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SPECTRAL GEOMETRY

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Contents

§0 Introduction

Let M be a Riemannian manifold of dimension m. Let

$$\Delta_M = \delta d \tag{0.1}$$

be the Laplacian acting on the space of smooth complex valued functions. We have chosen the geometers sign convention; $\Delta_e = -\Sigma_i \partial_{x_i}^2$ is the Euclidean Laplacian on $\mathbf{R}^m$. More generally, if $x = (x^1, ..., x^m)$ is a system of local coordinates on M, let

$$g_{ij} = (\partial_i, \partial_j) \text{ and } g^{ij} = (dx^i, dx^j) \tag{0.2}$$

be the representation of the metric locally. The Riemannian element of volume is $dvol = gdx$ where $g = \sqrt{\det(g_{ij})}$. We compute the adjoint δ of d in a system of local coordinates:

$$\begin{aligned}(\delta\Sigma_j f_j dx^j, f)_{L^2} &= (\Sigma_j f_j dx^j, df)_{L^2}\\ &= \Sigma_{ij}\textstyle\int (f_j dx^j, \partial_i f dx^i) dvol\\ &= \Sigma_{ij}\textstyle\int g g^{ij} f_i \partial_i(\bar f) dx\\ &= -\Sigma_{ij}\textstyle\int \partial_i(g g^{ij} f_i)\bar f dx\\ &= -\Sigma_{ij}\textstyle\int g^{-1}\partial_i(g g^{ij} f_i)\bar f dvol.\end{aligned} \tag{0.3}$$

This shows $\delta(\Sigma_i f_i dx^i) = -\Sigma_{ij} g^{-1}\partial_i(g g^{ij} f_i)$ and consequently

$$\Delta_M = -\Sigma_{ij} g^{-1}\partial_i(g g^{ij}\partial_j). \tag{0.4}$$

If M is an oriented two dimensional manifold, the metric defines a conformal structure on M and hence a complex structure. If $z = x + iy$ are local holomorphic coordinates, $ds^2 = g(dx^2 + dy^2)$ and $\Delta = g^{-1}(\partial_x^2 + \partial_y^2)$.

We suppose M is compact and with empty boundary to avoid imposing boundary conditions for the moment. Standard elliptic theory, see [G-3], shows there is a spectral resolution $\{\lambda_\nu, \phi_\nu\}$ for Δ_M. The $\{\phi_\nu\}$ are a complete orthonormal basis for $L^2(M)$ of smooth functions ϕ_ν so that $\Delta\phi_\nu = \lambda_\nu\phi_\nu$. Let

$$E(\lambda, \Delta_M) = \operatorname{span}_{\lambda_\nu=\lambda}\{\phi_\nu\} \tag{0.5}$$

be the corresponding finite dimensional eigenspaces; $\{\lambda, E(\lambda, \Delta_M)\}$ is also called a spectral resolution of Δ_M. Let $\operatorname{spec}(\Delta, M) = \{\lambda_\nu\}$ be the eigenvalues where each eigenvalue is repeated according to multiplicity.

We will examine the relationship between $\operatorname{spec}(\Delta, M)$ and the geometry and topology of M. The question for a domain in $\mathbf{R}^2$ with Dirichlet boundary conditions was posed by Kac [Ka] and has an attractive formulation due to Protter [Pr]:

> "Suppose a drum is being played in one room and a person with perfect pitch hears but cannot see the drum. Is it possible for her to deduce the precise shape of the drum just from hearing the fundamental tone and all the overtones?"

In §1, we use spherical harmonics to determine $\operatorname{spec}(\Delta, M)$ if M is a spherical space form i.e. has constant sectional curvature +1. In §2, we use results of Ikeda to construct spherical space forms of dimensions 5, 7, and 9 which are isospectral but not diffeomorphic. This shows one can not hear the shape of a drum in general. At present, all known examples of isometric non isospectral manifolds without boundary have non trivial fundamental group. In §3, we discuss some relationships between the fundamental group and the spectrum using spherical space forms as examples.

Heat equation asymptotics are a powerful tool in spectral geometry. In §4, we discuss the heat equation asymptotics for manifolds without boundary and in §5 the corresponding generalization to manifolds with boundary. In §6, we conclude by discussing briefly the corresponding results for first order operators of Dirac type.

There is a vast literature on the subject; we refer to Bérard and Berger [BB] and Bérard [Be] for further references in addition to those appearing at the end of this paper.

§1 Spherical Space Forms

We begin our investigation by studying spherical harmonics. Denote a point of $\mathbf{R}^{m+1}$ by $\vec{x} = (x^0, ..., x^m)$. Let $S^m = \{\vec{x} : |\vec{x}|^2 = 1\}$ be the unit sphere. Let

$$S(m+1,j) = \{f \in \mathbf{C}[x^0, ..., x^m] : f(t\vec{x}) = t^j f(\vec{x}) \text{ for } t \in \mathbf{C}\} \tag{1.1}$$

be the vector space of polynomials in the $\{x^i\}$ variables which are homogeneous of degree j. Let

$$H(m+1,j) = \{f \in S(m+1,j) : \Delta_e f = 0\} \tag{1.2}$$

be the subspace of harmonic polynomials; identify a harmonic polynomial with its restriction to S^m. Let $r = |x|^2 = x_0^2 + ... + x_m^2$.

Theorem 1.1: *Let Δ_{S^m} be the spherical Laplacian on S^m.*
(a) $\dim\{S(m+1,j)\} = \binom{m+j}{m}$.
(b) $S(m+1,j) = r^2 S(m+1,j-2) \oplus H(m+1,j)$.

(c) $\dim\{H(m+1,j)\} = \binom{m+j}{m} - \binom{m+j-2}{m}$.
(d) $\{j(j+m-1), H(m+1,j)\}_{j=0}^{\infty}$ *is the spectral resolution of* Δ_{S^m} *on* S^m.

Remark: If $m = 1$, let $z = x_0 + ix_1 \in S(2,1)$. Then $H(2,j) = \operatorname{span}\{z^j, \bar{z}^j\}$ and the spectral resolution of $\Delta_{S^1} = -\partial_\theta^2$ decomposes $L^2(S^1) = \oplus_j e^{ij\theta} \cdot \mathbf{C}$ in terms of Fourier series.

Proof: Since $S(m+1,j) = x_m \cdot S(m+1,j-1) \oplus S(m,j)$,

$$\dim\{S(m+1,j)\} = \dim\{S(m+1,j-1)\} + \dim\{S(m,j)\}. \tag{1.3}$$

Since $\dim\{S(m+1,0)\} = 1$ and $\dim\{S(1,j)\} = 1$, (a) follows by induction.

If $p = \Sigma_\alpha p_\alpha x^\alpha \in S(m+1,j)$, let $P(p) = \Sigma_\alpha p_\alpha \partial_\alpha$. Define a positive definite symmetric bilinear inner product $\langle\cdot,\cdot\rangle$ on $S(m+1,j)$ by:

$$\langle p,q\rangle = P(p)(\bar{q}) = \Sigma_{\alpha,\beta}\ p_\alpha \partial_\alpha\{\bar{q}_\beta x^\beta\} = \Sigma_\alpha\ \alpha! p_\alpha \bar{q}_\alpha. \tag{1.4}$$

Let $p \in S(m+1,j-2)$ and $q \in S(m+1,j)$. Since $P(r^2) = -\Delta_e$,

$$-\langle p, \Delta_e q\rangle = \langle r^2 p, q\rangle. \tag{1.5}$$

Multiplication by r^2 is injective. Since $\operatorname{coker}(r^2) = \ker(\Delta_e)$, (b) and (c) follow.

We have identified a harmonic function with its restriction to S^m. Let

$$\mathcal{A} = \Sigma_j H(m+1,j) \subset C^\infty(S^m) \tag{1.6}$$

be the subspace generated by the $H(m+1,j)$. Since $r^2|_{S^m} = 1$, we use (b) to see:

$$\begin{aligned} \Sigma_{\nu\le j} H(m+1,\nu) &= \{S(m+1,j) + S(m+1,j-1)\}|_{S^m} \\ \mathcal{A} &= \cup_j\{S(m+1,j) + S(m+1,j-1)\}|_{S^m}. \end{aligned} \tag{1.7}$$

Since

$$S(m+1,j) \cdot S(m+1,k) \subset S(m+1,j+k), \tag{1.8}$$

$\mathcal{A}$ is a sub-algebra of $C^\infty(M)$. Since $1 \in H(m+1,0)$, $\mathcal{A}$ is unital. Since $x^i \in H(m+1,1)$, $\mathcal{A}$ separates points. Thus by the Stone-Weierstrauss theorem, $\mathcal{A}$ is dense in $C^\infty(S^m)$ so

$$L^2(S^m) = \bar{\mathcal{A}}. \tag{1.9}$$

We introduce polar coordinates $x = (r, \theta)$ for $r \in [0, \infty)$ and $\theta \in S^m$ on $\mathbf{R}^{m+1}$. In polar coordinates, the Euclidean Laplacian has the form

$$\Delta_e = -\partial_r^2 - mr^{-1}\partial_r + \Delta_{S^m}. \tag{1.10}$$

If $f \in H(m+1, j)$, then $\Delta_e(f) = 0$ so (1.10) implies

$$\Delta_{S^m} f(\theta) = j(j+m-1)f(\theta). \tag{1.11}$$

Since Δ_{S^m} is self-adjoint, $E(\lambda, \Delta_{S^m}) \perp E(\mu, \Delta_{S^m})$ for $\lambda \neq \mu$. Since

$$H(m+1, \nu) \subseteq E(j(j+m-1), \Delta_{S^m}), \tag{1.12}$$

$H(m+1, j)$ and $H(m+1, k)$ are orthogonal in $L^2(S^m)$ for $j \neq k$. This shows

$$\begin{aligned} L^2(S^m) &= \oplus_j H(m+1, j) \\ H(m+1, j) &= E(j(j+m-1), \Delta_{S^m}). \end{aligned} \tag{1.13}$$

■

Let $\tau : G \to O(m+1)$ be a real orthogonal representation of a finite group G. We say τ is fixed point free if

$$\det(\tau(g) - I) \neq 0 \text{ for } g \neq I. \tag{1.14}$$

For such a τ, let $M_\tau = S^m/\tau(G)$; M_τ is called a spherical space form. The manifold M_τ is compact with constant sectional curvature $+1$; all complete manifolds with constant sectional curvature $+1$ arise in this way. The spherical space forms have been classified; see Wolf [WO]. Let

$$H(m+1, j)^\tau = \{f \in H(m+1, j) : f(x) = f(\tau(g)x) \ \forall g \in G\}. \tag{1.15}$$

If $f \in H(m+1, j)^\tau$, f is G equivariant and may be regarded as a smooth function on M_τ; this embeds $H(m+1, j)^\tau$ in $C^\infty(M_\tau)$. We define a generating function

$$F_\tau(t) = \Sigma_j \dim\{H(m+1, j)^\tau\} t^j; \tag{1.16}$$

$F_\tau(t)$ is holomorphic for $|t| < 1$ since $\dim\{H(m+1, j)^\tau\} \leq \binom{m+j}{m}$.

Theorem 1.2: *Let Δ_{M_τ} be the Laplacian on $M_\tau = S^m/\tau(G)$.*
(a) $\{j(j+m-1), H(m+1,j)^\tau\}_{j=0}^\infty$ *is the spectral resolution of* Δ_{M_τ}.
(b) $F_\tau(t) = |G|^{-1}\Sigma_g(1-t^2)\det(I - t\tau(g))^{-1}$.

