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MORSE THEORY VIA MODULI SPACES

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Introduction

In recent years the classical techniques of Morse theory on compact manifolds have been applied to various infinite dimensional settings and have yielded many important results. For example Floer's important work in symplectic geometry [5] revolved around a Morse theoretic analysis on the loop space LM^{2n} of a closed symplectic manifold M^{2n}. Also his "instanton homology" invariants of a homology 3 - sphere Σ were defined using a similar Morse theory on the space of connections on the trivial principal bundle $\Sigma \times SU(2)$ [6].

In an ongoing joint project with J.D.S Jones and G.B. Segal we have attempted to understand the underlying algebraic topological aspects of this type of infinite dimensional Morse theory. One feature that these examples have in common is that the indices of the critical points are infinite, although the *relative index* between any two critical points is finite. Here the relative index can be viewed as the dimension of the space of gradient flow lines connecting one critical point to the other. These spaces can be viewed as moduli spaces. In the above mentioned infinite dimensional examples considered by Floer, these spaces of flow lines turn out to be the moduli spaces of pseudo - holomorphic spheres in a symplectic manifold, and of asymptotically flat instantons on $\Sigma \times \mathbf{R}$, respectively. Thus in our project we were led to the question of reformulating the basic ideas of classical Morse theory purely in

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terms of the moduli spaces of flow lines in such a way that it would apply to various infinite dimensional settings, including those described above.

This was the theme of the minicourse I gave at the Campinas workshop on Gauge theory. In this paper I will describe some of the ideas and results of this project. In section 1 I will recall how, given a Morse function on a compact Riemannian manifold, one can associate a CW - complex with the same homotopy type as the manifold. I will then discuss the work of Franks [7] that describes the relative attaching maps of this complex in terms of the framed moduli spaces of flow lines. In section 2 I will describe the category of critical points and flow lines associated to a Morse function. I will then outline the main theorem of [4] which describes a method of completely recovering the topology of the manifold from this categorical data. In section 3 I will discuss various generalizations of this result, and in particular a theorem that applies in a rather large class of infinite dimensional examples. The basic properties required for this theorem to apply are a finiteness condition on the relative indices and a gluing procedure for flow lines (see (3.3)). In section 4 I will describe results concerning the stability of moduli spaces and will show how they can be proven using this machinery. Finally in section 5 I will discuss the notion of "Floer homotopy type" and compute it explicitly in the infinite dimensional example of the symplectic action on the loop space of the Riemann sphere. Most of the results described in sections 3 through 5 represent joint work in progress with Jones and Segal. Details of this work will appear in due course.

I would like to take this opportunity to thank A. Rigas, F. Burstall, and the other organizers of the workshop for the lively mathematical atmosphere at the meeting and their marvelous hospitality.

1. The CW complex of a Morse function

In this section we recall some basic constructions from classical Morse theory. Specifically we describe the CW - complex associated to a generic Morse function on a compact, Riemannian manifold, and recall Franks' theorem [7] describing the attaching maps of the cells in terms of the framed bordism classes of the moduli spaces of flows. In particular this gives a description of the boundary maps in the Morse - Smale chain complex.

We begin by establishing some notation and terminology. Let M be a C^∞, compact, Riemannian manifold of dimension n and

$$f : M \longrightarrow \mathbf{R}$$

a C^∞ map. A point $p \in M$ is a *critical point* of f if the differential $df_p : T_pM \longrightarrow \mathbf{R}$ is zero. (Here T_pM denotes the tangent space of M at p.) A critical point $p \in M$ is said to be *nondegenerate* if the Hessian $Hess_p(f)$ is nonsingular. $Hess_p(f)$ is a symmetric, bilinear form on the tangent space

$$Hess_p(f) : T_pM \times T_pM \longrightarrow \mathbf{R}$$

which, in terms of local coordinates $\{x_1, \cdots, x_n\}$ of a neighborhood of $p \in M$, is represented by the matrix of second order partial derivatives

$$Hess_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right).$$

The *index* of a critical point p, $\lambda(p)$, is defined to be the dimension of the negative eigenspace of $Hess_p(f)$. A *gradient flow line* (or *integral curve*) is a curve

$$\gamma : \mathbf{R} \longrightarrow M$$

that satisfies the following differential equation (the *flow equation*):

$$\frac{d\gamma}{dt} + \nabla_\gamma(f) = 0.$$

Here $\nabla(f)$ is the gradient vector field determined by f. Given the Riemannian metric, $\nabla(f)$ is determined by the property that

$$\langle \nabla_p(f); v \rangle = df_p(v)$$

where v is any tangent vector $v \in T_pM$.

