditions

$$f^{-1} \frac{\partial f}{\partial z}(z, \lambda) = \frac{1}{2} \left(1 - \frac{1}{\lambda}\right) A(z), \quad f^{-1} \frac{\partial f}{\partial \bar{z}}(z, \lambda) = \frac{1}{2} (1 - \lambda) B(z)$$

for some functions  $A, B$ . (Here,  $z \in S^2 = \mathbb{C} \cup \infty$  is the domain parameter, and  $\lambda \in S^1$  is the loop parameter.)

An extended solution is automatically holomorphic, for a map  $f$  is holomorphic if and only if the "tangent vectors"  $f^{-1} \partial f / \partial z$ ,  $f^{-1} \partial f / \partial \bar{z}$  belong respectively to  $\text{Map}(S^2, T_{1,0} \Omega U_n)$ ,  $\text{Map}(S^2, T_{0,1} \Omega U_n)$ ; it can be shown that this is so if and only if they extend respectively to the discs  $\infty \geq |\lambda| \geq 1$ ,  $0 \leq |\lambda| \leq 1$ , as holomorphic matrix valued functions in the interiors of the discs.

**Theorem [Uh].** Let  $\phi : S^2 \rightarrow U_n$  be harmonic. Let  $\gamma \in \Omega U_n$ . Then there exists a unique extended solution  $\Phi : S^2 \rightarrow \Omega U_n$  such that  $\Phi(z, -1) = \phi(z)$  for all  $z \in S^2$ , and  $\Phi(\infty, \lambda) = \gamma(\lambda)$  for all  $\lambda \in S^1$ . Conversely, if  $\Phi$  is an extended solution, then the map  $\phi$  defined by  $\phi(z) = \Phi(z, -1)$  is harmonic.

(This result carries over to any compact Lie group  $G$ , in fact.)

### §1.8 The Uhlenbeck theorems on harmonic maps.

Uhlenbeck proved a "finiteness" theorem for extended solutions: given any extended solution  $\Phi$ , there is a loop  $\gamma$  such that  $\gamma\Phi$  is of the form  $\sum_{i=-m}^m A_i(z) \lambda^i$ , i.e. is polynomial in  $\lambda, \lambda^{-1}$ . The least such value of  $m$  is called the *minimal uniton number* of  $\Phi$ . An extended solution of minimal uniton number  $m$  is called an *m-uniton*. She gave two applications of this, which we shall now review.

**Theorem A.** Any extended solution may be factored as a product  $\Phi = \gamma \Phi_1 \dots \Phi_m$ , where  $\gamma \in \Omega U_n$  and each  $\Phi_i$  is of the form  $\Phi_i(z, \lambda) = P_{f_i(z)} + \lambda P_{f_i(z)}^\perp$  for some

map  $f_i$  into a complex Grassmannian.

It is easy to check that a map of the form  $P_f + \lambda P_f^\perp$  is an extended solution if and only if  $f$  is holomorphic. However, it is not true in general that each factor  $\Phi_i$  in the theorem is itself an extended solution. All that can be said is that each subproduct  $\Phi_1 \dots \Phi_i$ ,  $1 \leq i \leq m$ , is an extended solution. Nevertheless, it is still true that the theorem expresses a general harmonic map in terms of "holomorphic data" (see [Wo]), so it can be regarded as a demonstration of the lowest index principle, at least for harmonic maps into Grassmannians.

The second application concerns the so called "dressing action" of loops  $\gamma \in \Omega U_n$  on extended solutions  $\Phi$ . The definition of this action, which comes from the theory of integrable systems (see [ZM],[ZS],[Wil]) is unfortunately rather complicated. Suppose that the loop  $\gamma$  is the restriction of a holomorphic  $GL_n(\mathbb{C})$ -valued function on a region containing the two small discs  $D_0, D_\infty$  given respectively by  $|\lambda| \leq \epsilon, |\lambda| \geq 1/\epsilon$ . Suppose that  $\Phi$  is the restriction of a holomorphic map  $A \rightarrow \Omega U_n$ , where  $A$  is the annulus given by  $\epsilon \leq |\lambda| \leq 1/\epsilon$ . Consider the function  $\gamma\Phi : C_0 \cup C_\infty \rightarrow \Omega U_n$  obtained by restricting  $\gamma, \Phi$  to the pair of circles  $C_0 \cup C_\infty = A \cap (D_0 \cup D_\infty)$ . Assume that this admits a factorization  $\gamma\Phi = \Phi_1\Phi_2$  of the same kind but in "reverse order", i.e. where  $\Phi_1$  extends to  $A$  and  $\Phi_2$  extends to  $D_0 \cup D_\infty$ . Then we define the action of  $\gamma$  on  $\Phi$  by

$$\gamma \circ \Phi = \Phi_1.$$

It can be shown that  $\gamma \circ \Phi$  is also an extended solution.

The biggest problem with this definition is that it is not clear when the required factorization can be performed (i.e., when the Riemann-Hilbert problem can be solved). However, Uhlenbeck showed that it can be done in the following situation:

**Theorem B.** *Let  $\gamma$  be the restriction of a  $GL_n(\mathbb{C})$ -valued rational function on  $\mathbb{C} \cup \infty$ , which is nonsingular on  $D_0 \cup D_\infty$ . Let  $\Phi$  be an extended solution. Then  $\gamma \circ \Phi$  is well defined.*

The proof uses a factorization theorem for “rational loops”, analogous to theorem A. It can be shown by direct calculation that the action of each factor is well defined, from which the theorem follows.

One might hope that this theorem could be used to give information about the space of harmonic maps  $S^2 \rightarrow U_n$  (sometimes referred to as the “moduli space” of harmonic maps). The definition of the action is rather difficult to work with, however, as the author discovered to his chagrin in [BG].

### §1.9 Further remarks.

There is in fact an explicit (if superficial) connection between the Yang-Mills problem and the harmonic maps problem, at least at the level of instantons. Atiyah [At1] shows that “axially symmetric  $G$ -instantons” on  $S^4$  correspond to holomorphic maps  $S^2 \rightarrow G/H$ , where  $G/H$  is a generalized flag manifold embedded in  $\Omega G$  (determined by the choice of axis).

Another related example — the Yang-Mills-Higgs problem—should be mentioned here. This leads to very interesting mathematics, although we shall not pursue it in these notes. The monopoles of the Yang-Mills-Higgs problem (analogous to the instantons of the Yang-Mills problem) are known to correspond to holomorphic maps from  $S^2$  to a generalized flag manifold (see [Do3],[Hu2],[HM]). In this case, the Morse theory has been studied by Taubes [Ta3] and found to be better behaved than in the Yang-Mills situation. For example, in the case of  $SU_2$ -monopoles, which correspond to holomorphic maps from  $S^2$  to  $S^2$ , he was able to obtain a proof of the theorem of Segal described above in 1.6.

As a general survey on the material of this lecture, as well as on monopoles, we recommend the article of Donaldson [Do2].

## Lecture II: The Mitchell-Segal filtration

### §2.1 The Grassmannian model of $\Omega U_n$ .

Let  $e_1, \dots, e_n$  be an orthonormal basis of  $\mathbb{C}^n$ . Let  $H$  be the Hilbert space  $L^2(S^1, \mathbb{C}^n) = \langle \lambda^i e_j \mid i \in \mathbb{Z}, j = 1, \dots, n \rangle$ , and let  $H_+$  be the subspace

$\langle \lambda^i e_j \mid i \geq 0, j = 1, \dots, n \rangle$ . The group  $\Omega U_n$  acts naturally on  $H$  by multiplication, and we have a map from  $\Omega U_n$  to the Grassmannian  $\text{Grass}(H)$  of all closed linear subspaces of  $H$ , given by  $\gamma \mapsto \gamma H_+ = \{\gamma f \mid f \in H_+\}$ . It is easy to see that this map is injective. Regarding the image, one has:

**Theorem [PS].** *The image of the map  $\Omega U_n \rightarrow \text{Grass}(H)$  is the subspace  $\text{Gr}_\infty(H)$  of  $\text{Grass}(H)$  consisting of linear subspaces  $W$  which satisfy*

- (1)  $\lambda W \subseteq W$ ,
- (2) the orthogonal projections  $W \rightarrow H_+$  and  $W \rightarrow (H_+)^\perp$  are respectively Fredholm and Hilbert Schmidt, and
- (3) the images of the orthogonal projections  $W^\perp \rightarrow H_+$  and  $W \rightarrow (H_+)^\perp$  consist of smooth functions.

Moreover, if  $\gamma \in \Omega U_n$  and  $W = \gamma H_+$ , then  $\deg(\det \gamma)$  is minus the index of the orthogonal projection operator  $W \rightarrow H_+$ .

This is known as the "Grassmannian model of  $\Omega U_n$ ". (Similar models exist for other differentiability classes of loops — see [PS].) This theorem is proved by showing that the (unbased) loop group  $\Lambda U_n = \text{Map}(S^1, U_n)$  acts transitively on  $\text{Gr}_\infty(H)$ , with isotropy subgroup  $U_n$ . It follows easily from this that the complex group  $\Lambda GL_n(\mathbb{C})$  also acts transitively; the isotropy subgroup is the subgroup  $\Lambda^+ GL_n(\mathbb{C})$  consisting of loops which are the boundary values of holomorphic maps  $\{\lambda \mid |\lambda| < 1\} \rightarrow GL_n(\mathbb{C})$ . Thus

$$\Omega U_n \cong \Lambda U_n / U_n \cong \Lambda GL_n(\mathbb{C}) / \Lambda^+ GL_n(\mathbb{C}).$$

This is analogous to the description of the ordinary Grassmannian  $\text{Gr}_k(\mathbb{C}^n)$  as a homogeneous space either of  $U_n$  or of  $GL_n(\mathbb{C})$ . Hence, like  $\text{Gr}_k(\mathbb{C}^n)$ , the loop group  $\Omega U_n$  acquires a natural complex structure (as a quotient of two complex Lie groups).

The "algebraic loop group" is defined by:

**Definition.**  $\Omega_{\text{alg}} U_n = \{\gamma \in \Omega U_n \mid \gamma(\lambda) \text{ is polynomial in } \lambda, \lambda^{-1}\}.$

The following Grassmannian model for  $\Omega_{\text{alg}} U_n$  may be deduced from the theorem (see [Pr] for a self-contained exposition of this):

**Corollary.** *Under the map  $\Omega U_n \rightarrow \text{Grass}(H)$ , the image of  $\Omega_{\text{alg}} U_n$  is the subspace  $\text{Gr}_{\text{alg}}(H)$  of  $\text{Grass}(H)$  consisting of linear subspaces  $W$  which satisfy*

- (1)  $\lambda W \subseteq W$  and
- (2)  $\lambda^k H_+ \subseteq W \subseteq \lambda^{-k} H_+$  for some  $k$ .

Moreover, if  $\gamma \in \Omega_{\text{alg}} U_n$  and  $W = \gamma H_+$ , then  $\deg(\det \gamma) = \frac{1}{2}(\dim \lambda^{-k} H_+ / W - \dim W / \lambda^k H_+)$ .