Proof: Let π be the projection from S^m to M_τ. Pullback π^* defines an injection

$$0 \to L^2(M_\tau) \xrightarrow{\pi^*} L^2(S^m); \text{ image}(\pi^*) = L^2(S^m)^\tau. \tag{1.17}$$

Consequently $L^2(M_\tau) = \oplus_j H(m+1,j)^\tau$; (a) follows from Theorem 1.1 since

$$\pi^*\Delta_{M_\tau} = \Delta_{S^m}\pi^*. \tag{1.18}$$

We use Ikeda [IK] to prove (b). Let $(g\cdot f)(x) = f(\tau(g)x)$ define representations $\tau_{H,j}$ and $\tau_{S,j}$ of G on $H(m+1,j)$ and $S(m+1,j)$. By Theorem 1.1,

$$\tau_{S,j} = \tau_{H,j} \oplus \tau_{S,j-2}. \tag{1.19}$$

Let $g \in G$ and $j = 0, 1, \ldots$. We compute:

$$\begin{aligned} \dim\{H(m+1,j)^\tau\} &= |G|^{-1}\Sigma_g \operatorname{tr}(\tau_{H,j}(g)) \\ F_\tau(t) &= |G|^{-1}\Sigma_{g,j} \operatorname{tr}(\tau_{H,j}(g))t^j \\ &= |G|^{-1}\Sigma_{g,j}\{\operatorname{tr}(\tau_{S,j}(g)) - \operatorname{tr}(\tau_{S,j-2}(g))\}t^j \\ &= (1-t^2)|G|^{-1}\Sigma_{g,j} \operatorname{tr}(\tau_{S,j}(g))t^j. \end{aligned} \tag{1.20}$$

Let $(A\cdot f)(x) = f(Ax)$ define a representation S_j of $O(\nu)$ on $S(\nu, j)$. Let

$$F(A,t) = \Sigma_j \operatorname{tr}\{S_j A\}t^j. \tag{1.21}$$

We complete the proof of Theorem 1.2 by showing $F(A,t) = \det(I - At)^{-1}$.

If $\nu > 1$, there is a non-trivial orthonormal decomposition of $\mathbf{R}^\nu$ so:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}. \tag{1.22}$$

Since $S(\nu_1 + \nu_2, j) = \oplus_k S(\nu_1, k) \otimes S(\nu_2, j-k)$,

$$F(A,t) = F(A_1,t)F(A_2,t). \tag{1.23}$$

Since the determinant is multiplicative, we need only consider the special cases $\nu = 1$ and $\nu = 2$. If $\nu = 1$, then A is scalar and

$$F(A,t) = \Sigma_j (At)^j = (1-At)^{-1} = \det(1-At)^{-1}. \tag{1.24}$$

If $\nu = 2$, A is a rotation through an angle θ. Let $\mathbf{R}^2 = \mathbf{C}$; $A = e^{i\theta} \in \mathbf{C}$. Since

$$\begin{aligned} S(2,j) &= \operatorname{span}\{z^a \bar{z}^b\}_{a+b=j} \\ F(A,t) &= \Sigma_{a,b} e^{ia\theta} e^{-ib\theta} t^{a+b} = (1 - e^{i\theta}t)^{-1}(1 - e^{-i\theta}t)^{-1} \\ &= \det(I - tA)^{-1}. \end{aligned} \tag{1.25}$$

■

§2 Isospectral Manifolds

There are many examples showing neither the geometry nor the topology of M is determined by $\operatorname{spec}(\Delta, M)$. We review some results:

Theorem 2.1:

(a) (Milnor [Mi]) *There exist isospectral non isometric flat tori of dimension 16.*

(b) (Vigneras [Vi]) *There exist isospectral non-isometric Riemann surfaces.*

(c) (Vigneras [Vi]) *There exist isospectral manifolds with different fundamental groups if $m \geq 3$.*

(d) (Ikeda [Ik]) *There exist isospectral non isometric spherical space forms.*

(e) (Urakawa [Ur]) *There exist regions $\Omega_i \subset \mathbf{R}^m$ for $m \geq 4$ which are isospectral for the Laplacian with both Dirichlet and Neumann boundary conditions but which are not isometric.*

Remark: See also [DG, Su] for general methods of constructing such examples. Gordon, Webb, and Wolpert (private communication) have constructed polygonal regions in the plane which are isospectral but not isometric for the Laplacian with Dirichlet boundary conditions.

These examples come in finite families. There are non trivial isospectral deformations:

Theorem 2.2:

(a) (Gordon-Wilson [GW]) *There exists a non trivial family of isospectral me-*

trics which are not conformally equivalent.

(b) (Brooks-Gordon [BrGo]) *There exists a non trivial family of isospectral metrics which are conformally equivalent.*

In addition to these examples, there are some compactness results:

Theorem 2.3:

(a) (Osgood, Phillips, and Sarnak [OPS]) *Families of isospectral metrics on Riemann surfaces are compact modulo gauge equivalence.*

(b) (Brooks, Chang, Perry, and Yang [BPY, CY]) *If $m = 3$, families of isospectral metrics within a conformal class are compact modulo gauge equiva lence.*

(c) (Brooks, Perry, and Petersen [BPP]) *Isospectral negative curvature manifolds contain only a finite number of topological types.*

In the remainder of §2, we present some results of Ikeda [IK] giving isospectral non isometric spherical space forms. We introduce some notation. If G is a group, let $\mathrm{Rep}(G)$ be the set of equivalence classes of complex finite dimensional representations of G. Let $\mathrm{Irr}(G)$ be the subset of irreducible complex representations. Let $\mathrm{Aut}(G)$ be the group of automorphisms of G; $\mathrm{Aut}(G)$ acts naturally on both $\mathrm{Rep}(G)$ and $\mathrm{Irr}(G)$. Let $\mathrm{Conj}(G)$ be the set of conjugacy classes. Let $\mathrm{Class}(G)$ be the space of complex class functions. If $|G| < \infty$ and $f_i \in \mathrm{Class}(G)$, let

$$\langle f_1, f_2 \rangle = |G|^{-1} \Sigma_g f_1(g) \bar{f}_2(g) \tag{2.1}$$

define a non-degenerate Hermitian inner product. Identify $\rho \in \mathrm{Rep}(G)$ with the character $Tr(\rho)$ to embed $\mathrm{Rep}(G)$ in $\mathrm{Class}(G)$; by the orthogonality relations, $\mathrm{Irr}(G)$ is an orthonormal basis for Rep(G). If $f \in \mathrm{Class}(G)$, $f \equiv 0$ if and only if $\langle f_1, \rho \rangle = 0 \ \forall \rho \in \mathrm{Irr}(G)$.

We begin by relating isometries of spherical space forms to group theory. Let $m > 1$. If $\tau : G \to O(m+1)$ is fixed point free, let $M_\tau = S^m/\tau(G)$. Since S^m is simply connected, τ defines an isomorphism between G and $\pi_1(M_\tau)$.

Lemma 2.4: M_τ and M_σ are isometric if and only if there exists $\psi \in Aut(G)$ and $A \in O(m+1)$ so $\sigma = A^{-1}(\tau \circ \psi)A$.

Remark: DeRham showed [De] that M_σ and M_τ are diffeomorphic if and only if M_σ and M_τ are isometric.

Proof: Suppose $A \cdot \sigma = (\tau \circ \psi) \cdot A$ so we have a commutative diagram:

$$\begin{array}{ccc} S^m & \xrightarrow{A} & S^m \\ \downarrow\sigma(g) & & \downarrow\tau(\psi(g)) \\ S^m & \xrightarrow{A} & S^m. \end{array} \tag{2.2}$$

Let $\phi(x) = Ax$. Since ϕ preserves equivalence classes, ϕ defines an isometry

$$\Phi : S^m/\sigma(G) \to S^m/\tau(\psi(G)) = S^m/\tau(G). \tag{2.3}$$

Conversely, let Φ be an isometry from M_σ to M_τ. Define $\psi \in \mathrm{Aut}(G)$ by

$$\psi = \Phi_* : \pi_1(M_\sigma) = G \to \pi_1(M_\tau) = G. \tag{2.4}$$

Lift Φ to an isometry ϕ of the universal cover S^m. Then $\phi(x) = Ax$ for some $A \in O(m+1)$ and (2.2) holds. Thus $\sigma = A^{-1}(\tau \circ \psi)A$. ∎

We now construct isospectral spherical space forms with non Abelian fundamental group which are not isometric. This construction is very group theoretic in nature. The construction generalizes easily, but we present it for a single example to simplify the discussion. Let

$$G = \langle A, B : A^{11} = B^{25} = 1,\ BAB^{-1} = A^3\rangle;\ |G| = 275. \tag{2.5}$$

Let $H_{11} = \langle A\rangle$; since H_{11} is normal in G, H_{11} is the unique Sylow 11-subgroup of G. The group G is metacyclic:

$$1 \to H_{11} \to G \to \mathbf{Z}_{25} \to 1. \tag{2.6}$$

Let $H_5 = \langle B^5\rangle$ be the center of G; $H_{11} \oplus H_5 = \langle AB^5\rangle$ is the maximal normal subgroup. We define 35 subsets of G by:

$$\begin{array}{rcl} C_i & = & B^{5i} \cdot \{A, A^3, A^9, A^5, A^4\} \text{ for } 0 \le i \le 4, \\ C_i & = & B^{5i} \cdot \{A^2, A^6, A^7, A^{10}, A^8\} \text{ for } 5 \le i \le 9, \\ C_i & = & \{B^{5i}\} \text{ for } 10 \le i \le 14, \\ D_i & = & B^i \cdot \{1, A, ..., A^{10}\} \text{ for } 1 \le i \le 24 \text{ and } i \ne 0\ (5). \end{array} \tag{2.7}$$

Let $\lambda = e^{2\pi i/11}$ and $\gamma = e^{2\pi i/25}$. Let $\{e_\nu\}$ be the standard basis for $\mathbf{C}^5$ where ν is defined mod 5. Define $\tau(a,b) : G \to U(5)$ by:

$$\tau(a,b)(e_\nu) = \lambda^{3^\nu a} e_\nu \text{ and } \tau(a,b)(e_\nu) = \gamma^b e_{\nu-1}. \tag{2.8}$$

Define $\rho(c) : G \to U(1)$ by $\rho(c)(A) = 1$ and $\rho(c)(B) = \gamma^c$. Since $\tau(a,b)$ and $\rho(c)$ preserve the defining relations, they define representations of G.

Lemma 2.5:
(a) $\mathrm{Conj}(G) = \{Ci, Dj\}$.
(b) $\mathrm{Irr}(G) = \{\tau(a,b), \rho(c)\}$ *for* $1 \le a \le 2$, $0 \le b \le 4$, *and* $0 \le c \le 24$.
(c) $\tau(a,b)$ *is fixed point free if* $a \not\equiv 0$ *mod 11 and* $b \not\equiv 0$ *mod 5.*

Proof: Since $BA^\nu B^{-1} = A^{3\nu}$, all the elements of C_i are conjugate. Since $A^{-1}B^iA = A^\nu B^i$ for $\nu = 3^i - 1$ and since $3^i - 1$ is a unit of $\mathbf{Z}_{11}$ for $i \not\equiv 0$ (5), all the elements of D_i are conjugate. Thus $|\mathrm{Conj}(G)| \le 35$. The representations in (b) are inequivalent and irreducible so $|\mathrm{Irr}(G)| \ge 35$. Since $|\mathrm{Conj}(G)| = |\mathrm{Irr}(G)|$, (a) and (b) follow. The eigenvalues of $\tau(a,b)(B^j)$ and $\tau(a,b)(A^kB^{5\ell})$ are

$$\{\gamma^{bj(1+5\nu)}\}_{\nu=0}^4 \text{ and } \{\lambda^{ak3^\nu}\gamma^{b\ell}\}_{\nu=0}^4. \tag{2.9}$$

These are not 1 for $j \not\equiv 0$ (25), and $k \not\equiv 0(11)$ or $\ell \not\equiv 0(5)$ since $a \not\equiv 0(11)$ and $b \not\equiv 0$ (5). ■

We study the automorphisms of G and isometry classes of the M_τ.

Lemma 2.6: Let $M(a,b) = S^9/\tau(a,b)(G)$. Let $\psi(A) = A^\alpha B^\beta$ and $\psi(B) = A^\gamma B^\delta$.
(a) $\psi \in \mathrm{Aut}(G) \Leftrightarrow \alpha \not\equiv 0$ (11), $\beta \equiv 0$ (25), *and* $\delta \equiv 1$ (5).
(b) $M(a_1,b_1)$ *is isometric to* $M(a_2,b_2) \Leftrightarrow b_1 \equiv \pm b_2$ (5).