A function $f : M \longrightarrow \mathbf{R}$ whose critical points are all nondegenerate is called a **Morse function.** From a topological point of view, the main goal of Morse theory is to obtain global topological information about the manifold M from the structure of the set of critical points of f and from the dynamics of the gradient vector field of f. An important example of this is the association of a CW - complex $C(f)$ of the same homotopy type as M, which has one cell of dimension λ for every critical point of index λ. We now quickly recall this construction. We refer the reader to [11] for details.

We begin by recalling the *regular neighborhood theorem.*

Theorem 1.1 *Let $f : M \longrightarrow \mathbf{R}$ be a smooth map on a compact Riemannian manifold with boundary. Suppose that f has no critical points and that $f(\partial M) = \{a, b\}$. Then there is a diffeomorphism*

$$F : f^{-1}(a) \times [a,b] \longrightarrow M$$

making the following diagram commute:

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & M \\ \text{proj.} \downarrow & & \downarrow f \\ [a,b] & \xrightarrow{=} & [a,b]. \end{array}$$

In particular all the level surfaces are diffeomorphic.

Now assume M is a compact manifold and let $f : M \longrightarrow \mathbf{R}$ be a smooth function. For $a \in \mathbf{R}$ we write

$$M^a = f^{-1}(-\infty, a] = \{x \in M \, : \, f(x) \leq a\}.$$

The next result follows immediately from the regular interval theorem.

Corollary 1.2 *Let $a < b$ and suppose that $f^{-1}[a,b] \subset M$ contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.*

This corollary says that the topology of the submanifolds M^a does not change with $a \in \mathbf{R}$ so long as a does not pass through a critical value. We now examine what happens to the topology of these submanifolds when one does pass through a critical value.

Fix a Morse function $f : M \longrightarrow \mathbf{R}$ on an n - dimensional closed manifold. For any point $x \in M$ let γ_x be the flow line through x. That is, it satisfies the flow equation

$$\frac{d\gamma}{dt} + \nabla_\gamma(f) = 0$$

and the initial condition $\gamma(0) = x$. Since M is compact one knows that $\gamma_x(t)$ tends to critical points of f as $t \to \pm\infty$. So for any critical point a of f we define the **stable manifold** $W^s(a)$ and the **unstable manifold** $W^u(a)$ as follows:

$$\begin{aligned} W^s(a) &= \{x \in M : \lim_{t\to+\infty} \gamma_x(t) = a\} \\ W^u(a) &= \{x \in M : \lim_{t\to-\infty} \gamma_x(t) = a\}. \end{aligned}$$

The following is referred to as the *Stable Manifold Theorem*. We refer the reader to [9] for a proof.

Theorem 1.3 *$W^u(a)$ is diffeomorphic to the disk D^λ, and $W^s(a)$ is diffeomorphic to the disk $D^{n-\lambda}$, where λ is the index of a.*

Let $c \in \mathbf{R}$ be a critical value of the Morse function $f : M \longrightarrow \mathbf{R}$ with $a_1, \cdots, a_k$ the set of critical points having $f(a_i) = c$. Let $\epsilon > 0$ be such that c is the only critical value in the open interval $(c - \epsilon, c + \epsilon)$. Finally, as above, let $W^u(a_i)$ be the unstable manifold of a_i, which is diffeormorphic to a disk D^{λ_i}, where λ_i is the index of the critical point a_i.

Theorem 1.4 *The inclusion of the subspace*

$$M^{c-\epsilon} \cup W^u(a_1) \cup \cdots \cup W^u(a_k) \hookrightarrow M^{c+\epsilon}$$

is a strong deformation retract.

We note also that the retraction ρ in this theorem can be taken to have the property that in a suitable coordinate system in a small neighborhood of the critical point a_i, in which we think of $f(x,y) = -|x|^2 + |y|^2$, where $x \in \mathbf{R}^{\lambda_i} \cong W^u(a_i)$, and $y \in \mathbf{R}^{n-\lambda_i} \cong W^s(a_i)$, then

$$\rho(x,y) = (x,0).$$

Assume $f : M \longrightarrow \mathbf{R}$ is a Morse function that satisfies the extra condition that for any two critical points a and b the unstable and stable manifolds $W^u(a)$ and $W^s(b)$ intersect transversally. This is the **Morse - Smale condition**, and it was shown by Smale in [17] that this is a generic condition. That is, an open, dense set of Morse functions satisfy this condition.