If we define

$$\Lambda_{\text{alg}} Gl_n(\mathbb{C}) = \{\gamma \in \Lambda Gl_n(\mathbb{C}) \mid \gamma(\lambda), \gamma(\lambda)^{-1} \text{ are polynomial in } \lambda, \lambda^{-1}\}$$

then we obtain the identifications

$$\Omega_{\text{alg}} U_n \cong \Lambda_{\text{alg}} U_n / U_n \cong \Lambda_{\text{alg}} Gl_n(\mathbb{C}) / \Lambda_{\text{alg}}^+ Gl_n(\mathbb{C}),$$

where  $\Lambda_{\text{alg}} U_n, \Lambda_{\text{alg}}^+ Gl_n(\mathbb{C})$  are defined in the obvious way.

The importance of  $\Omega_{\text{alg}} U_n$  is that it is a “good approximation” to  $\Omega U_n$ , in particular it is homotopy equivalent to  $\Omega U_n$  (see [Pr], [PS], [Mi2]), yet it is much simpler, being the union of a sequence of *finite dimensional complex projective varieties* (indexed by  $k$ ). A similar statement holds for  $\Omega_{\text{alg}} SU_n$  and  $\Omega SU_n$ .

## §2.2 The Mitchell-Segal filtration.

Mitchell [Mi1] and Segal [Se2] introduced the following subspaces of  $\text{Gr}_{\text{alg}}(H), \Omega_{\text{alg}} U_n$ :

**Definition.**

- (1)  $F_k = \{W \in \text{Grass}(H) \mid H_+ \subseteq W \subseteq \lambda^{-k} H_+, \lambda W \subseteq W, \dim W / H_+ = k\}$
- (2)  $M_k = \{\gamma \in \Omega U_n \mid \gamma(\lambda) \text{ is polynomial in } \lambda^{-1}, \deg(\det \gamma) = -k\}$

From the theorem, it is easy to see that  $F_k$  is mapped diffeomorphically to  $M_k$  under the identification  $\Omega U_n \rightarrow \text{Gr}_{\text{alg}}(H)$ . But one should not use this

identification too casually, as  $F_k$  and  $M_k$  reflect quite different properties of the loop group.

As  $\Omega_{\text{alg}} SU_n$  is equal to the identity component of  $\Omega_{\text{alg}} U_n$ , it is clear that

$$\Omega_{\text{alg}} SU_n = \bigcup_{k \geq 0} \lambda^k M_{kn}.$$

We call this the *Mitchell-Segal filtration*.

Usually we shall work with  $F_k$ , converting to  $M_k$  only when it is convenient to do so. From the definition, we see that  $F_k$  is an algebraic subvariety of the Grassmannian  $Gr_k(\mathbb{C}^{kn})$ . More precisely, if we make the identification  $\mathbb{C}^{kn} \cong \lambda^{-k} H_+ / H_+ = \langle [\lambda^{-i} e_j] \mid 1 \leq i \leq k, j = 1, \dots, n \rangle$ , then

$$F_k \cong \{E \in Gr_k(\mathbb{C}^{kn}) \mid NE \subseteq E\},$$

where  $N$  is the nilpotent operator on  $\mathbb{C}^{kn}$  given by multiplication by  $\lambda$ . Obviously  $F_0$  is a point, and  $F_1 \cong \mathbb{C}P^{n-1}$ . But for  $k \geq 2$ ,  $F_k$  is a *singular* variety. Varieties of this type have been studied from the point of view of algebraic geometry, and have various nice properties. For example, they admit a Schubert cell decomposition analogous to the Schubert cell decomposition of a Grassmannian (see [HS] and §3 of [GP] for this, and for further references). On the other hand, the Grassmannians  $Gr_{\text{alg}}(H), Gr_{\infty}(H)$  also admit cell decompositions of the Schubert type (see [PS])— and it turns out that the intersections of these with  $F_k$  give the Schubert cell decomposition of  $F_k$ . Mitchell [Mi1] used this in order to relate the homology of  $F_k$  to that of the loop group.

The group  $Gl_{kn}(\mathbb{C})$  acts transitively on  $Gr_k(\mathbb{C}^{kn})$ . Its subgroup

$$G_k = \{X \in Gl_{kn}(\mathbb{C}) \mid XN = NX\},$$

consisting of transformations which commute with  $N$ , acts on  $F_k$ . Although  $G_k$  does not act transitively on  $F_k$ , it has one open dense orbit. The orbits may be described using the Jordan Normal Form of a nilpotent transformation (on a finite dimensional complex vector space): recall that this expresses the transformation (up to similarity) as a sum of cyclic transformations, of lengths

$k_1, \dots, k_r$ . Moreover, the invariants  $k_1, \dots, k_r$  determine the transformation, up to similarity.

**Proposition.** *Two elements  $E, E' \in F_k$  are in the same  $G_k$ -orbit if and only if the nilpotent transformations  $N|_E, N|_{E'}$  have the same invariants.*

**Proof:** If  $N|_E, N|_{E'}$  have the same invariants  $k_1, \dots, k_r$ , we can choose bases  $\{N^i x_j\}, \{N^i x'_j\}$  of  $E, E'$  with  $1 \leq j \leq r, 0 \leq i \leq k_j - 1$ . These may be extended to bases  $\{N^i y_j\}, \{N^i y'_j\}$  of  $\mathbb{C}^{kn}$  with  $1 \leq j \leq n, 0 \leq i \leq k - 1$ . (This is elementary; see the proof of lemma 4.3 of [GP].) The linear transformation defined by  $X(N^i y_j) = N^i y'_j$  is then an element of  $G_k$  taking  $E$  to  $E'$ . Conversely, if  $E$  and  $E'$  are in the same  $G_k$ -orbit, it is obvious that  $N|_E$  and  $N|_{E'}$  have the same invariants. ■

The proposition was first noted by Mitchell [Mil], in the following form. We have  $\mathbb{C}^{kn} \cong \lambda^{-k}H_+/H_+$ , which has the structure of an  $A_k$ -module of rank  $n$ , where  $A_k$  is the truncated polynomial ring  $\mathbb{C}[\lambda]/(\lambda^k)$  (a principle ideal domain). An element  $E \in F_k$  defines a submodule of  $\lambda^{-k}H_+/H_+$ . The fundamental theorem on finitely generated modules over a P.I.D. says that  $E$  decomposes into a sum of cyclic modules, and that the isomorphism class of  $E$  is determined by the "type" of this decomposition. The group  $Gl_n(A_k)$  acts naturally on  $\lambda^{-k}H_+/H_+$ , and the orbits of this action give the decomposition of  $F_k$  into isomorphism classes of  $A_k$ -modules.

It is well known that the Schubert cell decomposition of  $Gr_k(\mathbb{C}^{kn})$  arises from a Morse function. The decomposition of  $F_k$  also has a Morse theoretic interpretation: it arises from taking the intersection with  $F_k$  of a decomposition of  $Gr_k(\mathbb{C}^{kn})$  given by a certain Morse-Bott function. For the details of this we refer to [Ri]; see also [Ko].

Turning now to  $M_k$ , we shall see that this is useful in situations where the group structure (of the loop group) plays a role. For example,  $M_k$  has the following simple description:

**Proposition [Se2].** Any  $\gamma \in M_k$  has a factorization into loops of the form  $\gamma_V(\lambda) = P_V^\perp + \lambda^{-1}P_V$ , where  $V$  is a subspace of  $\mathbb{C}^n$  and  $P_V, P_V^\perp$  denote the orthogonal projections onto  $V, V^\perp$ . That is, given  $\gamma$ , there exist subspaces  $V_1, \dots, V_l$  of  $\mathbb{C}^n$  with  $\gamma(\lambda) = (P_{V_1}^\perp + \lambda^{-1}P_{V_1}) \dots (P_{V_l}^\perp + \lambda^{-1}P_{V_l})$ .

Since  $\deg(\det P_V^\perp + \lambda^{-1}P_V) = -\dim V$ , we must have  $\sum_{i=1}^l \dim V_i = k$ , so  $l \leq k$ .

This factorization may be proved directly, by an induction argument (see [Cr]), but it is instructive to give a proof using  $F_k$ , as in [Se2]. Suppose we have a flag

$$H_+ \subseteq V_{(1)} \subseteq V_{(2)} \subseteq \lambda^{-k}H_+$$

with  $\lambda V_{(i)} \subseteq V_{(i)}$  for  $i = 1, 2$ . By the Grassmannian model, we have  $V_{(i)} = \gamma_i H_+$  for some  $\gamma_i \in M_{k_i}$ , where  $k_i = \dim V_{(i)}/H_+$ . Since  $\lambda$  induces a nilpotent transformation of  $V_{(2)}/V_{(1)}$ , we must have  $\lambda^r V_{(2)} \subseteq V_{(1)}$  for some  $r$ .

**Lemma [Se2].** Flags  $H_+ \subseteq V_{(1)} \subseteq V_{(2)} \subseteq \lambda^{-k}H_+$  with  $V_{(i)} \in F_{k_i}$  are in one to one correspondence with factorizations  $\gamma_2 = \gamma_1 \delta$  with  $\gamma_i \in M_{k_i}$ .

**Proof:** It is clear that a factorization gives rise to a flag. Conversely, given a flag of the above type, we have  $\lambda^l \gamma_2 H_+ \subseteq \gamma_1 H_+ \subseteq \gamma_2 H_+$ . From this it follows that  $\gamma_1^{-1} \gamma_2$  must be polynomial in  $\lambda^{-1}$  of degree less than or equal to  $l$ . ■

More generally, this shows that factorizations  $\gamma = \gamma_1 \dots \gamma_l$  with each  $\gamma_i$  linear in  $\lambda^{-1}$  correspond to flags  $H_+ \subseteq W_{(1)} \dots \subseteq W_{(l)} = W = \gamma H_+$  with  $\lambda W_{(i)} \subseteq W_{(i-1)}$  for all  $i$ , where  $W_{(i)} = \gamma_1 \dots \gamma_i$ .

Given  $W \in F_k$ , it is easy to produce such a flag (and hence complete the proof of the proposition). We shall make the following choice:

**Definition.** The canonical flag of  $W \in F_k$  is the flag

$$H_+ \subseteq \lambda^{k-1}W + H_+ \subseteq \lambda^{k-2}W + H_+ \subseteq \dots \subseteq \lambda W + H_+ \subseteq W \subseteq \lambda^{-k}H_+.$$

(It is obvious that this flag has the required properties.) We shall refer to the

corresponding factorization of  $\gamma$  as the *canonical factorization*.

Another manifestation of the group structure is the natural map

$$M_{k-i} \times M_i \rightarrow M_k,$$

given simply by multiplication of loops. We shall regard this in the following way: each point of  $M_{k-i}$  determines an embedding of  $M_i$  in  $M_k$ . Given  $\delta \in M_{k-i}$ , we have the embedding  $\delta : M_i \rightarrow M_k$ ,  $\gamma \mapsto \delta\gamma$ . The corresponding multiplication  $F_{k-i} \times F_i \rightarrow F_k$  will be written  $(V, W) \mapsto V.W$ . For  $V \in F_{k-i}$ , the subspace  $V.F_i$  of  $F_k$  is given by  $V.F_i = \{W \in F_k \mid V \subseteq W\}$ . (This is a direct consequence of the lemma.)