Proof: Let $\psi \in \mathrm{Aut}(G)$. As $\langle A \rangle = H_{11}$ is the only subgroup of order 11 in G, $\psi(H_{11}) = H_{11}$; thus $B^\beta = 1$ so $\beta \equiv 0$ (25) and $A^\alpha \ne 1$ so $\alpha \not\equiv 0$ (11). Also

$$\psi(BAB^{-1}) = A^\gamma B^\delta A^\alpha B^{-\delta} A^{-\gamma} = A^{3^\delta \alpha}. \tag{2.10}$$

Since $\psi(BAB^{-1}) = \psi(A^3) = A^{3\alpha}$, $3^\delta \equiv 3$ (11) so $\delta \equiv 1$ (5). This establishes one implication. Conversely, if $\{\alpha, \beta, \gamma, \delta\}$ satisfy the conditions of (a), ψ preserves the defining relations and extends to an automorphism of G. Let

$$\begin{aligned} S_1 &= \{M(1,1),\ M(2,1),\ M(1,4),\ M(2,4)\} \\ S_2 &= \{M(1,2),\ M(2,2),\ M(1,3),\ M(2,3)\}. \end{aligned} \tag{2.11}$$

Define $\psi \in \mathrm{Aut}(G)$ by $\psi(A) = A^2$ and $\psi(B) = B$. Then $\tau(1,b) \circ \psi = \tau(2,b)$ so $M(1,b)$ is isometric to $M(2,b)$ for any b. Since the complex conjugate $\tau(2-a, 5-b)$ is equivalent to $\tau(a,b)$ in $O(10)$, $M(2,b)$ is isometric to $M(1,5-b)$. Thus all the manifolds in S_i are isometric for $i = 1,2$. Conversely, if $\psi \in \mathrm{Aut}(G)$, $\psi(B^5) = B^5$. Let $\tau_r(a,b) : G \to O(10)$ be the underlying real representation;

$$\mathrm{tr}\{\tau_r(a,b)\psi(B^5)\} = \mathrm{tr}\{\tau_r(a,b)B^5\} = 5\cos(2\pi b/5). \tag{2.12}$$

This shows $\tau(1,1)$ and $\tau(1,2) \circ \psi$ are not conjugate in $O(10)$ so $M(1,1)$ and $M(1,2)$ are not isometric. ∎

Let D be a natural operator of Riemannian geometry; for example we could take $D = \Delta_p = (d\delta + \delta d)$ on the bundle of smooth p forms.

Theorem 2.7: *$M(1,1)$ and $M(1,2)$ are D isospectral for all natural operators D but $M(1,1)$ and $M(1,2)$ are not isometric.*

Proof: We use ideas from the proof of Theorem 1.2. Let $\{\lambda, E(\lambda, D)\}$ be the spectral resolution of D on S^9. By hypothesis, the isometry group $O(10)$ of S^9 acts equivariantly with respect to D so there is a natural representation

$$E(\lambda) : O(10) \to Gl(E(\lambda, D)) \tag{2.13}$$

of $O(10)$ on the eigenspaces. Let

$$E(\lambda, D)^\tau = \{f \in E(\lambda, D) : E(\lambda)(\tau(g)) \cdot f = f\ \forall g \in G\} \tag{2.14}$$

be the invariant subspace; $\{\lambda, E(\lambda, D)^\tau\}$ is the spectral resolution of D on M_τ. Let $\psi(A^jB^k) = A^jB^{2k}$; ψ is a bijective set theoretic correspondence preserving conjugacy classes which is not a group homomorphism. We verify:

$$\operatorname{tr}(\tau(1,1)(\psi g)) = \operatorname{tr}(\tau(1,2)(g)). \tag{2.15}$$

Thus $\tau(1,1)(\psi g)$ is conjugate in $U(5)$ and hence in $O(10)$ to $\tau(1,b)(g)$ so

$$\operatorname{tr}\{E(\lambda)\tau(1,1)(\psi g)\} = \operatorname{tr}\{E(\lambda)\tau(1,2)(g)\}. \tag{2.16}$$

Therefore:

$$\begin{aligned} \dim E(\lambda, D)^{\tau(1,1)} &= |G|^{-1}\Sigma_g \operatorname{tr}\{E(\lambda)\tau(1,1)(g)\} \\ &= |G|^{-1}\Sigma_g \operatorname{tr}\{E(\lambda)\tau(1,1)(\psi g)\} \\ &= |G|^{-1}\Sigma_g \operatorname{tr}\{E(\lambda)\tau(1,2)(g)\} \\ &= \dim E(\lambda, D)^{\tau(1,2)}. \end{aligned} \tag{2.17}$$

■

Remark: Ikeda [IK] and Gordon [Go] have constructed examples of manifolds which are Δ_0 isospectral but not Δ_1 isospectral; thus Δ_0 isospectral does not imply isospectral for all natural operators of Riemannian geometry in general.

We now turn our attention to Abelian fundamental groups. In contrast to the preceeding example, the construction here is primarily number theoretic in nature. Let $n \geq 2$ and let $S = \{s_1, ..., s_k\}$ be a collection of integers coprime to n. Let $\mathbf{Z}_n = \{\lambda \in \mathbf{C} : \lambda^n = 1\}$ be the cyclic group of n^{th} roots of unity and let

$$\rho_S(\lambda) = \operatorname{diag}(\lambda^{s_1}, ..., \lambda^{s_k}) : \mathbf{Z}_n \to U(k) \subset O(2k) \tag{2.18}$$

define a fixed point free representation of $\mathbf{Z}_n$. Let

$$L(n; S) = S^{2k-1}/\rho_S(\mathbf{Z}_n) \tag{2.19}$$

be a lenspace. The following follows directly from Lemma 2.4:

Lemma 2.8: *$L(n;S)$ and $L(n;\tilde{S})$ are isometric if and only if there is a permutation σ, an integer ℓ coprime to n, and signs $\epsilon_i = \pm 1$ so $s_{\sigma(i)} = \epsilon_i \ell \tilde{s}_i$ for $1 \leq i \leq k$.*

We construct isospectral lens spaces which are not isometric. Let $p = 2\nu - 1$ be prime. Let $S = \{s_1, ..., s_a\}$ and $R = \{r_1, ..., r_b\}$ for $a + b = \nu$ be complementary collections of distinct indices so $S \dot{\sqcup} R = \{1, 2, ..., \nu\}$.

Theorem 2.9:

(a) *$L(p;S)$ and $L(p;\tilde{S})$ are isometric $\Leftrightarrow$ $L(p;R)$ and $L(p;\tilde{R})$ are isometric.*

(b) *Let $b = 2$ and $p \geq 11$. $L(p; 3, 4, 5, .., \nu)$ is not isometric to $L(p; 2, 4, 5, ..., \nu)$.*

(c) *Let $b = 2$ and $p \geq 11$. All the $L(p; S)$ are all isospectral.*

Proof: Let $\mathbf{Z}_p^*$ be the set of primitive p^{th} roots of unity. Let $\mathbf{Z}_2 = \{\pm 1\}$ act on $\mathbf{Z}_p^*$ by multiplication; $\{1, ..., \nu\}$ is a set of representatives for $\mathbf{Z}_p^*/\mathbf{Z}_2$. Let

$$\psi_\ell(s) = s \cdot \ell : \mathbf{Z}_p^*/\mathbf{Z}_2 \to \mathbf{Z}_p^*/\mathbf{Z}_2. \tag{2.20}$$

By Lemma 2.8, $L(p; S)$ and $L(p; \tilde{S})$ are isometric $\Leftrightarrow$ $\psi_\ell(S) = \tilde{S}$ for some ℓ. Since $R = \{1, ..., \nu\} - S$ and $\tilde{R} = \{1, ..., \nu\} - \tilde{S}$; $\psi_\ell(S) = \tilde{S}$ $\Leftrightarrow$ $\psi_\ell(R) = \tilde{R}$. This proves (a). $L(p; 1, 2)$ is isometric to $L(p; 1, 3)$ $\Leftrightarrow$ $2 \cdot 3 \equiv \pm 1$ (p); this fails for $p \geq 11$. Thus by (a), the complementary lens spaces $L(p; 3, 4, 5, .., \nu)$ and $L(p; 2, 4, 5, ..., \nu)$ are not isometric. We complete the proof by showing the generating function

$$F_S(t) = p^{-1}(1 - t^2)\Sigma_\lambda \det(I - t \cdot \rho_S(\lambda))^{-1} \tag{2.21}$$

of Theorem 1.2 is independent of S or equivalently

$$\tilde{F}_S(t) = \Sigma_{\lambda \neq 1} \det(I - t \cdot \rho_S(\lambda))^{-1} \tag{2.22}$$

is independent of S. Since $\{1, ..., \nu\} = S \sqcup R$, if $\lambda \neq 1$,

$$\begin{aligned} g(t) &= \Pi_\alpha(1 - t\lambda^{s_\alpha})(1 - t\lambda^{-s_\alpha}) \cdot \Pi_\beta(1 - t\lambda^{r_\beta})(1 - t\lambda^{-r_\beta}) \\ &= \Pi_{\ell \neq 1}(1 - t\lambda^\ell) = \Pi_{\xi \neq 1}(1 - t\xi) \end{aligned} \tag{2.23}$$

is independent of S and λ. We clear denominators:

$$\begin{aligned} g(t)\tilde{F}_S(t) &= \Sigma_{\lambda \neq 1}(1 - t\lambda^{r_1})(1 - t\lambda^{-r_1})(1 - t\lambda^{r_2})(1 - t\lambda^{-r_2}) \\ &= a_0(S) + a_1(S)t + a_2(S)t^2 + a_3(S)t^3 + a_4(S)t^4. \end{aligned} \tag{2.24}$$

We note $\pm r_\nu$ and $\pm r_1 \pm r_2$ are not divisible by p. We compute:

$$\begin{array}{rcl} a_0(S) & = & a_4(S) = 1 \\ a_1(S) & = & a_3(S) = -\Sigma_{\lambda\neq 1}\lambda^{r_1} + \lambda^{-r_1} + \lambda^{r_2} + \lambda^{-r_2} \\ & = & -4\Sigma_{\lambda\neq 1}\lambda \\ a_2(R) & = & \Sigma_{\lambda\neq 1}2 + \lambda^{\pm r_1 \pm r_2} = \Sigma_{\lambda\neq 1}(2+4\lambda). \end{array} \tag{2.25}$$

■

Remark: Two lens spaces L_i are isometric if and only if they are diffeomorphic; they are homotopic if and only if there exists ℓ so $\Pi_i s_i = \pm\ell^k \Pi_i \tilde{s}_i$; see [Co]. Thus $L(11;3,4,5)$ and $L(11;2,4,5)$ are isospectral 5 dimensional lens spaces which are homotopic but not diffeomorphic; $L(13;3,4,5,6)$ and $L(13;2,4,5,6)$ are isospectral 7 dimensional lens spaces which are not homotopic.

§3 π_1 Isospectral Manifolds

Let M be connected. Let $\tilde{M}$ be the universal cover of M. Let V be the representation space of $\rho \in \mathrm{Rep}(\pi_1(M))$ and let V_ρ be the associated bundle:

$$V_\rho = \tilde{M} \times V/\pi_1(M) \tag{3.1}$$

where we identify $(x,w) = (gx, \rho(g)w)$. The transition functions of V_ρ are locally constant so V_ρ is flat; V_ρ inherits a natural connection ∇_ρ with zero curvature. The holonomy of the connection ∇_ρ is the representation ρ so we may identify representations of $\pi_1(M)$ and flat bundles. Since the transition functions of V_ρ are locally constant, we can define the Laplacian Δ_ρ on $C^\infty(V_\rho)$ to be locally isomorphic to $\dim(\rho)$ copies of the standard Laplacian Δ_0.

Let $M = S^m/\tau(G)$ be a spherical space form. We generalize Theorem 1.2 to the equivariant setting as follows; we omit the proof as it is analogous. Let

$$F_{\tau,\rho}(t) = \Sigma_j \dim\{(H(m+1,j)\otimes V)^G\}t^j. \tag{3.2}$$

Theorem 3.1: *Let $M = S^m/\tau(G)$ and let $\rho \in Rep(G)$.*
(a) $\{j(j+m-1), (H(m+1,j)\otimes V)^G\}_{j=0}^{\infty}$ *is the spectral resolution of* Δ_ρ *on* M_τ.
(b) $F_{\tau,\rho}(t) = |G|^{-1}\Sigma_g(1-t)^2\det(I - t\tau(g))^{-1}\operatorname{tr}(\rho(g))$.