If $f : M \longrightarrow \mathbf{R}$ is a Morse - Smale function, then the deformation retractions of Corollary 1.2 and Theorem 1.4 allow one to define in the obvious way a CW - complex $C(f)$ homotopy equivalent to the manifold M whose cells correspond to the unstable manifolds of the critical points of f. See [11] for details. Hence by Theorem 1.3 $C(f)$ has one cell of dimension λ for each critical point of index λ.

Given a CW - complex X let $X^{(q)}$ denote the q - skeleton. For a q - cell D^q_α in X, let $\phi_\alpha : S^{q-1} \longrightarrow X^{(q-1)}$ be the attaching map. Let $r < q$ be the minimal positive integer so that the attaching map ϕ_α factors up to homotopy through the r - skeleton

$$\phi_\alpha : S^{q-1} \longrightarrow X^{(r)}.$$

For an r - cell D^r_β in X, define the *relative attaching map* of α and β

$$\phi_{\alpha,\beta} : S^{q-1} \longrightarrow S^r$$

to be the composition

$$\phi_{\alpha,\beta} : S^{q-1} \longrightarrow X^{(r)} \longrightarrow X^{(r)}/X^{(r-1)} \simeq \bigvee_{Cell_r} S^r \longrightarrow S^r_\beta \tag{1.5}$$

where $Cell_r$ denotes the set of r - cells of X and the last map is the projection onto the sphere given by the one point compactification of the cell D^r_β.

This relative attaching map, viewed as an element of the $(q-1)^{st}$ homotopy group of S^r, is represented, via the Thom - Pontryagin construction, by a $q-r-1$ - dimensional framed submanifold $N_{\alpha,\beta}$ of S^{q-1}. Perturbing $\phi_{\alpha,\beta}$ if necessary so that $0 \in \mathbf{R}^r \cup \infty = S^r$ is a regular value then

$$N_{\alpha,\beta} = \phi_{\alpha,\beta}^{-1}(0) \subset S^{q-1}$$

and has the induced framing on its normal bundle. This framed manifold is well defined up to framed cobordism.

In [7] Franks identified the relative attaching maps in the CW - complex $C(f)$ in terms of the corresponding framed cobordism classes. We now describe one of his results.

Continuing in the setting of a Morse - Smale function $f : M \longrightarrow \mathbf{R}$ we consider the intersection manifold

$$W(a,b) = W^u(a) \cap W^s(b)$$

where a and b are critical points of f. This is a manifold of dimension $p-r$, where $p = index(a)$ and $r = index(b)$. We refer to this number as the *relative index* of a and b. $W(a,b)$ is the space of all points that lie on flow lines starting from a and ending at b.

The space $W(a,b)$ has a natural free action of the real line $\mathbf{R}$ given by the flow of $\nabla(f)$. That is, the action

$$W(a,b) \times \mathbf{R} \longrightarrow W(a,b)$$

is defined by

$$(x,t) \longrightarrow \gamma_x(t)$$

where, as above, γ_x is the unique flow through x satisfying $\gamma_x(0) = x$. Notice that if we pick any point t between $f(a)$ and $f(b)$ and set $W(a,b)^t$ to be the submanifold $W(a,b) \cap f^{-1}(t)$ then this action restricts to give a diffeomorphism

$$W(a,b)^t \times \mathbf{R} \xrightarrow{\cong} W(a,b).$$

Therefore we may form the quotient orbit space

$$\mathcal{M}(a,b) = W(a,b)/\mathbf{R}.$$

This space will be referred to as the **moduli space of flow lines** from a to b. This terminology is justified by the fact that two points x and $y \in W(a,b)$ represent the same element in $\mathcal{M}(a,b)$ if and only if they lie on the same flow line. Notice furthermore that the composition

$$W(a,b)^t \hookrightarrow W(a,b) \longrightarrow W(a,b)/\mathbf{R} = \mathcal{M}(a,b)$$

is a diffeomorphism for any value t between $f(a)$ and $f(b)$, and hence $\mathcal{M}(a,b)$ is a manifold of dimension $p-r-1$ and can be viewed as a subspace of $W(a,b)$.