Observe that the embedding  $\lambda^j M_i \subseteq M_{i+jn}$ , which appears in the Mitchell-Segal filtration, corresponds to the embedding  $C^{jn}.F_i \subseteq F_{i+jn}$  where  $C^{jn} = \text{Ker } N^j$ .

### §2.3 Further properties of the Mitchell-Segal filtration.

We shall give some further information on the spaces  $F_k$ . From now on (until lecture IV), however, we shall assume that  $n = 2$ . Analogous properties hold for arbitrary  $n$ , but are a little more complicated.

Throughout this section we shall consider  $F_k$  to be the set of  $N$ -invariant elements of  $Gr_k(\mathbb{C}^{kn})$ , where  $\mathbb{C}^{kn}$  is identified with  $\lambda^{-k}H_+/H_+$ , and where  $N$  is given by multiplication by  $\lambda$ . The canonical flag of  $E \in F_k$  is given by  $\{0\} \subseteq N^{k-1}E \subseteq N^{k-2}E \subseteq \dots \subseteq NE \subseteq E$ . Unless stated otherwise, we shall assume that  $k$  is even.

**Lemma.** *The canonical flag of  $E$  is of the form*

$$\{0\} \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E_{i-1} \subseteq E_i \subseteq E_{i+2} \subseteq \dots \subseteq E_{k-2} \subseteq E_k = E$$

for some  $i$ , where  $\dim E_j = j$ .

**Proof:** Since  $N$  is nilpotent and  $\text{Ker } N = \mathbb{C}^2 = \langle \lambda^{-1}e_1, \lambda^{-1}e_2 \rangle$  has dimension 2, we have  $\dim N^i E = \dim N^{i-1} E - t$ , where  $t = 1$  if  $\mathbb{C}^2 \not\subseteq N^{i-1} E$  and  $t = 2$

if  $C^2 \subseteq N^{i-1}E$ . If  $C^2 \subseteq N^iE$  for some  $i$ , then we have  $C^2 \subseteq N^jE$  for all  $j \geq i$ . ■

The (even) integer  $i$  in the canonical flag of  $E$  above will be called the *height* of  $E$ .

**Definition.** Let  $F_k^{(i)}$  denote the subset of  $F_k$  consisting of elements of height  $i$ .

By Mitchell's observation,  $F_k^{(i)}$  constitutes a single  $G_k$ -orbit. It consists of those elements  $E$  of  $F_k$  for which  $C^{k-i} \subseteq E$ ,  $C^{k-i+2} \not\subseteq E$ , where  $C^j$  denotes  $\text{Ker } N^{j/2}$ . The space  $F_k^{(i)}$  has the structure of a complex vector bundle of rank  $i-1$  over  $CP^1$ , the projection map being given by  $E \mapsto E_1$ . We have  $F_k = F_k^{(k)} \cup C^2.F_{k-2}$  (disjoint union).

From these remarks it should be clear that  $F_k$  is closely analogous to  $CP^k$ , which has a similar algebraic/Morse theoretic decomposition. In fact  $F_k$  and  $CP^k$  have isomorphic cohomology groups, but their cohomology rings differ. As noted in [Mil],  $F_k$  is the  $2k$ -skeleton of  $\Omega SU_2$ , whereas  $CP^k$  is the  $2k$ -skeleton of  $CP^\infty$ . Despite this analogy,  $\text{Hol}_d^*(S^2, \Omega SU_2)$  behaves rather differently to  $\text{Hol}_d^*(S^2, CP^\infty)$ . For example, from 1.2 and 1.5 we see that the former is a manifold of (real) dimension  $8d$ ; on the other hand it is easy to see that the latter is infinite dimensional!

## §2.4 The space $\text{Hol}_d(S^2, F_k)$ .

From our remarks on the topology of  $F_k$ , it follows that  $\pi_2 F_k \cong \pi_2 \Omega SU_2 \cong \mathbb{Z}$  (for  $k \neq 0$ ). So the components of  $\text{Map}_d(S^2, F_k)$  are indexed by the "degree"  $[f] \in \pi_2 F_k \cong \mathbb{Z}$  of a map  $f \in \text{Map}_d(S^2, F_k)$ . We denote the  $d$ -th component by  $\text{Map}_d(S^2, F_k)$ . Similarly,  $\text{Hol}_d(S^2, F_k)$  is the space of holomorphic maps of degree  $d$ .

Let  $\mathcal{E}_k$  be the tautologous  $k$ -plane bundle on  $F_k$ . For  $f \in \text{Hol}_d(S^2, F_k)$ , let  $\mathcal{F} = f^* \mathcal{E}_k$ .

**Definition.** The *canonical flag* of  $f \in \text{Hol}_d(S^2, F_k)$  is the sequence  $\{0\} \subseteq N^{k-1}\mathcal{F} \subseteq N^{k-2}\mathcal{F} \subseteq \dots \subseteq N\mathcal{F} \subseteq \mathcal{F}$ .

A priori,  $N^i \mathcal{F}$  defines a bundle only over a dense open subset of  $S^2$ , as  $\dim N^i f(z)$  is not necessarily independent of  $z$ . However, because  $f$  is holomorphic, and its domain is  $S^2$ , the bundle may in fact be extended to a holomorphic bundle on  $S^2$ . The notation  $N^i \mathcal{F}$  refers to this extended bundle. Thus  $N^i f(z) \subseteq (N^i \mathcal{F})(z)$ , with equality except possibly for a finite number of points  $z \in S^2$ .

By the lemma of 2.3, the canonical flag of  $\mathcal{F}$  is of the form

$$\{0\} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_{i-1} \subseteq \mathcal{F}_i \subseteq \mathcal{F}_{i+2} \subseteq \dots \subseteq \mathcal{F}_{k-2} \subseteq \mathcal{F}_k = \mathcal{F}$$

for some  $i$ , where  $\mathcal{F}_j$  has rank  $j$ . As in 2.3, we call  $i$  the *height* (of  $\mathcal{F}$ , or of  $f$ ).

**Definition.** Let  $\text{Hol}_d^{(i)}(S^2, F_k)$  denote the subset of  $\text{Hol}_d(S^2, F_k)$  consisting of elements of height  $i$ .

The degree  $d$  of  $f \in \text{Hol}_d(S^2, F_k)$  may be interpreted as  $-c_1(\mathcal{F})$ , and it is necessarily nonnegative, as  $\mathcal{F}$  is a holomorphic subbundle of the trivial bundle  $S^2 \times \mathbb{C}^{kn}$ . Extending this, we have:

**Definition.** Let  $\{\mathcal{F}_j\}$  be the canonical flag of  $f \in \text{Hol}_d^{(i)}(S^2, F_k)$ . We define  $d_j = -c_1(\mathcal{F}_j)$  ( $\geq 0$ ), and  $e_j = d_j - d_{j-1}$  (with  $d_0 = 0, d_i = d$ ).

Thus,  $\text{Hol}_d^{(i)}(S^2, F_k)$  may be decomposed according to the multidegree  $(d_1, \dots, d_i)$  of  $f \in \text{Hol}_d^{(i)}(S^2, F_k)$ . (This is not a decomposition into path components.)

The next lemma, concerning the bundle  $N^{-1}\mathcal{F}_j/\mathcal{F}_j$ , will be crucial for our results on  $\text{Hol}_d(S^2, F_k)$  in the next lecture. Note that this bundle is trivial as a smooth bundle, as multiplication by  $f_j^{-1}$  gives an isomorphism  $N^{-1}\mathcal{F}_j/\mathcal{F}_j \rightarrow N^{-1}H_+/H_+$ , where  $\mathcal{F}_j(z) = f_j(z)H_+$ .

**Lemma.** Let  $f \in \text{Hol}_d^{(i)}(S^2, F_k)$ . For  $1 \leq j \leq i-1$  we have  $e_j \leq e_{j+1}$ , and there is a holomorphic splitting  $N^{-1}\mathcal{F}_j/\mathcal{F}_j \cong \mathcal{L} \oplus \mathcal{L}'$ , where  $\mathcal{L} = N^{-1}\mathcal{F}_{j-1}/\mathcal{F}_j \cong \mathcal{O}(e_j)$ ,  $\mathcal{L}' \cong \mathcal{O}(-e_j)$ .

**Proof:** We proceed by induction on  $j$ , using the exact sequences

$$\{0\} \rightarrow \mathcal{F}_{j+1}/\mathcal{F}_j \rightarrow N^{-1}\mathcal{F}_j/\mathcal{F}_j \rightarrow N^{-1}\mathcal{F}_j/\mathcal{F}_{j+1} \rightarrow \{0\} \quad (1)$$

$$\{0\} \rightarrow N^{-1}\mathcal{F}_{j-1}/\mathcal{F}_j \rightarrow N^{-1}\mathcal{F}_j/\mathcal{F}_j \rightarrow N^{-1}\mathcal{F}_j/N^{-1}\mathcal{F}_{j-1} \rightarrow \{0\}. \quad (2)$$

Since  $N^{-1}\mathcal{F}_j/\mathcal{F}_j$  is topologically trivial for all  $j$ , we have  $\mathcal{F}_{j+1}/\mathcal{F}_j \cong \mathcal{O}(-e_{j+1})$ ,  $N^{-1}\mathcal{F}_j/\mathcal{F}_{j+1} \cong \mathcal{O}(e_{j+1})$ , and  $N^{-1}\mathcal{F}_{j-1}/\mathcal{F}_j \cong \mathcal{O}(e_j)$ ,  $N^{-1}\mathcal{F}_j/N^{-1}\mathcal{F}_{j-1} \cong \mathcal{O}(-e_j)$ .

For  $j = 1$ , sequence (2) is  $\{0\} \rightarrow \mathcal{O}(d_1) \rightarrow N^{-1}\mathcal{F}_1/\mathcal{F}_1 \rightarrow \mathcal{O}(-d_1) \rightarrow \{0\}$ , and this splits holomorphically as the extension group is  $H^0(\mathcal{O}(-2d_1-2)) = 0$ . From sequence (1) we see that  $N^{-1}\mathcal{F}_1/\mathcal{F}_1 \cong \mathcal{O}(e_1) \oplus \mathcal{O}(-e_1)$  has a subbundle  $\mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{O}(-e_2)$ , which is possible only if  $e_2 \geq e_1$ . This starts the induction.

Now assume that the lemma holds for  $1, \dots, j-1$ . The extension group for sequence (2) is  $H^0(\mathcal{O}(-2e_j-2))$ , which is zero as we are assuming  $0 \leq e_1 \leq \dots \leq e_j$ . So sequence (2) splits holomorphically. From sequence (1), we see that  $N^{-1}\mathcal{F}_j/\mathcal{F}_j \cong \mathcal{O}(-e_j) \oplus \mathcal{O}(-e_j)$  has a subbundle  $\mathcal{F}_{j+1}/\mathcal{F}_j \cong \mathcal{O}(-e_{j+1})$ , which is possible only if  $e_j \leq e_{j+1}$ . This completes the inductive step. ■

Thus, the integers  $e_1, \dots, e_i$  satisfy the conditions  $0 \leq e_1 \leq \dots \leq e_i$  and  $\sum_{j=1}^i e_j = d$ .