There are two different ways in which to relate flat bundles over different Riemannian manifolds. The first is to assume a given isomorphism ψ of fundamental groups or marking.

Definition : Two Riemannian manifolds M_i are marked π_1 isospectral if there exists an isomorphism $\psi : \pi_1(M_1) \to \pi_1(M_2)$ so

$$\operatorname{spec}(\Delta_\rho, M_2) = \operatorname{spec}(\Delta_{\rho\circ\psi}, M_1)\ \forall\rho \in \operatorname{Rep}(G). \tag{3.3}$$

Theorem 3.2: *If M_1 and M_2 are marked π_1 isospectral spherical space forms, then M_1 and M_2 are isometric.*

Remark: Carolyn Gordon informs us the examples of [GW] are marked π_1 isospectral so Theorem 3.2 fails in general.

Proof: We shall see in §4 that $\operatorname{spec}(\Delta, M)$ determines the dimension m of the underlying manifold. We use isomorphism $\psi : \pi_1(M_1) \to \pi_2(M_2)$ to identify the fundamental groups and to express $M_i = S^m/\tau_i(G)$ where the τ_i are real representations of G; let $\tau_{i,c}$ be the complexifications. Let

$$\mu_1(M_i, \rho) = \dim\{\operatorname{kernel}(\Delta_\rho - j(j+m-1))_{M_i}\} \tag{3.4}$$

be the multiplicity of the first non-zero eigenvalue of Δ_ρ. By Theorem 3.1,

$$\mu_1(M_i, \rho) = \dim\{(H(m+1,1)\otimes V)^G\} = \langle\tau_{i,c}, \rho^*\rangle. \tag{3.5}$$

Since the M_i are marked π_1 isospectral, we use the orthogonality relations to see $\tau_{1,c} = \tau_{2,c}$ so τ_1 and τ_2 are equivalent real representations of G. ∎

The notion of marked π_1 isospectral depends upon the isomorphism ψ chosen between the fundamental groups. It is convenient for work in equivariant bordism to have a notion which is independent of ψ.

Definition : Two connected manifolds M_i are π_1 isospectral if there exists an isomorphism $\psi : \pi_1(M_1) = G_1 \to \pi_1(M_2) = G_2$ so

$$\operatorname{spec}(\Delta_\rho, M_2) = \operatorname{spec}(\Delta_{\rho\circ\psi}, M_1)\ \forall \rho \in \operatorname{Rep}(G_1)^{\operatorname{Aut}(G_2)}. \tag{3.6}$$

Since ρ is $\operatorname{Aut}(G_2)$ invariant, (3.6) is independent of ψ.

The following Theorem shows the nature of the fundamental group involved plays a crucial role, even for spherical space forms.

Theorem 3.3:

(a) *Let p be prime and let $M_i = L(p; S)$ be isospectral lens spaces. Then the M_i are π_1 isospectral. Thus there exist π_1 isospectral non-isometric lens spaces.*

(b) *Let $G = \langle A, B : A^{11} = B^{25} = 1,\ BAB^{-1} = A^3\rangle$ as discussed in §2. Let $M_i = S^m/\tau_i(G)$. If the M_i are π_1 isospectral, then the M_i are isometric.*

Proof: Let $\mathbf{Z}_p = \{\lambda \in \mathbf{C} : \lambda^p = 1\}$ for p is prime. Let $M = L(p; S)$. Let $\rho_s(\lambda) = \lambda^s$; $\{\rho_s\}_{s=0}^{p-1}$ parametrize the irreducible representations of G. Let

$$\delta = \rho_0 + \ldots + \rho_{p-1}. \tag{3.7}$$

be the regular representation. If $\rho \in \operatorname{Rep}(G)^{\operatorname{Aut}(G)}$, then $\rho = a \cdot \rho_0 + b \cdot \delta$. Since

$$\begin{aligned} \operatorname{spec}(\Delta_{\delta_0}, M) &= \operatorname{spec}(\Delta, S^m) \\ \operatorname{spec}(\Delta_{\rho_0}, M) &= \operatorname{spec}(\Delta, M), \end{aligned} \tag{3.8}$$

isospectral implies π_1 isospectral. This proves (a).

Before proving (b), we must examine the structure of the representation ring. We adopt the notation of §2:

Lemma 3.4:

(a) $\tau_{1,b}+\tau_{2,b}\in \mathrm{Rep}(G)^{\mathrm{Aut}(G)}$ and $\tau_{0,b}\in \mathrm{Rep}(G)^{\mathrm{Aut}(G)}$ $\forall b$.

(b) $\tau_{1,b}\otimes\tau_{1,\beta}=2\tau_{1,b+\beta}+3\tau_{2,b+\beta}$ and $\tau_{2,b}\otimes\tau_{2,\beta}=2\tau_{2,b+\beta}+3\tau_{1,b+\beta}$.

(c) $\tau_{1,b}\otimes\tau_{2,\beta}=2\tau_{1,b+\beta}+2\tau_{2,b+\beta}+\tau_{0,b+\beta}$.

(d) $S_2(\tau_{1,b})=2\tau_{2,2b}+\tau_{1,2b}$ and $S_2(\tau_{2,b})=2\tau_{1,2b}+\tau_{2,2b}$.

Remark: $\tau_{0,b}=\rho_b+\rho_{5+b}+\rho_{10+b}+\rho_{15+b}+\rho_{20+b}$ is not irreducible.

Proof: (a) follows from Lemma 2.6. We compute characters to prove the remaining assertions. If g has order 25,

$$\mathrm{tr}\{\tau_{a,b}\otimes\tau_{\alpha,\beta}\}(g)=\mathrm{tr}\{S_2(\tau_{a,b})\}(g)=0. \tag{3.9}$$

Similarly $\mathrm{tr}(\tau_{\gamma,\delta}(g))=0$. Let $\lambda=e^{2\pi i/11}$, $\mu=e^{2\pi i/5}$, and let $g=A^uB^{5v}$. The eigenvalues of $(\tau_{a,b}\otimes\tau_{a,\beta})(g)$ are $\lambda^{au\nu}\mu^{(b+\beta)v}$ for

$$\begin{aligned}\nu\in\ &\{1,3,9,5,4\}+\{1,3,9,5,4\}\\ =\ &(2,4,10,6,5,\ 4,6,1,8,7,\ 10,1,7,3,2,\ 6,8,3,10,9,\ 5,7,2,9,8);\end{aligned} \tag{3.10}$$

Similarly the eigenvalues of $(\tau_{1,b}\otimes\tau_{2,\beta})(A^uB^{5v})$ are $\lambda^{u\nu}\mu^{(b+\beta)v}$ for

$$\begin{aligned}\nu\ &\in\{1,3,9,5,4\}+\{2,6,7,10,8\}\\ =\ &(3,7,8,0,9,\ 5,9,10,2,0,\ 0,4,5,8,6,7,\ 0,1,4,2,6,\ 10,0,3,1);\end{aligned} \tag{3.11}$$

Finally, the eigenvalues of $S_2(\tau_{a,b})(A^uB^{5v})$ are $\lambda^{u\nu}\mu^{2bv}$ for

$$\nu\in\{2,6,7,10,8,4,10,6,5,1,8,7,3,2,9\}. \tag{3.12}$$

∎

Lemma 3.5: *Let $\tau_c=\Sigma_\nu a_\nu(\tau_{1,\nu}+\tau_{2,5-\nu})$ be the complexification of a real fixed point free representation τ_r of G.*

(a) $\langle\tau_c,\tau_{1,1}+\tau_{2,1}\rangle=a_1+a_4$.

(b) $\langle\tau_c,\tau_{1,2}+\tau_{2,2}\rangle=a_2+a_3$.

(c) $\langle S^2\tau_c,\tau_{0,0}\rangle=5(a_1^2+a_2^2+a_3^2+a_4^2)$.

(d) $\langle S^2\tau_c,\tau_{0,1}\rangle=5(a_1a_2+a_2a_3+a_3a_4)$.

Proof: (a) and (b) are immediate. We decompose:

$$\begin{aligned} S_2\tau_c = \quad & \Sigma a_i\{S_2(\tau_{1,i}) + S_2(\tau_{2,5-i})\} + \Sigma_i a_i^2 \tau_{1,i} \otimes \tau_{2,5-i} \\ & + \Sigma_i \tfrac{1}{2} a_i(a_i - 1)\{\tau_{1,i} \otimes \tau_{1,i} + \tau_{2,5-i} \otimes \tau_{2,5-i}\} \\ & + \Sigma_{i<j} a_i a_j \{\tau_{1,i} \otimes \tau_{1,j} + \tau_{1,i} \otimes \tau_{2,5-j}\} \\ & + \Sigma_{i<j} a_i a_j \{\tau_{2,5-i} \otimes \tau_{1,j} + \tau_{2,5-i} \otimes \tau_{2,5-j}\} \end{aligned}$$

We use Lemma 3.4 to expand $S_2\tau_c$ in terms of the $\tau_{\alpha,\beta}$. We are interested in $\alpha = 0$ and supress the terms where $\alpha \neq 0$. Consequently we ignore the $S_2(\tau_{i,j})$, $\tau_{1,i} \otimes \tau_{1,i}$, and $\tau_{2,i} \otimes \tau_{2,i}$ terms. We expand $\tau_{1,i} \otimes \tau_{2,j} = \tau_{0,i+j} + \ldots$ to see:

$$S_2\tau_c = \Sigma_i a_i^2 \tau_{0,0} + \Sigma_{i<j} a_i a_j (\tau_{0,i-j} + \tau_{0,j-i}) + \ldots \tag{3.13}$$

■

Let $\tau_c = \Sigma_\nu a_\nu(\tau_{1,\nu} + \tau_{2,5-\nu})$ determine a spherical space form $M(\vec{a})$ for the $\vec{a} \in \mathbf{N}^4$. Let $\vec{a}^t = (a_4, a_3, a_2, a_1)$ define an involution of $\mathbf{N}^4$. The argument of Lemma 2.6 shows $M(\vec{a}) = M(\vec{b}) \Leftrightarrow \vec{a} = \vec{b}$ or $\vec{a} = \vec{b}^t$; therefore $\mathbf{N}^4/\mathbf{Z}_2$ is the natural parameter space. We note

$$\dim\{E(j(j+m-1), \Delta_\rho)\} = \langle h_j(\tau), \rho^*\rangle. \tag{3.14}$$

Since $s_j = h_j + h_{j-2} + \ldots$,

$$\{\langle s_j(\tau), \rho\rangle\}_{\rho \in \mathrm{Rep}(G)^{\mathrm{Aut}(G)}} \tag{3.15}$$

are spectrally determined. We complete the proof of Theorem 3.3 by showing

$$\begin{gathered} \{\alpha = a_1 + a_4, \quad \beta = a_2 + a_3, \\ \gamma = a_1^2 + a_2^2 + a_3^2 + a_4^2, \quad \delta = a_1a_2 + a_2a_3 + a_3a_4\} \end{gathered} \tag{3.16}$$

distinguish points of $\mathbf{N}^4/\mathbf{Z}_2$. Introduce new variables:

$$\begin{gathered} x = a_1 - \tfrac{1}{2}\alpha, \quad y = a_2 - \tfrac{1}{2}\beta; \\ a_1 = \tfrac{1}{2}\alpha + x,\ a_2 = \tfrac{1}{2}\beta + y, \quad a_3 = \tfrac{1}{2}\beta - y,\ a_4 = \tfrac{1}{2}\alpha - x. \end{gathered} \tag{3.17}$$

In these coordinates $(x,y)^t = -(x,y)$. We express:

$$\begin{aligned} \gamma &= (\tfrac{1}{2}\alpha + x)^2 + (\tfrac{1}{2}\beta + y)^2 + (\tfrac{1}{2}\beta - y)^2 + (\tfrac{1}{2}\alpha - x)^2 \\ &= \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\beta^2 + 2x^2 + 2y^2 \\ \delta &= (\tfrac{1}{2}\alpha + x)(\tfrac{1}{2}\beta + y) + (\tfrac{1}{2}\beta + y)(\tfrac{1}{2}\beta - y) + (\tfrac{1}{2}\beta - y)(\tfrac{1}{2}\alpha - x) \\ &= 2xy - y^2 + \tfrac{1}{2}\alpha\beta + \tfrac{1}{4}\beta^2 \end{aligned} \tag{3.18}$$

Suppose $(\alpha,\beta,\gamma,\delta)(\vec{a}) = (\alpha,\beta,\gamma,\delta)(\vec{b})$. If $\vec{a}$ corresponds to (x,y) and $\vec{b}$ to $(\bar{x},\bar{y})$,

$$x^2+y^2=\bar{x}^2+\bar{y}^2 \text{ and } 2xy-y^2=2\bar{x}\bar{y}-\bar{y}^2. \tag{3.19}$$

We complete the proof by showing $(x,y)=\pm(\bar{x},\bar{y})$.