### Lecture III: The approximation principle for rational maps and instantons

In this lecture we shall see how the approximation principle for maps  $S^2 \rightarrow \mathbb{C}P^n$  (see 1.6) may be extended to the case of maps  $S^2 \rightarrow F_k$ . The method of [Sel] is somewhat mysterious (especially from the point of view of Morse theory), so we shall begin in 3.1 by reviewing it in some detail. We shall do so in such a way that the generalization to maps  $S^2 \rightarrow F_k$  will appear naturally.

#### §3.1 The approximation principle for $\text{Hol}_d^+(S^2, \mathbb{C}P^n)$ .

Recall the theorem of Segal, which was stated in 1.6:

**Theorem.** *The inclusion  $I_d : \text{Hol}_d^+(S^2, \mathbb{C}P^n) \rightarrow \text{Map}_d^+(S^2, \mathbb{C}P^n)$  induces isomorphisms in homology groups  $H_i$  and homotopy groups  $\pi_i$  for  $i < (2n-1)d$ ,*

and an isomorphism for  $i = (2n-1)d$ , where  $*$  indicates any basepoint in  $CP^n$

It suffices to prove the theorem for a single basepoint, because  $CP^n$  is a complex homogeneous space. Let us choose a basepoint of the form  $[z_0; \dots; z_n]$ , where  $z_i \neq 0$  for all  $i$ . Then a based holomorphic map of degree  $d$  may be represented uniquely by a sequence of monic polynomials  $p_0, \dots, p_n$  of degree  $d$ , which have no common root. In fact, we can say that  $\text{Hol}_d^*(S^2, CP^n)$  is biholomorphically equivalent to the manifold  $Q_d^{(n)}(C)$  consisting of sequences of positive divisors  $\alpha_0, \dots, \alpha_n$  of degree  $d$  in  $C$ , which have no common point.

The first step in the proof is to establish the "approximation principle" for the fundamental groups:

**Proposition 1.** *For  $d \geq 1$ , the inclusion  $I_d: \text{Hol}_d^*(S^2, CP^n) \rightarrow \text{Map}_d^*(S^2, CP^n)$  induces an isomorphism on fundamental groups, both of which are isomorphic to  $\mathbb{Z}$  (if  $n = 1$ ) or to 0 (if  $n > 1$ ). Moreover, in the case  $n = 1$ , the action of  $\pi_1$  on the homology group  $H_i$  of the universal covering space of  $\text{Hol}_d^*(S^2, CP^n)$  is nilpotent for  $i \leq (2n-1)d$ .*

**Sketch proof:** Only the statements concerning  $\text{Hol}_d^*(S^2, CP^n)$  are not obvious. The space  $Q_d^{(n)}(C)$  is obtained from  $C^{d(n+1)}$  by removing a closed subvariety of complex codimension  $n$ , hence  $\pi_1 Q_d^{(n)}(C) \cong \pi_1 C^{d(n+1)} = 0$  if  $n > 1$ . If  $n = 1$ , the resultant gives a map  $R: Q_d^{(n)}(C) \rightarrow C^*$  which induces an isomorphism on  $\pi_1$  (a result of Jones, see [Se1]), so  $\pi_1 Q_d^{(n)}(C) \cong \mathbb{Z}$ . The universal covering space of  $Q_d^{(n)}(C)$  may be described as the subset of  $C \times Q_d^{(n)}(C)$  consisting of pairs  $(z, (p, q))$  with  $e^* R(p, q) = 1$ . The action of  $\pi_1 Q_d^{(n)}(C)$  on the homology of this space may be described explicitly, and shown to be nilpotent (up to dimension  $(2n-1)d$ ). ■

This will allow the approximation principle in homotopy to be deduced from the approximation principle in homology, to which we now turn.

The idea of the proof is to relate the inclusion map  $I_d: Q_d^{(n)}(C) \rightarrow \text{Map}_d^*(S^2, CP^n)$  to a more geometrical "scanning map", to which standard

methods of configuration space theory can be applied. This is done by replacing  $CP^n$  by a homotopy equivalent but "fattened up" space. Let  $X$  be any open subset of  $C$ . Let  $F(X)$  be the set of maps  $X \rightarrow CP^n$  which extend to holomorphic maps  $S^2 \rightarrow CP^n$ , with no coordinate polynomial identically zero. The space  $CP^n$  will be replaced by the space  $F(U)$ , where  $U$  is the open unit disc in  $C$ . The map  $e: F(U) \rightarrow CP^n$  given by evaluation at 0 is a homotopy equivalence, but there is an important difference between the two spaces: the natural action of  $(C^*)^n$  given by  $(u_1, \dots, u_n) \cdot [z_0; \dots; z_n] = [z_0; u_1 z_1; \dots; u_n z_n]$  is *free* in the case of  $F(U)$ . The quotient map  $p: F(U) \rightarrow F(U)/(C^*)^n$  assigns to a map  $f = g|_U$  the corresponding sequence  $[g]$  of divisors obtained from  $g$ .

Fix some  $\epsilon > 0$ . There is a natural map  $F(C) \rightarrow \text{Map}(C, F(U))$  which assigns to a map  $p \in F(C)$  and a point  $z \in C$  the map  $w \mapsto p(\epsilon w + z)$ . (In other words, take the restriction of  $p$  to the open disc  $D(z, \epsilon)$  of radius  $\epsilon$  and centre  $z$ , then identify  $D(z, \epsilon)$  with  $U$  in the canonical way.) The standard metric on  $C$  is used here, so that from the point of view of the standard metric on  $S^2 = C \cup \{\infty\}$ , the disc  $D(z, \epsilon)$  shrinks to a point as  $z \rightarrow \infty$ . Hence, this natural map extends to a map

$$F(C) \rightarrow \text{Map}^*(S^2, F(U))$$

where the basepoint of  $F(U)$  is taken as the constant function given by the basepoint of  $CP^n$ . On restricting to  $Q_d^{(n)}(C) \subseteq F(C)$ , one obtains a map

$$s_d: Q_d^{(n)}(C) \rightarrow \text{Map}_d^*(S^2, F(U)).$$

The inclusion  $I_d$  is evidently the composition

$$Q_d^{(n)}(C) \xrightarrow{s_d} \text{Map}_d^*(S^2, F(U)) \xrightarrow{\Omega^2 \epsilon} \text{Map}_d^*(S^2, CP^n).$$

This factorization of the inclusion map suggests the following construction. If  $X$  is any subspace of  $S^2 = C \cup \infty$ , and  $Y$  is a closed subspace of  $X$ , let  $Q^{(n)}(X, Y)$  denote the set of sequences  $(\alpha_0, \dots, \alpha_n)$  of positive divisors in  $X$ , modulo the equivalence relation which identifies two sequences if they agree on  $X - Y$ . There is a map  $Q_d^{(n)}(C) \rightarrow \text{Map}(C, Q^{(n)}(S^2, \infty))$  defined by

"scanning": given  $(\alpha_0, \dots, \alpha_n)$  in  $Q_d^{(n)}(\mathbb{C})$  and  $z \in \mathbb{C}$ , one obtains an element of  $Q^{(n)}(S^2, \infty)$  by taking those points of  $(\alpha_0, \dots, \alpha_n)$  which lie in  $D(z, \epsilon)$ ; this defines an element of  $Q^{(n)}(\bar{D}(z, \epsilon), \partial \bar{D}(z, \epsilon)) \cong Q^{(n)}(S^2, \infty)$ . Moreover, this map extends to a map

$$S_d : Q_d^{(n)}(\mathbb{C}) \rightarrow \text{Map}_d^+(S^2, Q^{(n)}(S^2, \infty))$$

where the basepoint of  $Q^{(n)}(S^2, \infty)$  is given by the sequence of empty divisors. (The suffix  $d$  indicates the component of  $\text{Map}^+(S^2, Q^{(n)}(S^2, \infty))$  which contains the image of  $S_d$ .) It is easy to verify that this scanning map  $S_d$  is the composition

$$Q_d^{(n)}(\mathbb{C}) \xrightarrow{s_d} \text{Map}_d^+(S^2, F(U)/(C^*)^n) \xrightarrow{\Omega^2 u} \text{Map}_d^+(S^2, Q^{(n)}(S^2, \infty))$$

where  $\bar{s}_d$  is the map induced by  $s_d$ , and where  $u$  assigns to the sequence of divisors  $[g] \in F(U)/(C^*)^n$  its intersection with  $U$ . We therefore have the following commutative diagram, by means of which the inclusion map  $I_d$  (the top row) is related to the scanning map  $S_d$  (the bottom row):

$$\begin{array}{ccccc} Q_d^{(n)}(\mathbb{C}) & \xrightarrow{s_d} & \text{Map}_d^+(S^2, F(U)) & \xrightarrow{\Omega^2 e} & \text{Map}_d^+(S^2, \mathbb{C}P^n) \\ = \downarrow & & \Omega^2 p \downarrow & & \\ Q_d^{(n)}(\mathbb{C}) & \xrightarrow{s_d} & \text{Map}_d^+(S^2, F(U)/(C^*)^n) & \xrightarrow{\Omega^2 u} & \text{Map}_d^+(S^2, Q^{(n)}(S^2, \infty)) \end{array}$$

We are aiming to show that the inclusion  $I_d = (\Omega^2 e) \circ s_d$  is a homology equivalence up to dimension  $(2n-1)d$ . Now, it is straightforward to show that  $\Omega^2 e$ ,  $\Omega^2 p$ , and  $\Omega^2 u$  are homotopy equivalences. Hence it suffices to show that the scanning map  $S_d = (\Omega^2 u) \circ \bar{s}_d$  is a homology equivalence up to dimension  $(2n-1)d$ .

The method proceeds in two stages. First one defines a "stabilized" map  $S = \lim_{d \rightarrow \infty} S_d$ , and shows that this is a homotopy equivalence (proposition 2 below). Then one shows that  $S_d$  approximates  $S$  in homology up to dimension  $(2n-1)d$  (proposition 3).

To define the stabilized space, replace  $Q_d^{(n)}(\mathbb{C})$  by the homeomorphic subspace  $Q_d^{(n)} = Q_d^{(n)}(\{ \text{Re } z < d \})$  consisting of sequences of divisors, all of

whose points satisfy the condition  $\text{Re } z < d$ . On choosing fixed distinct points  $x_0^{d+1}, \dots, x_n^{d+1}$  in the region  $d \leq \text{Re } z < d+1$ , one obtains an inclusion  $i_d: Q_d^{(n)} \rightarrow Q_{d+1}^{(n)}$  by adding  $x_0^{d+1}, \dots, x_n^{d+1}$  to the divisors  $\alpha_0, \dots, \alpha_n$ . The direct limit  $\lim_{d \rightarrow \infty} Q_d^{(n)}$  is then defined, and may be identified with the space  $\hat{Q}^{(n)}$  consisting of sequences  $\alpha_0, \dots, \alpha_n$  of positive divisors of infinite degree, with  $\deg(\alpha_i - \sum_{d=1}^{\infty} x_i^d)$  finite for all  $i$ . The component of  $\hat{Q}^{(n)}$  consisting of those sequences with  $\deg(\alpha_i - \sum_{d=1}^{\infty} x_i^d) = 0$  for all  $i$  will be denoted by  $\hat{Q}_0^{(n)}$ . There are maps  $j_d: \text{Map}_d^*(S^2, Q^{(n)}(S^2, \infty)) \rightarrow \text{Map}_{d+1}^*(S^2, Q^{(n)}(S^2, \infty))$  such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Q_d^{(n)} & \xrightarrow{i_d} & Q_{d+1}^{(n)} \\ I_d \downarrow & & I_{d+1} \downarrow \\ \text{Map}_d^*(S^2, Q^{(n)}(S^2, \infty)) & \xrightarrow{j_d} & \text{Map}_{d+1}^*(S^2, Q^{(n)}(S^2, \infty)) \end{array}$$

In the limit  $d \rightarrow \infty$  one obtains (up to homotopy) a map  $S: \hat{Q}_0^{(n)} \rightarrow \text{Map}_0^*(S^2, Q^{(n)}(S^2, \infty))$ .

**Proposition 2.** *S is a homotopy equivalence.*

**Sketch proof:** Let  $B$  be the unit square in  $\mathbb{R}^2 (\cong \mathbb{C})$  given by  $0 < x < 1$ ,  $0 < y < 1$ . Let  $(\eta_0, \dots, \eta_n)$  be a fixed  $(n+1)$ -tuple of positive divisors in  $B$  which have no common point, each of which converges to  $(\frac{1}{2}, 0)$  as  $d \rightarrow \infty$ . Then the stabilization  $\hat{Q}^{(n)}(B)$  of  $Q_d^{(n)}(B)$  may be defined with respect to  $(\eta_0, \dots, \eta_n)$ , and  $Q_d^{(n)}(B) \simeq Q_d^{(n)}$ ,  $\hat{Q}^{(n)}(B) \simeq \hat{Q}^{(n)}$ . One also has  $Q^{(n)}(\bar{B}, \partial \bar{B}) \simeq Q^{(n)}(S^2, \infty)$ . (To simplify notation we shall in future write  $Q^{(n)}(X, \partial X)$  instead of  $Q^{(n)}(\bar{X}, \partial \bar{X})$ .)

The scanning map  $S$  may be decomposed into a composition of "horizontal" and "vertical" scanning maps  $S^H$  and  $S^V$ , each of which will be shown to be a homotopy equivalence. In order to define  $S^H$  and  $S^V$ , let  $\{V_t \mid 0 < t < 1\}$  be the family of "vertical" rectangles in  $B$  defined by  $t - \epsilon_t < x < t + \epsilon_t$ ,  $0 < y < 1$ , where  $\epsilon_t > 0$  and  $\epsilon_t \rightarrow 0$  as  $t \rightarrow 0$  or as  $t \rightarrow 1$ . Let  $\{H_t \mid 0 < t < 1\}$  be the family of "horizontal" rectangles in  $B$  defined by  $0 < x < 1$ ,  $t - \epsilon_t < y < t + \epsilon_t$ . For any rectangle  $X$  we shall use

the notation  $\sigma X$  to denote the union of the sides of  $X$  which are parallel to the  $y$ -axis.

The map  $S$  is determined (up to homotopy) by the stabilization of the map  $Q_d^{(n)}(B) \times (0, 1) \times (0, 1) \rightarrow Q^{(n)}(B, \partial B)$ ,  $(C, t_1, t_2) \mapsto C \cap V_{t_1} \cap H_{t_2} \in Q^{(n)}(V_{t_1} \cap H_{t_2}, \partial(V_{t_1} \cap H_{t_2})) \cong Q^{(n)}(B, \partial B)$ . Let  $S_d^H : Q_d^{(n)}(B) \rightarrow \Omega Q^{(n)}(B, \sigma B)$  be the map determined by  $(C, t) \mapsto C \cap V_t \in Q^{(n)}(V_t, \sigma V_t) \cong Q^{(n)}(B, \sigma B)$ . Similarly let  $S^V : Q^{(n)}(B, \sigma B) \rightarrow \Omega Q^{(n)}(B, \partial B)$  be the map determined by  $(C, t) \mapsto C \cap H_t \in Q^{(n)}(B \cap H_t, \partial(B \cap H_t)) \cong Q^{(n)}(B, \partial B)$ . It suffices to show that  $S^V$  and the stabilization of  $S_d^H$  are homotopy equivalences.

We begin with  $S^V$ . Up to homotopy this may be defined by  $(C, t) \mapsto C \cap B_t$ , where  $B_t$  is the square given by  $0 < x < 1$ ,  $2t - 1 < y < 2t$ . Let  $B^*$  be the rectangle given by  $0 < x < 1$ ,  $-1 < y < 2$ . Consider the following two "restriction" maps:

$$\begin{aligned} (1) \quad & Q^{(n)}(B^*, \partial B^*) \rightarrow Q^{(n)}(B^*, \partial B^* \cup B) \cong Q^{(n)}(B, \partial B) \times Q^{(n)}(B, \partial B) \\ (2) \quad & \text{Map}([0, 1], Q^{(n)}(B, \partial B)) \rightarrow \text{Map}(\{0, 1\}, Q^{(n)}(B, \partial B)) \cong Q^{(n)}(B, \partial B) \times Q^{(n)}(B, \partial B). \end{aligned}$$

The first is a quasifibration, as one sees from the criterion of Dold and Thom. It is elementary that the second is a fibration. Moreover, there is a fibre preserving homotopy equivalence  $Q^{(n)}(B^*, \partial B^*) \rightarrow \text{Map}([0, 1], Q^{(n)}(B, \partial B))$  defined by scanning. Our map  $S^V$  is just the restriction of this map to the fibre over the configuration of empty divisors, hence it is a homotopy equivalence, as required.

The case of  $S_d^H$  is similar. This is homotopic to the map determined by  $(C, t) \mapsto C \cap \tilde{B}_t$ , where  $\tilde{B}_t$  is the square given by  $2t - 1 < x < 2t$ ,  $0 < y < 1$ . Let  $\tilde{B}^*$  be the rectangle given by  $-1 < x < 2$ ,  $0 < y < 1$ . Then we have restriction maps as follows:

$$\begin{aligned} (3) \quad & Q_d^{(n)}(\tilde{B}^*, \sigma \tilde{B}^*) \rightarrow Q^{(n)}(\tilde{B}^*, \sigma \tilde{B}^* \cup \tilde{B}) \cong Q^{(n)}(B, \sigma B) \times Q^{(n)}(B, \sigma B) \\ (4) \quad & \text{Map}([0, 1], Q^{(n)}(B, \sigma B)) \rightarrow \text{Map}(\{0, 1\}, Q^{(n)}(B, \sigma B)) \cong Q^{(n)}(B, \sigma B) \times Q^{(n)}(B, \sigma B). \end{aligned}$$

where  $Q_d^{(n)}(\tilde{B}^*, \sigma \tilde{B}^*)$  denotes the subset of  $Q^{(n)}(\tilde{B}^*, \sigma \tilde{B}^*)$  consisting of divisors whose intersections with  $B$  have degree  $d$ . Again we have a fibre preserving

map from (3) to (4), and our map  $S_d^H$  is the restriction of this to the fibre over the configuration of empty divisors. After stabilization, (3) becomes a quasifibration, and the fibre preserving map becomes a homotopy equivalence, so the result follows. ■

This completes the first stage. Now we come to the second stage, which involves showing that  $S_d$  approximates  $S$  in homology up to dimension  $(2n - 1)d$ .

**Proposition 3.** *The inclusion  $Q_d^{(n)} \rightarrow Q_{d+1}^{(n)}$  induces an isomorphism in homology groups  $H_i$  for  $i < (2n - 1)d$ , and a surjection for  $i = (2n - 1)d$ .*

**Sketch proof:** Denote the statement of the proposition by  $(A_d)$ . It will be proved by induction on  $d$ . Certainly  $(A_0)$  holds, so the induction begins. Let us assume  $(A_c)$  for  $0 \leq c < d$ , and attempt to prove  $(A_d)$ . Define  $P_{d,k}^{(n)}$  to be the set of  $(n+1)$ -tuples of divisors of degree  $d$ , all of whose points satisfy the condition  $\text{Rez} < d$ , and such that the divisors have at least  $k$  common points. Let  $X_{d,k}^{(n)} = P_{d,k}^{(n)} - P_{d,k+1}^{(n)} \cong Q_{d-k}^{(n)} \times \mathbb{C}^k$ . By Poincaré duality the statement  $(A_{d-k})$  is equivalent to

$$H_c^j X_{d,k}^{(n)} \rightarrow H_c^{j+2n+2} X_{d+1,k}^{(n)} \text{ is an isomorphism for } j > 3d - k, \quad (B_{d-k})$$

and a surjection for  $j = 3d - k$ ,

the map being that induced by the restriction to  $X_{d,k}^{(n)}$  of the open embedding  $P_{d,k}^{(n)} \times V_0 \times \dots \times V_n \rightarrow P_{d+1,k}^{(n)}$ , where each  $V_i$  is a small open neighbourhood of  $x_i^{d+1}$ . A subsidiary induction on  $k$  may now be carried out, to prove

$$H_c^j P_{d,k}^{(n)} \rightarrow H_c^{j+2n+2} P_{d+1,k}^{(n)} \text{ is an isomorphism for } j > 3d - k, \quad (C_k)$$

and a surjection for  $j = 3d - k$ .

The induction starts with  $k = d$ , as  $P_{d,d}^{(n)} \cong \mathbb{C}^d$  and  $P_{d+1,d}^{(n)}$  is of dimension  $2(d + n + 1)$ . The inductive step uses the diagram

$$\begin{array}{ccccccccc}
H_c^{j-1}P_{d,k+1}^{(n)} & \rightarrow & H_c^jX_{d,k}^{(n)} & \rightarrow & H_c^jP_{d,k}^{(n)} & \rightarrow & H_c^jP_{d,k+1}^{(n)} & \rightarrow & H_c^{j+1}X_{d,k}^{(n)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_c^{j+N-1}P_{d+1,k+1}^{(n)} & \rightarrow & H_c^{j+N}X_{d+1,k}^{(n)} & \rightarrow & H_c^{j+N}P_{d+1,k}^{(n)} & \rightarrow & H_c^{j+N}P_{d+1,k+1}^{(n)} & \rightarrow & H_c^{j+N+1}X_{d+1,k}^{(n)}
\end{array}$$

where the horizontal sequences are the exact sequences in cohomology with compact supports for the pairs  $(P_{d,k}^{(n)}, P_{d,k+1}^{(n)})$ ,  $(P_{d+1,k}^{(n)}, P_{d+1,k+1}^{(n)})$ , and the vertical maps are those induced by the open embeddings already described, with  $N = \dim V_0 \times \dots \times V_n = 2n + 2$ . If  $(C_{k+1})$  is true, then by  $(B_{d-k})$  and the five lemma, we obtain  $(C_k)$ . Hence, by downwards induction, we arrive at  $(C_1)$ . For  $k = 0$ , the map is an isomorphism for all  $j \geq 0$ , as  $P_{d,0}^{(n)} \cong C^{(n+1)d}$ . Finally, as  $P_{d,0}^{(n)} - P_{d,1}^{(n)} \cong Q_d^{(n)}$ , the desired statement  $(A_d)$  follows by one more application of the five lemma. ■

It follows from this (and the definition of  $S$ ) that  $S_d$  approximates  $S$  up to dimension  $(2n-1)d$ . This completes the second stage of the proof.