If $y=\bar{y}=0$, then $x=\pm\bar{x}$. We therefore assume $y\neq 0$ henceforth. We acknowledge with gratitude the assistance of N. Spaltenstein with the following argument. We use (3.19) to see:

$$x^2=\bar{x}^2+\bar{y}^2-y^2 \text{ and } 4x^2y^2=(2\bar{x}\bar{y}-\bar{y}^2+y^2)^2. \tag{3.20}$$

We eliminate x to see $4(\bar{x}^2+\bar{y}^2-y^2)y^2=(2\bar{x}\bar{y}-\bar{y}^2+y^2)^2$. This yields

$$5y^4+y^2(4\bar{x}\bar{y}-6\bar{y}^2-4\bar{x}^2)+(2\bar{x}\bar{y}-\bar{y}^2)^2=0. \tag{3.21}$$

We factor this quadratic expression to see:

$$(y^2-\bar{y}^2)(5y^2-(2\bar{x}-\bar{y})^2)=0. \tag{3.22}$$

Since $\{y,\bar{y},\bar{x}\}$ are integers and $y\neq 0$,

$$5y^2-(2\bar{x}-\bar{y})^2\neq 0. \tag{3.23}$$

Thus $y^2=\bar{y}^2$. By changing the sign if necessary, we may suppose $y=\bar{y}>0$. Since $2xy=2\bar{x}y$ and $y\neq 0$, $x=\bar{x}$. ■

§4 Heat Equation Asymptotics for manifolds without boundary.

So far we have seen that isospectral manifolds are not necessarily isometric or even topologically equivalent. We now change our focus and investigate what geometric properties are determined by the spectrum. Heat equation asymptotics are a fundamental tool. We refer to Gilkey [Gi] for proofs and details unless otherwise indicated in this and subsequent sections. Let

$$\mathrm{Tr}_{L^2}\, e^{-t\Delta}=\Sigma_\nu e^{-t\lambda_\nu} \tag{4.1}$$

be the global trace of the fundamental solution of the heat equation. As $t \to 0^+$, there is an asymptotic series

$$\mathrm{Tr}_{L^2}\, e^{-t\Delta} \sim \Sigma_{n=0}^{\infty} a_n(\Delta) t^{(n-m)/2}. \tag{4.2}$$

The $a_n(\Delta)$ are spectral invariants which are determined by the local geometry of M. Let Roman indices i, j etc. range from 1 through m and index a local orthonormal frame $\{e_1, ..., e_m\}$ for the tangent space $T(M)$. We use the metric to identify $T(M)$ with the cotangent space T^*M. Let R_{ijkl} be the curvature tensor of the Levi-Civita connection with the sign convention $R_{1212} = -1$ on the standard sphere. We sum over repeated indices to define

$$\begin{aligned} \tau &= -R_{ijij},\ \rho_{ij} = -R_{ikjk}, \\ \rho^2 &= R_{ikjk}R_{iljl}, R^2 = R_{ijkl}R_{ijkl}. \end{aligned} \tag{4.3}$$

Theorem 4.1: (Sakai [Sa], Gilkey [Gi]). *Let the boundary of M be empty.*

(a) $a_n(\Delta) = 0$ *for n odd .*

(b) $a_0(\Delta) = (4\pi)^{-m/2} \int_M 1$.

(c) $a_2(\Delta) = 6^{-1}(4\pi)^{-m/2} \int_M \tau$.

(d) $a_4(\Delta) = 360^{-1}(4\pi)^{-m/2} \int_M 5\tau^2 - 2\rho^2 + 2R^2$.

(e) $a_6(\Delta) = 45360^{-1}(4\pi)^{-m/2} \int_M -142(\nabla\tau)^2 - 26(\nabla\rho)^2 - 7(\nabla R)^2 + 35\tau^3$
$-42\tau\rho^2 + 42\tau R^2 - 36\rho_{ij}\rho_{jk}\rho_{ki} - 20\rho_{ij}\rho_{kl}R_{ikjl} - 8\rho_{ij}R_{ikln}R_{jkln}$
$-24R_{ijkl}R_{ijnp}R_{klnp}$.

Remark: Avramidi [Av] and Amsterdamski, Berkin, and O'Connor [ABC] have computed $a_8(\Delta)$; a_8 has formidable combinatorial complexity.

We can use Theorem 4.1 to draw some positive conclusions in spectral geometry:

Theorem 4.2: *Let $spec(\Delta, M_1^{m_1}) = spec(\Delta, M_2^{m_2})$.*

(a) *$m_1 = m_2$ and $vol(M_1) = vol(M_2)$.*

(b) *If $m_1 = 2$, $\chi(M_1) = \chi(M_2)$.*

(c) *Let $m_1 \leq 5$. If M_1 has constant sectional curvature c so does M_2. If $M_1 =$*

S^m then $M_2 = S^m$.

Proof: (a) follows from Theorem 4.1 (b). If $m = 2$, we use the Gauss-Bonnet theorem to see the Euler-Poincare characteristic is a spectral invariant:

$$\chi(M) = (4\pi)^{-1}\textstyle\int_M \tau = 6a_2(\Delta). \tag{4.4}$$

In particular, if the M_i are Riemann surfaces, they are diffeomorphic. The proof of (c) is a bit more delicate. Let c and ϵ be real parameters. Define:

$$\begin{aligned} \tilde{\rho} = {}& \rho - (m-1)c\delta \\ W_{ijkl}(\epsilon) = {}& R_{ijkl} - \epsilon\{\tilde{\rho}_{il}\delta_{jk} + \tilde{\rho}_{jk}\delta_{il} - \tilde{\rho}_{ik}\delta_{jl} - \tilde{\rho}_{jl}\delta_{ik}\} \\ & -c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \end{aligned} \tag{4.5}$$

Lemma 4.3:

(a) *M has constant sectional curvature c $\Leftrightarrow$ $|W|^2 = |\tilde{\rho}|^2 = 0$.*

(b) $|\tilde{\rho}|^2 = |\rho|^2 + O(c)$.

(c) $|W|^2 = |R|^2 - (8\epsilon + (8-4m)\epsilon^2)|\tilde{\rho}|^2 + 4\epsilon^2\tau^2 + O(c)$.

Proof: We follow the argument of Berger and Tanno [Ber, Ta]. If M has constant sectional curvature c,

$$R_{ijkl} = c\delta_{il}\delta_{jk} - c\delta_{ik}\delta_{jl} \tag{4.6}$$

so $\rho_{ij} = (m-1)c\delta_{ij}$ and $\tilde{\rho} = 0$. Thus $W = R_{ijkl} - c\delta_{il}\delta_{jk} - c\delta_{ik}\delta_{jl} = 0$. Conversly, if $\tilde{\rho} = 0$, then $W = R_{ijkl} - c\delta_{il}\delta_{jk} - c\delta_{ik}\delta_{jl} = 0$ so R has constant sectional curvature c. This proves (a); (b) is immediate. We prove (c) by computing:

$$\begin{aligned} |W|^2 = {}& W_{ijkl}W_{ijkl} \\ = {}& |R|^2 - 2\epsilon\{\tilde{\rho}_{il}R_{ijjl} + \tilde{\rho}_{jk}R_{ijki} - \tilde{\rho}_{ik}R_{ijkj} - \tilde{\rho}_{jl}R_{ijil}\} \\ & +\epsilon^2\{\tilde{\rho}_{il}\tilde{\rho}_{il}\delta_{jk}\delta_{jk} + \tilde{\rho}_{jk}\tilde{\rho}_{jk}\delta_{il}\delta_{il} + \tilde{\rho}_{ik}\tilde{\rho}_{ik}\delta_{jl}\delta_{jl} + \tilde{\rho}_{jl}\tilde{\rho}_{jl}\delta_{ik}\delta_{ik} \\ & +2\tilde{\rho}_{il}\tilde{\rho}_{jk}\delta_{jk}\delta_{il} - 2\tilde{\rho}_{il}\tilde{\rho}_{ik}\delta_{jk}\delta_{jl} - 2\tilde{\rho}_{il}\tilde{\rho}_{jl}\delta_{jk}\delta_{ik} \\ & -2\tilde{\rho}_{jk}\delta_{il}\tilde{\rho}_{ik}\delta_{jl} - 2\tilde{\rho}_{jk}\tilde{\rho}_{jl}\delta_{il}\delta_{ik} + 2\tilde{\rho}_{ik}\tilde{\rho}_{jl}\delta_{jl}\delta_{ik}\} + O(c) \\ = {}& |R|^2 - (8\epsilon + (8-4m)\epsilon^2)|\tilde{\rho}|^2 + 4\epsilon^2\tau^2 + O(c). \end{aligned} \tag{4.7}$$

■

Let $g(\epsilon, m) = 8\epsilon + (8-4m)\epsilon^2 - 1$. We use Lemma 4.3 to decompose:

$$\begin{aligned} a_4(\Delta) &= 360^{-1}(4\pi)^{-1}\textstyle\int_M \ 5\tau^2 - 2\rho^2 + 2R^2 \\ &= 360^{-1}(4\pi)^{-1}\textstyle\int_M \ 2|W|^2 + 2g(\epsilon, m)|\tilde\rho|^2 \\ &\qquad +(5-8\epsilon^2)(\tau - c)^2 + O(c). \end{aligned} \tag{4.8}$$

The missing terms are multiples of $\int c\tau$ and $\int c^2$ which are controlled by a_0 and a_2. Consequently, we can find universal constants $\alpha_i = \alpha_i(c, \epsilon)$ so:

$$(\alpha_0 a_0 + \alpha_2 a_2 + \alpha_4 a_4)(\Delta) = \textstyle\int_M 2|W|^2 + 2g(\epsilon, m)|\tilde\rho|^2 + (5-8\epsilon^2)(\tau - c)^2. \tag{4.9}$$

If we can choose $\epsilon \in [0, \frac{1}{2}]$ so $g(\epsilon, m)$ is positive, this spectral invariant will vanish if and only if M has constant sectional curvature c. We note:

$$g(\tfrac{1}{2}, 2) = 3 \ > \ g(\tfrac{1}{2}, 3) = 2 \ > \ g(\tfrac{1}{2}, 4) = 1 \ > \ 0. \tag{4.10}$$

The function g is maximal when $\epsilon = (m-2)^{-1}$; $(m-2)^{-1} \in [0, \frac{1}{2}]$ for $m \geq 4$ and

$$g((m-2)^{-1}, m) = \tfrac{4}{m-2} - 1 > 0 \text{ for } m < 6. \ \blacksquare$$

Remark: Tanno [Ta] has proved Theorem 4.2 (c) also holds in the limiting case $m = 6$. He has also shown isospectral deformations of spherical space form are trivial; this analysis uses a_6.

Let $\Delta_p = d_{p-1}\delta_{p-1} + \delta_p d_p$ be the Laplacian on p forms. We may expand

$$\mathrm{Tr}_{L^2}(e^{-t\Delta_p}) \sim \Sigma_n a_n(\Delta_p) t^{(n-m)/2} \tag{4.11}$$

where the $a_n(\Delta_p)$ are spectral invariants of the Laplacian on p forms. We can generalize Theorem 4.1 to study the geometry of the form valued Laplacian. The $a_n(\Delta_p)$ are locally computable. If n is odd and the boundary of M is empty, $a_n(\Delta_p) = 0$. Introduce constants:

$$\begin{aligned} c(m,p) &= \tbinom{m}{p} = \tfrac{m!}{p!(m-p)!}, \\ c_0(m,p) &= c(m,p) - 6c(m-2, p-1), \\ c_1(m,p) &= 5c(m,p) - 60c(m-2,p-1) + 180c(m-4,p-2), \\ c_2(m,p) &= -2c(m,p) + 180c(m-2,p-1) - 720c(m-4,p-2), \\ c_3(m,p) &= 2c(m,p) - 30c(m-2,p-1) + 180c(m-4,p-2). \end{aligned} \tag{4.12}$$

Set $c(m,p) = c_\nu(m,p) = 0$ for $p < 0$ or $p > m$. The invariants $a_n(\Delta_p)$ for $n \leq 4$ were computed by Patodi [Pa]:

Theorem 4.4: (Patodi) *Suppose the boundary of M is empty.*
(a) $a_0(\Delta_p) = (4\pi)^{-m/2}\int_M c(m,p)\cdot 1$.
(b) $a_2(\Delta_p) = (4\pi)^{-m/2}\frac{1}{6}\int_M c_0(m,p)\cdot\tau$
(c) $a_4(\Delta_p) = (4\pi)^{-m/2}\frac{1}{360}\int_M\{c_1(m,p)\tau^2 + c_2(m,p)\rho^2 + c_3(m,p)R^2\}$.

Remark: We refer to [Gi] for formulas which would permit a calculation of $a_6(\Delta_p)$ and to Avramidi [Av] for similar formulas relating to a_8.

Corollary 4.5: (Patodi) *Suppose* $spec(\Delta_p, M_1) = spec(\Delta_p, M_2)$ *for* $p = 0,1,2$.
(a) *If M_1 has constant scalar curvature $\tau = c$, so does M_2.*
(b) *If M_1 is Einstein, so is M_2.*
(c) *If M_1 has constant sectional curvature c, so does M_2.*

Proof: The matrix of coefficients $c_\nu(m,p)$ for $1 \le \nu \le 3$ and $0 \le p \le 2$ is non singular. Thus

$$\{\int_M 1,\ \int_M \tau,\ \int_M \tau^2,\ \int_M \rho^2,\ \int_M R^2\} \tag{4.13}$$

are invariants of spec(Δ_p, M) for $0 \le p \le 2$. The Corollary now follows. ∎

Remark: We will discuss two generalizations of this result to manifolds with bounary in §5.

It is natural at this point to study general operators with leading symbol given by the metric tensor. Let Greek indices ν, μ, etc. range from 1 through m and index the coordinate frame for $T(M)$. Let $ds^2 = g_{\nu\mu}dx^\nu \circ dx^\mu$ be the metric tensor and let

$$\Gamma_{\nu\mu}{}^{\sigma} = \tfrac{1}{2}g^{\sigma\epsilon}(\partial_\nu g_{\epsilon\mu} + \partial_\mu g_{\epsilon\nu} - \partial_\epsilon g_{\nu\mu}) \tag{4.14}$$

be the Christoffel symbols of the Levi-Civita connection. Let V be a smooth vector bundle over M and let D be a second order partial differential operator on $C^\infty(V)$ with leading symbol given by the metric tensor. In local coordinates, D has the form:

$$D = -(g^{\nu\mu} I \cdot \partial_\nu \partial_\mu + A^\nu \partial_\nu + B) \tag{4.15}$$

where A^ν and B are endomorphisms of V. We can work more invariantly. Let ∇ be a connection on V and let E be an auxilary endomorphism of V. Let

$$D(g, \nabla, E) = -(\Sigma_{\nu,\mu} g^{\nu\mu} \nabla_\nu \nabla_\mu + E) \tag{4.16}$$

on $C^\infty(V)$ be an operator with leading symbol given by the metric tensor.

Theorem 4.6: *If D is a second order operator on $C^\infty(V)$ with leading symbol given by the metric tensor, there exists a unique connection ∇ on V and a unique endomorphism E of V so $D = D(g, \nabla, E)$. If ω_ν is the connection 1-form of ∇,*

$$\begin{aligned} \omega_\nu &= \tfrac{1}{2} g_{\mu\nu}(A^\mu + g^{\sigma\epsilon}\Gamma_{\sigma\epsilon}{}^\mu) \\ E &= B - g^{\mu\sigma}(\partial_\sigma \omega_\mu + \omega_\mu \omega_\sigma - \omega_\epsilon \Gamma_{\mu\sigma}{}^\epsilon). \end{aligned}$$

Example : Let Δ_p be the Laplacian on p forms. The associated connection is the Levi-Civita connection and E is given by the Weitzenböch formulas. Let $\mathrm{ext}^\ell(\cdot)$ be left exterior multiplication and let $\mathrm{int}^\ell(\cdot)$ be the dual, left interior multiplication. Let

$$\gamma^\ell(\cdot) = \mathrm{ext}^\ell(\cdot) - \mathrm{int}^\ell(\cdot) \tag{4.17}$$

define a left Clifford module structure on the exterior algebra;

$$\gamma_i^\ell \gamma_j^\ell + \gamma_j^\ell \gamma_i^\ell = -2\delta_{ij}. \tag{4.18}$$

Define a right Clifford module structure γ^r similarly. Then

$$E = -\tfrac{1}{8} R_{ijkn} \gamma_i^l \gamma_j^l \gamma_k^r \gamma_n^r - \tfrac{1}{4}\tau. \tag{4.19}$$

By the Bianchi identities, E preserves the decomposition $\Lambda(M) = \oplus_p \Lambda^p(M)$ and restricts to endomorphisms E_p of $\Lambda^p(M)$. Let $*$ be Clifford multiplication. We illustrate these formulas by computing:

$$\begin{aligned} E_0(1) &= -\tfrac{1}{8} R_{ijkl} e_i * e_j * e_l * e_k - \tfrac{1}{4}\tau \\ &= \tfrac{1}{4} R_{ijkj} e_i * e_k - \tfrac{1}{4}\tau = \tfrac{1}{4}\tau - \tfrac{1}{4}\tau = 0 \\ E_1(e_n) &= -\tfrac{1}{8} R_{ijkl} e_i * e_j * e_n * e_l * e_k - \tfrac{1}{4}\tau e_n \\ &= (-\tfrac{1}{8} R_{ijkl} e_i * e_j * e_l * e_k - \tfrac{1}{4}\tau) * e_n - \tfrac{1}{2} R_{ijnl} e_i * e_j * e_l \\ &= E_0(1) e_n + R_{ijnj} e_i = -\rho_{in} e_i. \end{aligned} \tag{4.20}$$

Example : Let Δ^{spin} be the spin Laplacian. The associated connection is the spin connection and $E = -\frac{1}{4}\tau$.

Let $\{\lambda_\nu, \phi_\nu\}$ be the spectral resolution of $D = -(g^{\nu\mu}\nabla_\nu\nabla_\mu + E)$. The fundamental solution e^{-tD} of the heat equation is given by a smooth kernel function K :

$$\begin{aligned} K(t,x,y,D) &= \Sigma_\nu e^{-t\lambda_\nu}\phi_\nu(x)\otimes\phi_\nu^*(y), \\ e^{-tD}u(x) &= \textstyle\int K(t,x,y,D)u(y)dy, \\ \mathrm{tr}_V\{K(t,x,x,D)\} &= \Sigma_\nu e^{-t\lambda_\nu}|\phi_\nu(x)|^2, \\ \mathrm{Tr}_{L^2}(e^{-tD}) = \Sigma_\nu e^{-t\lambda_\nu} &= \textstyle\int_M \mathrm{tr}_V\, K(t,x,x,D). \end{aligned} \tag{4.21}$$

Theorem 4.7: *Let the boundary of M be empty. As $t \to 0^+$,*

$$\mathrm{tr}_V\, K(t,x,x,D) \sim \Sigma_n\ a_n(D)(x)t^{(n-m)/2}.$$

The $a_n(D)(x)$ are locally computable invariants which vanish for n odd.

$$\mathrm{Tr}_{L^2}(e^{-tD}) \sim \Sigma_n a_n(D)t^{(n-m)/2}\ for\ a_n(D) = \textstyle\int_M a_n(D)(x).$$

Theorem 4.8: *Let Ω be the curvature of the connection ∇ on V.*

(a) $a_0(D)(x) = (4\pi)^{-m/2}\,\mathrm{tr}(1)$.

(b) $a_2(D)(x) = (4\pi)^{-m/2}6^{-1}\,\mathrm{tr}(6E+\tau)$.

(c) $a_4(D)(x) = (4\pi)^{-m/2}360^{-1}\,\mathrm{tr}\{60E_{;kk} + 60\tau E + 180E^2 + 30\Omega_{ij}\Omega_{ij} + 12\tau_{;kk}$
$+5\tau^2 - 2\rho^2 + 2R^2\}$.

Remark: See [Gi] for $a_6(D)(x)$ and Avramidi [Av] for $a_8(D)(x)$. We specialize to the case $D = \Delta_p$ to derive Theorems 4.1 and 4.4.

This gives complete information concerning a_n for $n \leq 8$. Partial information about all the coefficients is also available.

Theorem 4.9: *Let the boundary of M be empty and let $n \geq 3$. Let*

$$c(n) = (4\pi)^{-m/2}/\{(-1)^n \cdot 2^{n+1} \cdot 1 \cdot 3 \cdot \ldots \cdot (2n+1)\}$$

Modulo cubic and higher degree terms which have fewer covariant derivatives,

$$a_{2n}(D) = c(n)\int_M \operatorname{tr}\{(n^2-n-1)|\nabla^{n-2}\tau|^2 + 2|\nabla^{n-2}\rho|^2$$
$$+4(2n+1)(n-1)\nabla^{n-2}\tau\cdot\nabla^{n-2}E + 2(2n+1)\nabla^{n-2}\Omega\cdot\nabla^{n-2}\Omega$$
$$+4(2n+1)(2n-1)\nabla^{n-2}E\cdot\nabla^{n-2}E + \ldots\}.$$

Remark: See Osgood, Phillips and Sarnak [OPS] if $m = 2$ and $D = \Delta_0$. This theorem plays a crucial role in the compactness results of Theorem 2.3.

§5 Heat Equation Asymptotics for manifolds with boundary.

We must impose suitable boundary conditions to generalize Theorems 4.7 and 4.8 to manifolds with boundary. Near the boundary, let e_m be the inward unit geodesic normal. Let Roman indices $\{a, b, \ldots\}$ range from 1 through $m-1$ and index an orthonormal frame for the tangent bundle of the boundary. Define the second fundamental form by

$$L_{ab} = (\nabla_{e_a} e_b, e_m). \tag{5.1}$$

Let $D = -(g^{\nu\mu}\nabla_\nu\nabla_\mu + E)$ on $C^\infty(V)$. Let χ be a smooth endomorphism of $V|_{\partial M}$ with $\chi^2 = 1$. Let

$$\Pi_N = \tfrac{1}{2}(1-\chi) \text{ and } \Pi_D = \tfrac{1}{2}(1+\chi) \tag{5.2}$$

be projection on the ± 1 eigenspaces of χ. Let S be a smooth endomorphism of range(Π_N). If $\phi \in C^\infty(V)$, let

$$\begin{aligned} B\phi &= \Pi_N\{(\nabla_{e_m} + S)\phi|_{\partial M}\} \oplus \Pi_D\{\phi|_{\partial M}\}; \\ \text{domain}(D_B) &= \{\phi \in C^\infty(V) : B\phi = 0\}. \end{aligned} \tag{5.3}$$

Let $\{\lambda_\nu, \phi_\nu\}$ be a spectral resolution of D_B. Let

$$K(t, x, y, D) = \Sigma_\nu e^{-t\lambda_\nu}\phi_\nu(x) \otimes \phi_\nu^*(y) \tag{5.4}$$

be the kernel of the heat operator e^{-tD_B}. On the diagonal K behaves like a distribution; we study this phenomena by localizing with a test function

$f \in C^\infty(M)$. Let $\nabla_m^\nu f$ be the ν^{th} normal covariant derivative of f. Let ";" be multiple covariant differentiation with respect to the Levi-Civita connection ∇^{LC} of M and let ":" be multiple covariant differentiation tangentially with respect to the Levi-Civita connection ∇^{lc} of the boundary.