### §3.2 The approximation principle for $\text{Hol}_d^*(S^2, F_k)$ .

In this section we shall indicate the proof of the following approximation theorem for maps  $S^2 \rightarrow F_k$ .

**Theorem.** *The inclusion  $\text{Hol}_d^*(S^2, F_k) \rightarrow \text{Map}_d^*(S^2, F_k)$  induces isomorphisms in homology groups  $H_i$  and homotopy groups  $\pi_i$  for  $i < d$ , and an isomorphism for  $i = d$ , where  $*$  indicates any basepoint in  $F_k^{(k)}$ .*

The restriction on the type of basepoint comes from the fact that  $F_k$  is not homogeneous. In Segal's theorem, for  $CP^n$ , the choice of basepoint is irrelevant because  $GL_{n+1}(\mathbb{C})$  acts transitively on  $CP^n$ ; this also implies that the theorem for based maps is equivalent to the corresponding theorem for unbased maps. For  $F_k$ , we have seen that  $G_k$  acts transitively only on  $F_k^{(k)}$ , so the above theorem does not immediately imply the corresponding statement for maps with other basepoints, nor does it imply the corresponding statement for unbased maps.

**Sketch proof:** The proof rests upon an identification of  $\text{Hol}_d^*(S^2, F_k)$  with a certain space of divisors, generalizing the earlier identification  $\text{Hol}_d^*(S^2, CP^n) \cong Q_d^{(n)}(C)$ .

Given  $f \in \text{Hol}_d^*(S^2, F_k)$ , consider the canonical flag  $\{\mathcal{F}_i\}$ . Because of the basepoint condition,  $f$  is necessarily of height  $k$ , so we have  $1 \leq i \leq k$ . Thus  $\text{Hol}_d^*(S^2, F_k)$  can be identified with the space of flags  $\mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_k$  of holomorphic bundles, which satisfy:

- 1)  $\text{rank } \mathcal{F}_i = i$
- 2)  $N\mathcal{F}_i \subseteq \mathcal{F}_{i-1}$
- 3)  $\mathcal{F}_i(\infty) = E_i$  (where  $\{E_i\}$  is the canonical flag of the basepoint  $E$ )
- 4)  $c_1 \mathcal{F}_k = -d$ .

We claim that such a flag may be represented by a sequence  $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ , where  $(\alpha_i, \beta_i)$  is a pair of disjoint divisors in  $C$ . For by the lemma of 2.4,  $\mathcal{F}_{i+1}$  is specified relative to  $\mathcal{F}_1, \dots, \mathcal{F}_i$  by a (based) holomorphic line subbundle of the bundle  $N^{-1}\mathcal{F}_i/\mathcal{F}_i$ . We may define  $\alpha_{i+1}, \beta_{i+1}$  respectively to be the divisors of points in  $C$  at which  $\mathcal{F}_{i+1}/\mathcal{F}_i$  agrees with  $N^{-1}\mathcal{F}_{i-1}/\mathcal{F}_i$ ,  $N^{-1}\mathcal{F}_i/\mathcal{F}_i \oplus N^{-1}\mathcal{F}_{i-1}/\mathcal{F}_i$ . (With a suitable choice of the basepoint  $E$ ,  $\alpha_{i+1}$  and  $\beta_{i+1}$  are always finite divisors. This is analogous to choosing a basepoint  $[z_0; \dots; z_n]$  in  $CP^n$  with all coordinates nonzero.) By the lemma, we have  $\deg \alpha_i = e_i + e_{i-1}$ ,  $\deg \beta_i = e_i - e_{i-1}$ , where  $0 \leq e_1 \leq \dots \leq e_k$  and  $\sum_{i=1}^k e_i = d$ .

Let  $Q_d^k(C)$  be the set of all sequences  $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$  arising in this way. Then we have a bijection  $\text{Hol}_d^*(S^2, F_k) \rightarrow Q_d^k(C)$ . This is a homeomorphism if the natural topology of  $Q_d^k(C)$  is modified so that a point of  $\alpha_i$  is allowed to "coalesce" with a point of  $\beta_i$ , whence they give a (new) double point of  $\beta_{i+1}$ .

Propositions 1, 2 and 3 may now be generalized to the case of  $F_k$ . We shall list below the (few) new features that arise.

- (1) For  $k \geq 2$  and  $d \geq 2$ , the fundamental group of  $\text{Hol}_d^*(S^2, F_k)$  is  $\mathbb{Z}/2\mathbb{Z}$ .

This may be deduced from Jones' result (see [Se1]) that  $\pi_1 Q_d^{(1)}(C) \cong \mathbb{Z}$ , a generator being given by a loop  $(\alpha_t, \beta_t)$  which "moves a point of  $\alpha_0$  once around

a point of  $\beta_0^n$ . Let  $\tilde{Q}_d^k(C)$  denote the subset of  $Q_d^k(C)$  given by the conditions that  $\alpha_1, \dots, \alpha_{k-1}$  and  $\beta_1, \dots, \beta_{k-1}$  are all empty. Then  $\tilde{Q}_d^k(C) \cong Q_d^{(1)}(C)$ , and the inclusion map  $Q_d^{(1)}(C) \cong \tilde{Q}_d^k(C) \rightarrow Q_d^k(C)$  is surjective on fundamental groups, as (by Jones' method) any loop in  $Q_d^k(C)$  is homotopic to a loop in  $\tilde{Q}_d^k(C)$ . Consider a generator  $(\alpha_t, \beta_t)$  of  $\pi_1 \tilde{Q}_d^k(C)$ . From the coalescing rule, it follows that  $2(\alpha_t, \beta_t)$  is zero in  $\pi_1 Q_d^k(C)$ , at least if  $k \geq 2$  and  $d \geq 2$ , since a double point of  $\beta_k$  may be separated into a pair of distinct points of  $\alpha_{k-1}$  and  $\beta_{k-1}$ . As  $(\alpha_t, \beta_t)$  is not itself zero in  $\pi_1 Q_d^k(C)$ , we deduce that  $\pi_1 Q_d^k(C) \cong \mathbb{Z}/2\mathbb{Z}$ .

One has a scanning map  $S_d : Q_d^k(C) \rightarrow \text{Map}_d^+(S^2, Q^k(S^2, \infty))$ , which is related to the inclusion map  $I_d : Q_d^k(C) \rightarrow \text{Map}_d^+(S^2, F_k)$  as in the case of  $CP^n$ . To stabilize  $Q_d^k(C)$  we may replace it by  $Q_d^k = Q_d^k(\{\text{Re } z < d\})$ , and then define an inclusion  $Q_d^k \rightarrow Q_{d+1}^k$  by adding fixed (distinct) points  $x_{d+1}, y_{d+1}$  (in the region  $d \leq \text{Re } z < d+1$ ) to the divisors  $\alpha_k, \beta_k$ . We obtain a stabilized space  $\hat{Q}^k$  and a map  $S : \hat{Q}_0^k \rightarrow \text{Map}_0^+(S^2, Q^k(S^2, \infty))$ .

The analogues of propositions 2 and 3 are:

(2)  $S$  is a homotopy equivalence.

This is proved in just the same way as proposition 2.

(3) The inclusion  $Q_d^k \rightarrow Q_{d+1}^k$  induces an isomorphism in homology groups  $H_i$  for  $i < d$ , and a surjection for  $i = d$ .

This is proved by the method of proposition 3. The only difference is that  $Q_d^k(C)$  is not in general a manifold, so one must use a more general version of Poincaré duality. That this can be done is an observation of A. Kozłowski. ■

Full details of the above proof will appear elsewhere.

### §3.3 The theorem of Atiyah-Jones re-visited

The theorem we have just described gives immediately another proof of the theorem of [AJ] on  $SU_2$ -instantons on  $S^4$ . In fact, we obtain a result in homotopy as well as homology, and the range of dimensions can be made precise:

**Theorem.** *The map  $I : M_d \rightarrow C_d$  induces a surjection in homology groups  $H_i$  and homotopy groups  $\pi_i$  for  $i \leq d$ .*

**Proof:** Consider the following commutative diagram, in which all maps are the natural inclusions:

$$\begin{array}{ccc} \mathrm{Hol}_d^*(S^2, \Omega SU_2) & \longrightarrow & \mathrm{Map}_d^*(S^2, \Omega SU_2) \\ \uparrow & & \uparrow \\ \mathrm{Hol}_d^*(S^2, F_k) & \longrightarrow & \mathrm{Map}_d^*(S^2, F_k) \end{array}$$

The right hand vertical map is an equivalence in homology and homotopy up to dimension  $2k - 2$ , as  $F_k$  is the  $2k$ -skeleton of  $\Omega SU_2$ . We have just proved that the lower horizontal map is an equivalence up to dimension  $d$ . Hence, by taking  $k$  large, we see that the upper horizontal map is surjective in homology and homotopy up to dimension  $d$ . ■

The diagram used here should be compared to that used by Atiyah and Jones (see 1.4). It is a refinement of that diagram, because  $\mathrm{Hol}_d^*(S^2, F_k)$  is evidently a closer approximation to  $\mathrm{Hol}_d^*(S^2, \Omega SU_2)$  than the configuration space  $C_d(\mathbb{R}^4)$ . In particular,  $\mathrm{Hol}_d^*(S^2, F_k)$  has the "correct" fundamental group,  $\mathbb{Z}/2\mathbb{Z}$ , whereas  $\pi_1 C_d(\mathbb{R}^4) \cong \Sigma_d$ .

A similar phenomenon occurs in the case of maps from  $S^2$  to  $S^2$ . There is an inclusion  $E : C_d(\mathbb{R}^2) \rightarrow \mathrm{Map}_d^*(S^2, S^2)$  (the two dimensional "electric field map") which induces a homology equivalence up to dimension  $[d/2]$ . On fundamental groups, however, this map induces a surjection from the braid group on  $d$  strings to the group  $\mathbb{Z}$ . If  $C_d(\mathbb{R}^2)$  is replaced by  $\mathrm{Hol}_d^*(S^2, S^2)$ , Segal's theorem says that we get a closer approximation, namely an equivalence in homology and homotopy up to dimension  $d$ .

In order to obtain the isomorphisms conjectured by Atiyah and Jones (rather than surjections), it is necessary to study  $\mathrm{Hol}_d^*(S^2, \Omega SU_2)$  rather than  $\mathrm{Hol}_d^*(\Omega_{\mathrm{alg}} SU_2)$ . The main problem is to obtain a description of the space  $\mathrm{Hol}_d^*(S^2, \Omega SU_2)$  in terms of divisors. Our description of  $\mathrm{Hol}_d^*(S^2, \Omega_{\mathrm{alg}} SU_2)$  was based on the canonical flag, or canonical factorization, of a holomorphic map  $S^2 \rightarrow \Omega_{\mathrm{alg}} SU_2$ . In 4.2 of the next lecture we shall see that there is a

similar flag or factorization for a holomorphic map  $S^2 \rightarrow \Omega SU_2$ , and it seems likely that this will give an approach to the Atiyah-Jones conjecture.

## Lecture IV: Applications to harmonic maps

In this lecture we shall denote  $F_k, M_k$  and  $G_k$  by  $F_{n,k}, M_{n,k}$  and  $G_{n,k}$ , as we shall need to distinguish different values of  $n$ .

### §4.1 Harmonic maps and the Grassmannian model.

The Grassmannian model provides a natural context for the theorems of Uhlenbeck on harmonic maps, as was pointed out by Segal [Se2]. Let  $\Phi : S^2 \rightarrow \Omega U_n$  be a smooth map. By the Grassmannian model, this may be identified with a map  $W : S^2 \rightarrow Gr_\infty(H)$ , where  $W(z) = \Phi(z)H_+$ . The conditions for  $\Phi$  to be an extended solution are

$$\frac{\partial}{\partial \bar{z}} W \subseteq W, \quad \frac{\partial}{\partial z} W \subseteq \lambda^{-1} W.$$

The first is simply the condition that  $\Phi$  is holomorphic; the second is a kind of "horizontality" condition. Segal's main result can be stated in the following form:

**Theorem [Se2].** *Let  $\Phi : S^2 \rightarrow \Omega U_n$  be holomorphic. Then there exists some loop  $\gamma \in \Omega U_n$  and some complex polynomial  $p$  such that*

$$H_+ \subseteq \tilde{W} \subseteq p(\lambda)^{-1} H_+$$

where  $\tilde{W} = \hat{\Phi} H_+$  and  $\hat{\Phi} = \gamma \Phi$ .

Moreover, if  $\Phi$  is an extended solution,  $\gamma$  can be chosen so that  $p(\lambda) = \lambda^m$  for some nonnegative integer  $m$ . This gives another proof of the finiteness theorem of Uhlenbeck. If  $m_\Phi$  is the minimal uniton number of  $\Phi$ , we have  $m_\Phi \leq m$ . (Segal showed that  $m \leq n - 1$ , hence  $n - 1$  is an upper bound for the minimal uniton number, a result which was also obtained by Uhlenbeck).

There are two basic integers associated with the extended solution  $\Phi$ , in addition to  $m_\Phi$ . First, the degree  $d = \deg \Phi$  of  $\Phi$  (i.e. the class  $[\Phi] \in \pi_2 \Omega U_n \cong$

$Z$ ) is known to be (up to normalization) the *energy* of the corresponding harmonic map. (This is a result of Valli [Va1]; see also [Se2]). Second, the component of  $\Omega U_n$  containing the image of  $\Phi$  (i.e. the class  $[\Phi(z)] \in \pi_0 \Omega U_n \cong \mathbb{Z}$ ) is given by  $k = \dim \tilde{W}/H_+$ . One has the relation  $m \leq k \leq mn$ .

#### §4.2 Factorization theorems.

The factorization theorem for extended solutions (theorem A of 1.8) is an immediate consequence of Segal's theorem in 4.1 and the description of  $F_{n,k}$ . Indeed, Segal's theorem tells us that an extended solution  $\Phi$  may be renormalized as  $\tilde{\Phi} = \gamma\Phi$ , with  $\tilde{\Phi}(S^2) \subseteq F_{n,k}$ , where  $k = \dim \tilde{W}/H_+$ . Hence, by 2.2, the canonical flag  $\tilde{W}_{(1)} \subseteq \dots \subseteq \tilde{W}_{(l)} = \tilde{W}$  defines a factorization  $\tilde{\Phi} = \Phi_1 \dots \Phi_l$ , where  $\Phi_i(z, \lambda)$  is of the form  $P_{V_i(z)}^\perp + \lambda^{-1}P_{V_i(z)}$ , and each  $V_i$  is a map from  $S^2$  to a Grassmannian.

It is clear that each subproduct  $\Phi_1 \dots \Phi_i$  is an extended solution. For this subproduct corresponds to  $\tilde{W}_{(i)}$ , which is (by definition)  $\lambda^{l-i}\tilde{W}$ , and the extended solution conditions for  $\tilde{W}$  imply those for  $\lambda^{l-i}\tilde{W}$ , since  $\partial/\partial z$  commutes with multiplication by  $\lambda$ .

In fact, Segal's theorem gives a factorization theorem for general holomorphic maps  $\Phi : S^2 \rightarrow \Omega U_n$ , by a similar argument. Again,  $\Phi$  can be renormalized as  $\tilde{\Phi} = \gamma\Phi$ , where  $H_+ \subseteq \tilde{W} \subseteq p(\lambda)^{-1}H_+$ . After choosing an ordering of the roots of  $p$ , we obtain a canonical flag

$$H_+ = p(\lambda)\tilde{W} \subseteq p_{l-1}(\lambda)\tilde{W} \subseteq \dots \subseteq p_1(\lambda)\tilde{W} \subseteq \tilde{W}$$

where  $p_i$  divides  $p_{i+1}$ ,  $\deg p_{i+1} = 1 + \deg p_i$ , and  $p = p_l$ . This corresponds to a factorization  $\tilde{\Phi} = \Phi_1 \dots \Phi_l$ , where each  $\Phi_i$  is a "linear fractional transformation" in  $\lambda$ . Such a factorization was first obtained by Valli [Va2], using an extension of Uhlenbeck's approach. A similar result was proved by Beggs [Be], also using a Grassmannian model. The case where  $\Phi$  is independent of  $z$  is essentially the factorization result for rational functions, used by Uhlenbeck in the proof of theorem B of 1.8.

It follows from the last paragraph and 1.5 that one has a factorization

theorem for instantons.

### §4.3 The dressing action.

We have seen in 2.1 that the loop group  $\Omega U_n$  may be identified with the quotient  $AGl_n(\mathbb{C})/\Lambda^+ Gl_n(\mathbb{C})$ . This implies that any  $\gamma \in AGl_n(\mathbb{C})$  can be written as  $\gamma = \gamma_u \gamma_+$ , where  $\gamma_u \in \Omega U_n$ ,  $\gamma_+ \in \Lambda^+ Gl_n(\mathbb{C})$ . The natural action of  $AGl_n(\mathbb{C})$  on the coset space  $AGl_n(\mathbb{C})/\Lambda^+ Gl_n(\mathbb{C}) \cong \Omega U_n$  can be written in terms of this: if  $\gamma \in AGl_n(\mathbb{C})$  and  $\delta \in \Omega U_n$ , then the coset of  $\gamma\delta$  corresponds to the element  $(\gamma\delta)_u$  of  $\Omega U_n$ .

If  $\gamma \in AGl_n(\mathbb{C})$  and  $\Phi : S^2 \rightarrow \Omega U_n$  is an extended solution, we shall write

$$\gamma \circ \Phi = (\gamma\Phi)_u.$$

It is clear from the extended solution condition that  $\gamma \circ \Phi$  is also an extended solution, for  $\gamma \circ \Phi$  corresponds to  $\gamma W$ , and  $\partial/\partial z(\gamma W) = \gamma \partial/\partial z W \subseteq \gamma \lambda^{-1} W = \lambda^{-1} \gamma W$ . It turns out that this action  $\circ$  is essentially the same as the "dressing action"  $\diamond$  which was described in 1.8:

**Theorem.** Let  $\gamma \in \Lambda^+ Gl_n(\mathbb{C})$ . Define a map  $\hat{\gamma} : D_0 \cup D_\infty \rightarrow Gl_n(\mathbb{C})$  by:  $\hat{\gamma}(\lambda) = \gamma(\lambda)$  for  $\lambda \in D_0$ ,  $\hat{\gamma}(\lambda) = [\gamma(\bar{\lambda}^{-1})^*]^{-1}$  for  $\lambda \in D_\infty$ . Let  $\Phi$  be a normalized extended solution. Then  $\gamma \circ \Phi = \hat{\gamma} \diamond \Phi$ .

(Strictly speaking, we defined  $\diamond$  only for "loops which extend to  $D_0 \cup D_\infty$ ". It is clear, though, that the definition makes sense for maps  $\gamma : D_0 \cup D_\infty \rightarrow Gl_n(\mathbb{C})$ , regardless of whether  $\gamma$  is defined on the circle  $|\lambda| = 1$ . Thus,  $\hat{\gamma} \diamond \Phi$  should be interpreted this way.)

**Sketch proof:** We have to find a factorization  $\gamma\Phi = \Phi_1\Phi_2$ , as described in 1.8, with  $\Phi_1 = (\gamma\Phi)_u$ . For this to be a valid choice of  $\Phi_1$ , we need to know that it extends to the annulus  $A$ ; in fact, it follows from the construction of extended solutions that a normalized extended solution extends holomorphically to the region  $0 < |\lambda| < \infty$ . We must find a function  $\Phi_2$ , which extends to  $D_0 \cup D_\infty$ , which satisfies

$$\hat{\gamma}(\lambda)\Phi(\lambda) = (\gamma\Phi)_+(\lambda)\Phi_2(\lambda) \quad (*)$$

on  $C_0 \cup C_\infty$ . On  $C_0$  we can take  $\Phi_2 = (\gamma\Phi)_+$ . This certainly extends to  $D_0$  and satisfies  $(*)$  on  $C_0$ . On  $C_\infty$  we can take  $\Phi_2(\lambda) = [(\gamma\Phi)_+(\bar{\lambda}^{-1})^*]^{-1}$ . This extends to  $D_\infty$ , and also satisfies  $(*)$  on  $C_\infty$ , as one sees by applying the transformation  $X(\lambda) \mapsto [X(\bar{\lambda}^{-1})^*]^{-1}$  to the equation  $(*)$  on  $C_0$ . ■

From this one sees that the tricky factorization (or Riemann-Hilbert problem) needed in the definition of  $\diamond$  is in fact incorporated in the Grassmannian model of  $\Omega U_n$ . Thus, the "hidden symmetry group"  $\Lambda Gl_n(\mathbb{C})$ , for harmonic maps  $S^2 \rightarrow U_n$ , is revealed naturally by the Grassmannian model formulation. It is now much easier to establish general properties of the action.

First, we see that

(i) the action of  $\Lambda^+ Gl_n(\mathbb{C})$  (and, indeed, an action of  $\Lambda Gl_n(\mathbb{C})$ ) is *always well defined*.

Next, as  $\Lambda^+ Gl_n(\mathbb{C})$  is connected, it follows that

(ii) the integer  $k = -\deg \det \Phi$  is preserved by the action of  $\Lambda^+ Gl_n(\mathbb{C})$ .

Because  $\Lambda Gl_n(\mathbb{C})$  acts by diffeomorphisms on  $\Omega U_n$ ,

(iii) the degree  $d$  of  $\Phi$  (or, the energy of the corresponding harmonic map), is preserved by the action  $\Lambda Gl_n(\mathbb{C})$ .