Theorem 5.1: *As $t \to 0^+$ there is an asymptotic series of the form:*

$$\mathrm{Tr}_{L^2}(fe^{-tD_B}) \sim \Sigma_n a_n(f, D_B) t^{(n-m)/2}.$$

There are local invariants $a_n(D)(x)$ defined on M and local invariants $a_{n,\nu}(D_B)(y)$ defined on the boundary ∂M so

$$a_n(f, D_B) = \textstyle\int_M f \cdot a_n(D)(x) + \Sigma_\nu \int_{\partial M} \nabla_m^\nu f \cdot a_{n,\nu}(D_B)(y).$$

Remark: The usual heat equation asymptotics arise by setting $f = 1$. They need not vanish if n is odd.

Theorem 5.2: (Branson and Gilkey [BG], Moss and Dowker [MD])

(a) $a_0(f, D_B) = (4\pi)^{-m/2} \int_M f\, \mathrm{tr}(1)$.

(b) $a_1(f, D_B) = \frac{1}{4}(4\pi)^{-(m-1)/2} \int_{\partial M} f\, \mathrm{tr}(\chi)$.

(c) $a_2(f, D_B) = \frac{1}{6}(4\pi)^{-m/2}\{\int_M f\, \mathrm{tr}(6E + \tau) + \int_{\partial M} f\, \mathrm{tr}(2fL_{aa} + 12fS)$
$+f_{;m}\, \mathrm{tr}(3\chi)\}$

(d) $a_3(f, D_B) = \frac{1}{384}(4\pi)^{-(m-1)/2}\{\int_{\partial M} f\, \mathrm{tr}\{96\chi E + 16\chi\tau + 8f\chi R_{amam}$
$+(13\Pi_N - 7\Pi_D)L_{aa}L_{bb} + (2\Pi_N + 10\Pi_D)L_{ab}L_{ab} + 96SL_{aa}$
$+192S^2 - 12\chi_{:a}\chi_{:a}\} + f_{;m}\, \mathrm{tr}\{(6\Pi_N + 30\Pi_D)L_{aa} + 96S\} + f_{;mm}\, \mathrm{tr}(24\chi)\}$

(e) $a_4(f, D_B) = \frac{1}{360}(4\pi)^{-m/2}\{\int_M f\, \mathrm{tr}\{60E_{;kk} + 60\tau E + 180E^2 + 30\Omega^2 + 12\tau_{;kk}$
$+5\tau^2 - 2\rho^2 + 2R^2\} + \int_{\partial M} f\, \mathrm{tr}\{(240\Pi_N - 120\Pi_D)E_{;m} + (42\Pi_N - 18\Pi_D)\tau_{;m}$
$+24L_{aa:bb} + 0L_{ab:ab} + 120EL_{aa} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab}$
$+4R_{abcb}L_{ac} + 0\Omega_{im:i} + 21^{-1}((280\Pi_N + 40\Pi_D)L_{aa}L_{bb}L_{cc} + 0\chi_{:a}\Omega_{am}$
$+(168\Pi_N - 264\Pi_D)L_{ab}L_{ab}L_{cc} + (224\Pi_N + 320\Pi_D)L_{ab}L_{bc}L_{ac}) + 720SE$

$$+120S\tau + 0SR_{amam} + 144SL_{aa}L_{bb} + 48SL_{ab}L_{ab} + 480S^2L_{aa} + 480S^3$$
$$+120S_{:aa} + 60\chi\chi_{:a}\Omega_{am} - 42\chi_{:a}\chi_{:a}L_{bb} + 6\chi_{:a}\chi_{:b}L_{ab} - 120\chi_{:a}\chi_{:a}S\}$$
$$+f_{;m}\operatorname{tr}(180\chi E + 30\chi\tau + 0R_{amam} + (84\Pi_N - 180\Pi_D)/7 \cdot L_{aa}L_{bb}$$
$$+(84\Pi_N + 60\Pi_D)/7 \cdot L_{ab}L_{ab} + 72SL_{aa} + 240S^2 - 18\chi_{:a}\chi_{:a})$$
$$+f_{;mm}\operatorname{tr}(24L_{aa} + 120S) + f_{;iim}\operatorname{tr}(30\chi)\}.$$

Remark: See also Dowker and Schofield [DS], Kennedy et al [KCD], Melmed [Me], Moss [Mo], and Smith [Sm] for related results. McAvity and Osborn [MO] have computed a_0, a_1, and a_2 for boundary conditions of the form $(\partial_n + Q)\phi|_{\partial P} = 0$ where Q is a suitable first order tangential differential operator; the invariants exhibit non polynomial dependence in the leading symbol of Q.

Absolute and relative boundary conditions are natural boundary conditions which are motivated by index theory and which fit into this framework. Let $y = (y^1, ..., y^{m-1})$ be local coordinates on the boundary and let x^m be the geodesic distance to the boundary; $x = (y, x^m)$ is a system of local coordinates near the boundary in which the metric has the form:

$$ds^2 = g_{\alpha\beta}dy^\alpha \circ dy^\beta + dx^m \circ dx^m. \tag{5.5}$$

If $I = \{1 \le \alpha_1 < \alpha_2 < ... < \alpha_p \le m-1\}$ is a multi-index, let

$$dy^I = dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_p} \in \Lambda^p(\partial M). \tag{5.6}$$

Decompose the exterior algebra $\Lambda M|_{\partial M} = \Lambda_N \oplus \Lambda_D$ where

$$\Lambda_N = \operatorname{span}\{dy^I\} \text{ and } \Lambda_D = \operatorname{span}\{dx^m \wedge dy^I\}; \tag{5.7}$$

Λ_N are the tangential differential forms and Λ_D are the forms which vanish on the boundary. Absolute boundary conditions $\mathcal{B}_a$ are defined by taking Neumann boundary conditions on Λ_N and Dirichlet boundary conditions on Λ_D. More precisely, if $\omega \in C^\infty \Lambda M$, decompose $\omega = \Sigma_I\{f_I dy^I + g_I dx^m \wedge dy^I\}$. Then

$$\mathcal{B}_a(\omega) = \{\Sigma_I\ \partial_m f_I dy^I\}|_{\partial M} \oplus \{\Sigma_I\ g_I dy^I\}|_{\partial M} \in \Lambda(\partial M) \oplus \Lambda(\partial M). \tag{5.8}$$

Relative boundary conditions $\mathcal{B}_r$ are defined using the Hodge duality operator $\star$;

$$\mathcal{B}_r(\omega) = \mathcal{B}_a(\star\omega). \tag{5.9}$$

Exterior differentiation d preserves the boundary conditions $\mathcal{B}_a$; interior differentiation δ preserves the boundary conditions $\mathcal{B}_r$. For $\epsilon \in \{r, a\}$, let $\Delta_{p,\epsilon}$ be defined by the boundary conditions $\mathcal{B}_\epsilon$; $\Delta_{p,\epsilon}$ has a self-adjoint non-negative extension to $L^2\Lambda^pM$. We note

$$a_n(\Delta_{p,r}) = a_n(\Delta_{m-p,a}) \tag{5.10}$$

so we need only consider absolute boundary conditions. Define constants:

$$\begin{aligned}
c(m,p) &= \binom{m}{p} = \tfrac{m!}{p!(m-p)!},\\
c_0(m,p) &= c(m,p) - 6c(m-2,p-1),\\
d_0(m,p) &= c(m-1,p) - c(m-1,p-1),\\
d_1(m,p) &= 16d_0(m,p) - 96d_0(m-2,p-1),\\
d_2(m,p) &= 8d_0(m,p) - 192d_0(m-2,p-1),\\
d_3(m,p) &= 3c(m,p) + 10d_0(m,p) - 96d_0(m-2,p-1),\\
d_4(m,p) &= 6c(m,p) - 4d_0(m,p) + 96d_0(m-2,p-1).
\end{aligned} \tag{5.11}$$

Set $c(m,p) = c_0(m,p) = d_\nu(m,p) = 0$ for $p < 0$ or $p > m$.

Theorem 5.3 (Blažić, Bokan, and Gilkey [BBG]) :
(a) $a_0(\Delta_{p,a}) = (4\pi)^{-m/2}\int_M c(m,p)$.
(b) $a_1(\Delta_{p,a}) = (4\pi)^{-(m-1)/2}\int_{\partial M} d_0(m,p)$.
(c) $a_2(\Delta_{p,a}) = (4\pi)^{-m/2}\frac{1}{6}c_0(m,p)\{\int_M \tau + \int_{\partial M} 2L_{aa}\}$.
(d) $a_3(\Delta_{p,a}) = (4\pi)^{-(m-1)/2}\frac{1}{384}\{\int_{\partial M} d_1(m,p)\tau + d_2(m,p)R_{amam}$
$\qquad + d_3(m,p)L_{aa}L_{bb} + d_4(m,p)L_{ab}L_{ab}\}$.

We can also study the total form valued Laplacian

$$\Delta_a = \oplus_p \Delta_{p,a};\ a_n(\Delta_a) = \Sigma_p a_n(\Delta_{a,p}). \tag{5.12}$$

Theorem 5.4 (Blažić, Bokan, and Gilkey) :

(a) $a_0(\Delta_a) = 2^m(4\pi)^{-m/2}\int_M 1$.

(b) $a_1(\Delta_a) = 0$.

(c) $a_2(\Delta_a) = -2^{m-1}\frac{1}{6}(4\pi)^{-m/2}\{\int_M \tau + \int_{\partial M} 2L_{aa}\}$.

(d) $a_3(\Delta_a) = 2^m(4\pi)^{-(m-1)/2}\frac{1}{384}\{\int_{\partial M} 3L_{aa}L_{bb} + 6L_{ab}L_{ab}\}$.

Although the calculation of $a_4(\Delta_{p,a})$ is in general quite difficult, there are two special cases which are important.

Theorem 5.5 (Blažić, Bokan, and Gilkey):

(a) $a_4(\Delta_{0,a}) = (4\pi)^{-m/2}\frac{1}{360}\{\int_M\{5\tau^2 - 2\rho^2 + 2R^2\}$
$+\int_{\partial M}\{30\tau_{;m} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac}$
$+\frac{1}{21}(280L_{aa}L_{bb}L_{cc} + 168L_{ab}L_{ab}L_{cc} + 224L_{ab}L_{bc}L_{ac})\}\}$

(b) $a_4(\Delta_{m,a}) = (4\pi)^{-m/2}\frac{1}{360}\{\int_M(5\tau^2 - 2\rho^2 + 2R^2)$
$+\int_{\partial M}\{-30\tau_{;m} + 20\tau L_{aa} + 4R_{amam}L_{bb} - 12R_{ambm}L_{ab} + 4R_{abcb}L_{ac}$
$+\frac{1}{21}(40L_{aa}L_{bb}L_{cc} - 264L_{ab}L_{ab}L_{cc} + 320L_{ab}L_{bc}L_{ac})\}\}$

We use these results to generalize Corollary 4.5 to manifolds with boundary:

Corollary 5.6 (Blažić, Bokan, and Gilkey): *Let* $spec(\Delta_{p,a}, M_1) = spec(\Delta_{p,a}, M_2)$ *for all* p.

(a) *If* ∂M_1 *is totally geodesic, so is* ∂M_2.

(b) *Assume* ∂M_1 *is totally geodesic.*

(i) *If* M_1 *has constant scalar curvature* $\tau = c$, *so does* M_2.

(ii) *If* M_1 *is Einstein, so is* M_2.

(iii) *If* ∂M_1 *has constant sectional curvature* c, *so does* M_2.

Proof: The following assertions are equivalent

(1) ∂M is totally geodesic.

(2) $L \equiv 0$.

(3) $\int_{\partial M} L_{ab}L_{ab} = 0$.

(4) $\int_{\partial M}\{3L_{aa}L_{bb} + 6L_{ab}L_{ab}\} = 0$.

We use Theorem 5.4 to see the spectrum of Δ_a determines if the boundary is totally geodesic and prove (a). We set $L = 0$ in (b) and use Theorem 5.5 to see

$$a_4(\Delta_{0,a}) - a_4(\Delta_{m,a}) = (4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_{\partial M} 60\tau_{;m}. \tag{5.13}$$

Consequently this integrand is a spectral invariant and vanishes under the hypothesis of (b-i)-(b-iii). We use Theorem 5.2 to see all the boundary integrands of $a_n(\Delta_{p,a})$ vanish for $n \leq 4$. Let the constants $c_i(m,p)$ be as in Theorem 4.4. By Theorems 4.4 and 5.2,

$$a_4(\Delta_{p,a}) = (4\pi)^{-m/2}\tfrac{1}{360}\textstyle\int_M \{c_1(m,p)\tau^2 + c_2(m,p)\rho^2 + c_3(m,p)R^2\}; \tag{5.14}$$

$\int_M \Delta\tau = \int_{\partial M}\tau_{;m} = 0$ so this invariant does not appear; the remainder of the proof is the same as that given for Corollary 4.5. ■

We conclude this section by discussing small geodesic balls. Let $B_r(x)$ be the geodesic ball of radius r about some point $x \in M$ and let $\operatorname{spec}(\Delta_{0,D}, B_r(x))$ be the spectrum of the scalar Laplacian with Dirichlet (relative) boundary conditions. Let $a_n(r) = a_n(r,x)$ be the asymptotics of the heat equation. Let

$$\begin{aligned} v(m) &= \operatorname{volume}(S^{m-1}) = m\cdot\pi^{m/2}\cdot\Gamma(\tfrac{1}{2}m+1)^{-1}\\ \alpha(m) &= v(m)/(6m)\\ \beta(m) &= v(m)/(360m(m+2)). \end{aligned} \tag{5.15}$$

Theorem 5.7 (Blažić, Bokan, and Gilkey):

(a) $a_0(r) = (4\pi)^{-m/2}\{r^m v(m)m^{-1} - r^{m+2}\alpha(m)(m+2)^{-1}\tau$
$+r^{m+4}\beta(m)(m+4)^{-1}(18\Delta\tau + 5\tau^2 + 8\rho^2 - 3R^2) + O(r^{m+6})\}$.

(b) $a_1(r) = -4^{-1}(4\pi)^{-(m-1)/2}\{v(m)r^{m-1} - r^{m+1}\alpha(m)\tau$
$+r^{m+3}\beta(m)(18\Delta\tau + 5\tau^2 + 8\rho^2 - 3R^2) + O(r^{m+5})\}$.

(c) $a_2(r) = 6^{-1}(4\pi)^{-m/2}\{v(m)r^{m-2}2(m-1) - r^m\alpha(m)(2m-4)\tau$
$+r^{m+2}\beta(m)(18(2m-4)\Delta(\tau) + 5(2m-6)\tau^2$
$+8(2m+6)\rho^2 - 3(2m+6)R^2 + O(r^{m+4})\}$.

(d) $a_3(r) = -384^{-1}(4\pi)^{-(m-2)/2}\{v(m)r^{m-3}(7m^2 - 24m + 17)$
$-r^{m-1}\alpha(m)(7m^2 - 92m - 3)\tau$

$$+r^{m+1}\beta(m)(18(7m^2-128m-279)\Delta\tau+5(7m^2-160m-351)\tau^2$$
$$+8(7m^2+32m+21)\rho^2-3(7m^2+32m+81)R^2+O(r^{m+3})\}.$$

Remark: There are similar formulas for Neumann (absolute) boundary conditions; the spectrum of the Laplacian on p forms has also been considered. One can also consider the conformal Laplacian with suitable boundary conditions.

The following is an immediate consequence:

Corollary 5.8 (Blažić, Bokan, and Gilkey): *Let M_1 be a homogeneous manifold. Suppose $spec(\Delta_{0,a}, B_r(x, M_1)) = spec(\Delta_{0,a}, B_r(y, M_2))$ for all $y \in M_2$ and any point x of M_1.*
(a) *If M_1 has constant scalar curvature $\tau = c$, so does M_2.*
(b) *If M_1 is Einstein, so is M_2.*
(c) *If M_1 has constant sectional curvature c, so does M_2.*

§6 Operators of Dirac Type

In some cases, it is possible to take a square root of D within the category of differential operators; for example the total form valued Laplacian has this property: $d\delta + \delta d = (d + \delta)^2$. This leads to additional spectral invariants. The Clifford algebra bundle is the natural setting for this discussion; it is the universal complex unital algebra bundle generated by the tangent bundle subject to the Clifford commutation relations

$$\partial_\nu * \partial_\mu + \partial_\mu * \partial_\nu = -2g_{\nu\mu}. \tag{6.1}$$

A Clifford module structure on V is a unital algebra morphism γ from the Clifford algebra of M to the bundle of endomorphisms of V. If $\gamma^\nu = \gamma(dx^\nu)$, then $\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = -2g^{\nu\mu}$. We may always choose a connection ∇ on V so $\nabla\gamma = 0$; fix such a connection henceforth. Let ψ be a smooth endomorphism of V and let

$$P = P(\nabla, \psi) = \gamma^\nu\nabla_\nu - \psi : C^\infty(V) \to C^\infty(V) \tag{6.2}$$

be an operator of Dirac type; any operator of Dirac type can be expressed in this form. We suppose the boundary of M empty for the moment.

Let $\{\mu_\nu, \phi_\nu\}$ be the spectral resolution of P and let $D = P^2$ be the associated Laplacian; $\{\lambda_\nu = \mu_\nu^2, \phi_\nu\}$ is the spectral resolution of D. The operator Pe^{-tD} is infinitely smoothing with kernel L;

$$\begin{aligned} L(t,x,y,P) &= P_x K(t,x,y,D) = \Sigma_\nu \mu_\nu e^{-t\mu_\nu^2}\phi_\nu(x)\otimes\phi_\nu^*(y) \\ Pe^{-tD}u(x) &= \textstyle\int L(t,x,y,P)u(y)dy, \\ \operatorname{tr}_V\{L(t,x,x,D)\} &= \Sigma_\nu \mu_\nu e^{-t\mu_\nu^2}|\phi_\nu(x)|^2, \\ \operatorname{Tr}_{L^2}(Pe^{-tD}) &= \Sigma_\nu \mu_\nu e^{-t\mu_\nu^2} = \textstyle\int_M \operatorname{tr}_V L(t,x,x,P). \end{aligned} \tag{6.3}$$

Theorem 6.1: *Let the boundary of M be empty. As $t \to 0^+$,*

$$\operatorname{tr}_V L(t,x,x,P) \sim \Sigma_n\, a_n(P)(x) t^{(n-m-1)/2}.$$

The $a_n(D)(x)$ are locally computable invariants which vanish for n even.

$$\operatorname{Tr}_{L^2}(Pe^{-tD}) \sim \Sigma_n a_n(P) t^{(n-m-1)/2} \ \textit{for}\ a_n(P) = \textstyle\int_M a_n(P)(x).$$

Let $\{e_i\}$ be a local orthonormal frame for $T(M)$ and let $\gamma_i = \gamma(e_i)$. Let $\Psi = \gamma_i\psi\gamma_i$ and let $W_{ij} = \Omega_{ij} - \frac{1}{4}R_{ijkl}\gamma_k\gamma_l$.

Theorem 6.2: (Branson and Gilkey [BG]). *If the boundary of M is empty,*

(a) $a_1(P)(x) = (4\pi)^{-m/2}(m-1)\operatorname{tr}\{\psi\}$.

(b) $a_3(P)(x) = -12^{-1}(4\pi)^{-m/2}\operatorname{tr}\{\{2(1-m)\psi_{;i} + 3(4-m)\psi\gamma_i\psi + 3\Psi\gamma_i\psi\}_{;i}$
$+(m-3)\{\tau\psi + 6\gamma_i\gamma_j W_{ij}\psi + 6\psi\psi_{;i}\gamma_i + (4-m)\psi\psi\psi + 3\psi\psi\Psi\}\}$.

If $\partial M \neq \emptyset$, we must impose boundary conditions analogous to those discussed in §5. Let χ be an endomorphism of $V|_{\partial M}$ with $\chi^2 = 1$ so that

$$\gamma_m\chi + \chi\gamma_m = \gamma_a\chi - \chi\gamma_a = 0; \tag{6.4}$$

χ always exists if m is even; if m is odd, χ need not exist. Let

$$\operatorname{domain}(P_\chi) = \{\phi \in C^\infty(V) : (1+\chi)(\phi|_{\partial M}) = 0\}. \tag{6.5}$$

As before, we may expand the trace in an asymptotic series:

$$\operatorname{Tr}_{L^2}(fP_\chi e^{-tD_\chi}) \sim \Sigma_n a_n(f,P,\chi) t^{(n-m)/2}. \tag{6.6}$$

Theorem 6.3 (Branson-Gilkey [BG]):

(a) $a_0(f,P,\chi)=0$.

(b) $a_1(f,P,\chi)=(4\pi)^{-m/2}\int_M(m-1)f\operatorname{tr}(\psi)$.

(c) $a_2(f,P,\chi)=4^{-1}(4\pi)^{-(m-1)/2}\int_{\partial M}(m-2)f\operatorname{tr}(\psi\chi)$.

(d) $a_3(f,P,\chi)=-12^{-1}(4\pi)^{-m/2}\{\int_M f\operatorname{tr}\{2(1-m)\psi_{;i}+3(4-m)\psi\gamma_i\psi+3\Psi\gamma_i\psi\}_{;i}$
$+(m-3)f\operatorname{tr}\{\tau\psi+6\gamma_i\gamma_jW_{ij}\psi+6\psi\psi_{;i}\gamma_i+(4-m)\psi\psi\psi+3\psi\psi\Psi\}$
$+\int_{\partial M}6(2-m)f_{;m}\operatorname{tr}\{\chi\psi\}+f\operatorname{tr}\{(18-6m)\chi\psi_{;m}+(2-2m)\psi_{;m}$
$-6\chi\gamma_m\gamma_a\psi_{;a}+6(m-2)\chi\psi L_{aa}+2(m-3)\psi L_{aa}+6(3-m)\chi\gamma_m\psi\psi$
$+3\gamma_m\psi\Psi_T+3(3-m)\chi\gamma_m\psi\chi\psi+6\chi\gamma_aW_{am}\}\}$.

These formulas play an important role in the index theorem for manifolds with boundary. We sketch just one application; there are others. For $Re(s)>>0$, define:

$$\eta(s,P,\chi)=\sum_{\lambda\neq 0}\operatorname{sign}(\lambda)|\lambda|^{-s}\dim\ E(\lambda,P_\chi). \tag{6.7}$$

Standard analytic arguments show η has a meromorphic extension to C with isolated simple poles on the real axis. Furthemore, modulo suitable normalizing constants,

$$\operatorname{Res}_{s=0}\eta(s,P,\chi)=a_m(1,P,\chi). \tag{6.8}$$

If the boundary of M is empty, η is regular at $s=0$ [APS, Gi] and we define

$$\eta(P)=\tfrac{1}{2}\{\eta(0,P)+\dim\ \ker(P)\} \tag{6.9}$$

as a measure of the spectral asymmetry of P; this invariant plays an important role in the index theorem for manifolds with boundary.

Theorem 6.2 shows the local eta invariant does not vanish so the regularity of η at $s=0$ is a global result. For example, if $m=3$,

$$a_3(1,P)(x)=-12^{-1}(4\pi)^{-m/2}\operatorname{tr}\{-4\psi_{;i}+3\psi\gamma_i\psi+3\Psi\gamma_i\psi\}_{;i} \tag{6.10}$$

is in divergence form so $\int a_3(1,P)(x)=0$. If the boundary of M is non empty, a similar computation may be made; we omit details in the interests of brevity.

§R References

Any list of references in this field is bound to be incomplete. We list below a selection of some of the papers in the field. We refer to [Be] and [BB] for a more comprehensive listing.

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