Finally, because  $\Lambda^+ Gl_n(\mathbb{C})$  is (by definition) the isotropy subgroup of  $H_+$ ,

(iv) the minimal uniton number  $m_\Phi$  is preserved by the action of  $\Lambda^+ Gl_n(\mathbb{C})$ .

This also shows that the action of  $\Lambda^+ Gl_n(\mathbb{C})$  on  $m$ -unitons reduces to the action of the finite dimensional group  $G_{n,k}$ .

Although the action  $\diamond$  of  $\Lambda^+ Gl_n(\mathbb{C})$  appears to give only a special case of the action  $\diamond$  defined in 1.8, it can be shown that the *effective* action on  $m$ -unitons agrees with the effective action of  $\Lambda^+ Gl_n(\mathbb{C})$ . In particular, this gives another proof of theorem B of 1.8, that the action of Uhlenbeck's rational functions is always defined.

The action of  $G_{n,k}$  on extended solutions is a natural generalization of an action of  $Gl_n(\mathbb{C})$  on harmonic maps  $S^2 \rightarrow CP^{n-1}$  which was introduced in [Gu2]. By the classification theorem, such harmonic maps are of the form

$\phi = \pi \circ \Phi$ , where  $\Phi : S^2 \rightarrow F_{r,r+1}(\mathbb{C}^n)$  is a holomorphic map which is horizontal with respect to the projection  $\pi : F_{r,r+1}(\mathbb{C}^n) \rightarrow \mathbb{C}P^{n-1}$ . Here,  $F_{r,r+1}(\mathbb{C}^n)$  is the space of flags of the form  $\{0\} \subseteq E_r \subseteq E_{r+1} \subseteq \mathbb{C}^n$ . If the flag corresponding to  $\Phi(z)$  is denoted by  $\{0\} \subseteq W_r(z) \subseteq W_{r+1}(z) \subseteq \mathbb{C}^n$ , then the holomorphicity condition is

$$\frac{\partial}{\partial \bar{z}} W_r \subseteq W_r, \quad \frac{\partial}{\partial \bar{z}} W_{r+1} \subseteq W_{r+1}$$

and the horizontality condition is

$$\frac{\partial}{\partial z} W_r \subseteq W_{r+1}.$$

From this one sees that the natural action of  $Gl_n(\mathbb{C})$  on  $F_{r,r+1}(\mathbb{C}^n)$  preserves both holomorphicity and horizontality. Hence if  $X \in Gl_n(\mathbb{C})$ , we obtain a new harmonic map  $X \circ \phi = \pi(X \cdot \Phi)$ . More generally, this phenomenon occurs for the “twistor fibration” (in the sense of [BR]) of any compact inner symmetric space  $G/H$ .

Just as the factorization theorem for extended solutions was a special case of a factorization theorem for general holomorphic maps  $S^2 \rightarrow \Omega U_n$ , the action of  $\Lambda Gl_n(\mathbb{C})$  on extended solutions is a special case of an action of  $\Lambda Gl_n(\mathbb{C})$  on general holomorphic maps (and hence on instantons).

#### §4.4 Deformations of harmonic maps.

We have seen that (up to left translation) extended solutions are maps  $\Phi \in \text{Hol}_d(S^2, F_{n,k})$  such that  $\partial/\partial z \Phi(z) \subseteq N^{-1}\Phi(z)$ . The significance of the integers  $d$  and  $k$  is that  $d (= \deg \Phi)$  represents the energy of the corresponding harmonic map, and  $k (= -\deg(\det \Phi(z)))$  represents the component of  $\Omega U_n$  containing  $\Phi(S^2)$ . Moreover, there is an integer  $m_\Phi$  associated to  $\Phi$ , the minimal uniton number. One has  $m_\Phi \leq k, n-1$ . The integers  $d$  and  $k$  are obviously preserved by continuous deformations through extended solutions, as they are preserved by deformations through continuous maps. However, the same does not hold for  $m_\Phi$  (see [Ej], [EK] for some examples).

One way of obtaining a continuous deformation of an extended solution

$\Phi : S^2 \rightarrow F_{n,k}$  is to define

$$\Phi^t = g_t \circ \Phi$$

where  $g : \mathbb{R} \rightarrow G_{n,k}$  is a curve in  $G_{n,k}$ . Such a deformation preserves  $m_\Phi$  as well as  $d$  and  $k$ . However, it may happen that  $\Phi^\infty = \lim_{t \rightarrow \infty} \Phi^t$  exists (in the terminology of [BG],  $\Phi^\infty$  is obtained from  $\Phi$  by "completion"). In this case,  $\Phi^\infty$  is an extended solution with the same values of  $d$  and  $k$ , but its minimal uniton number may be *lower* than that of  $\Phi$ .

A natural choice of the curve  $g$  is suggested by Morse theory. Recall that a parabolic subgroup  $P$  of  $GL_{kn}(\mathbb{C})$  determines a Morse-Bott decomposition of  $Gr_k(\mathbb{C}^{kn})$ ; the gradient flow of a corresponding Morse-Bott function is then of the form  $x_t = g_t \cdot x$ , where  $g$  is a one parameter subgroup of  $GL_{kn}(\mathbb{C})$ . (See [At2], for example.) If this flow preserves  $F_{n,k}$ , we obtain a Morse theoretic decomposition of  $F_{n,k}$ .

Let  $\Phi^t$  be the deformation of an extended solution  $\Phi : S^2 \rightarrow F_{n,k}$  obtained by applying a gradient flow of the above type. Then, since each flow line converges to a critical point, the limit  $\Phi^\infty(z) = \lim_{t \rightarrow \infty} \Phi^t(z)$  exists for each  $z \in S^2$ . The function  $\Phi^\infty$  is not in general continuous. But, since  $\Phi$  is holomorphic,  $\Phi^\infty$  defines a holomorphic map (in fact, an extended solution) on a subset  $S^2 - \{z_1, \dots, z_r\}$ , and the singularities  $z_1, \dots, z_r$  are removable. After removing them, we obtain an extended solution  $\tilde{\Phi}^\infty : S^2 \rightarrow F_{n,k}$ . (In the terminology of [BG],  $\tilde{\Phi}^\infty$  is obtained from  $\Phi$  by "modified completion"; in terms of harmonic maps, it is an example of "bubbling off".)

It is easy to check that there is a Morse function on  $Gr_k(\mathbb{C}^{kn})$  (i.e. a function with isolated non-degenerate critical points) whose gradient flow preserves  $F_{n,k}$ , and which preserves extended solutions. In fact, such a function gives rise to the Schubert cell decomposition of  $F_{n,k}$  which was referred to in 2.2. By applying the gradient flow to an extended solution, we obtain the following result (which answers the question posed in §7 of [BG]):

**Theorem.** *Let  $\Phi : S^2 \rightarrow F_{n,k}$  be any extended solution. Then  $\Phi$  may be deformed to a constant map by applying a one parameter family of dressing*

*transformations and taking the limit, in the manner described above.*

In order to obtain results on *continuous* deformations, one needs to investigate when the singularities  $z_1, \dots, z_r$  do not occur. We shall conclude by describing the simplest example of this phenomenon (which necessarily involves using a Morse-Bott function with at least one critical *manifold* of positive dimension).

Recall that harmonic maps  $\phi : S^2 \rightarrow S^4$  correspond to holomorphic maps  $\Phi : S^2 \rightarrow \mathbb{C}P^3$  which are horizontal with respect to the twistor fibration

$$\pi : \mathbb{C}P^3 \rightarrow S^4$$

(see [Ca],[Br]). This fibration may be identified with the natural map

$$\frac{Sp_2}{S^1 \times Sp_1} \rightarrow \frac{Sp_2}{Sp_1 \times Sp_1}.$$

A 2-plane  $V$  in  $\mathbb{C}^4 \cong \mathbb{H}^2$  is called *horizontal* if and only if  $V \perp jV$  (with respect to the Hermitian metric of  $\mathbb{C}^4$ ). (The name comes from the fact that if  $V$  is horizontal, then  $\mathbf{P}(V)$  is horizontal with respect to  $\pi$ .) The group  $Sp_2$  acts transitively on the space of horizontal 2-planes, which is thereby identified with the homogeneous space  $Sp_2/U_2$  (see [Ga]).

The integer  $\deg \Phi = [\Phi] \in \pi_2 \mathbb{C}P^3 \cong \mathbb{Z}$  is called the *twistor degree* of  $\phi$ . We shall give a short proof of the following theorem, which was proved recently by Loo [Lo] (see also [Ve1],[Ve2] for earlier results on the space of harmonic maps, and [Ej],[EK],[Kt] for an alternative point of view).

**Theorem.** *The space of harmonic maps  $S^2 \rightarrow S^4$  of fixed twistor degree is path connected.*

**Proof:** Using homogeneous coordinates, a holomorphic map  $\Phi : S^2 \rightarrow \mathbb{C}P^3$  may be written in the form  $\Phi = [p_0; p_1; p_2; p_3]$ , where  $p_0, p_1, p_2, p_3$  are polynomials. The horizontality condition is  $p_0 p'_1 - p_1 p'_0 + p_2 p'_3 - p_3 p'_2 = 0$  (see [Br]). Consider the path  $\Phi^t = [p_0; t p_1; p_2; t p_3]$ , where  $t \in [0, 1]$ ; this obviously preserves both holomorphicity and horizontality. Now, if we know that

$\Phi$  avoids the horizontal 2-plane  $V_{02}$  given by  $z_0 = z_2 = 0$ , then we have a continuous path joining  $\Phi = \Phi^1$  to an element  $\Phi^0$  of the path connected space  $\text{Hol}_d(S^2, \mathbf{P}(V_{13}))$ , where  $V_{13}$  is given by  $z_1 = z_3 = 0$ .

We claim that a transformation  $X \in Sp_2$  can be found such that  $X\Phi$  avoids  $V_{02}$ . Since the action of  $Sp_2$  preserves harmonicity, this will complete the proof, as  $\Phi$  can then be deformed to  $X\Phi$  by using a path in  $Sp_2$  connecting the identity to  $X$ . It suffices to show that  $\Phi$  avoids some horizontal 2-plane. But this is obvious, as the space of horizontal 2-planes is three dimensional, whereas, for each  $z \in S^2$ , the dimension of the space of horizontal 2-planes containing  $\Phi(z)$  is one (since such a 2-plane satisfies the condition  $\Phi(z) \subseteq V \subseteq (j\Phi(z))^\perp$ ). ■

Although we did not explicitly use Morse theory and dressing transformations in this proof, they provided our initial motivation. There is a natural Morse-Bott function on  $CP^3$ , with critical manifolds  $\mathbf{P}(V_{02}), \mathbf{P}(V_{01})$ , and the deformation  $\Phi^t$  is simply the result of applying the gradient flow to  $\Phi$ . The gradient flow is given by the action of a one parameter subgroup of the complexification  $Sp_2^C$ , which acts by dressing transformations on harmonic maps. Further details and results will appear in a joint article with Y. Ohnita.

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