One of the results in [17] is the transitivity property. That is, if a and b are critical points of a Morse - Smale function f, then we write $a > b$ if $W(a,b)$, or equivalently $\mathcal{M}(a,b) \neq \emptyset$. The transitivity property then says that if $\mathcal{M}(a,b)$ and $\mathcal{M}(b,c)$ are nonempty, then so is $\mathcal{M}(a,c)$. This implies that " $<$ " is a partial ordering of the critical points.

Lemma 1.6 *Let $\{\alpha_n\}$ be a Cauchy sequence in the moduli space $\mathcal{M}(a,b)$ and suppose that α_n does not converge in $\mathcal{M}(a,b)$ as $n \to \infty$. Then there is a finite sequence of critical points*

$$a = a_0 > a_1 > \ldots > a_l > a_{l+1} = b$$

with $l \geq 1$ and flow-lines γ_i joining a_{i-1} to a_i, where $1 \leq i \leq l$, with the following property. Given $\epsilon > 0$ there is an integer $N > 0$ such that

$$d(\gamma_i, \alpha_n) = \inf_t d(\gamma_i(0), \alpha_n(t)) < \epsilon$$

for all $n \geq N$.

Proof We will identify $\mathcal{M}(a,b)$ with the subspace $W^t(a,b)$ of M as described above. Choose a real number t_1 so that there is no critical value of f between $f(a)$ and t_1. The sequence α_n gives a Cauchy sequence x_n in $W^{t_1}(a,b)$

which does not converge in $W^{t_1}(a,b)$. However, since M is compact, this sequence has a limit y_1 in M. Let $\gamma_1(t)$ be the flow line through y_1 and let a_0 and a_1 be the limit points of this flow line,

$$\lim_{t\to-\infty}\gamma_1(t) = a_0, \qquad \lim_{t\to\infty}\gamma_1(t) = a_1.$$

We want to show that $a_0 = a$. The map $(x,t) \mapsto \gamma_x(t)$ is continuous so

$$f(\gamma_1(t)) = \lim_{n\to\infty} f(\gamma_{x_n}(t))$$

and since f decreases along flow lines, it follows that if $t \le 0$ then $f(a) > f(\gamma_{x_n}(t)) > t_1$, and so $f(a) \ge f(\gamma_1(t)) \ge t_1$. Letting $t \to -\infty$ we have that $f(a) \ge f(a_0) \ge t_1$. From the choice of t_1 there are two alternatives. Either $a = a_0$, or $a \neq a_0$ but $f(a) = f(a_0)$. Suppose the latter holds. By the facts that the critical points are isolated and that f decreases along gradient flows, then for sufficiently small ϵ there is a disk $D_\epsilon(a_0)$ of radius ϵ about a_0 with the property that if γ is any flow line beginning at a, then $\gamma(t) \notin D_\epsilon(a_0)$ for all t. Now because $\lim_{t\to-\infty}\gamma_1(t) = a_0$ there is some t_0 such that $\gamma_1(t_0) \in D_{\epsilon/2}(a_0)$. But there is no n such that $\gamma_{x_n}(t_0) \in D_\epsilon(a_0)$, contradicting the continuity of $(x,t) \longrightarrow \gamma_x(t)$. We therefore must have that $a = a_0$. Also we know that since y_1 is not in $W(a,b)$ it must follow that $a_1 \neq b$. Now repeat the process, with t_1 replaced by t_2, where t_2 is chosen so that there is no critical value between $f(a_1)$ and t_2, to get a flow line γ_2 connecting a_1 to some other critical point a_2 and so on. The process must terminate after a finite number of steps, say l, in the sense that we reach the stage where $a_l = b$. Now it is clear from the construction that the critical points a_i and the flow–lines γ_i satisfy the conclusion of the lemma.

Now let $a \in M$ be a critical point of index q. A critical point $b \in M$ is said to be a *successor* of a if $b < a$ and b is maximal with respect to this property. That is, there exists no critical point c with $b < c < a$. Notice that by Lemma 1.6 we see that if b is a successor of a, then $\mathcal{M}(a,b)$ is a compact, smooth manifold of dimension $p - r - 1$ where p and r are the indices of a and

b respectively. Indeed this lemma says that $\mathcal{M}(a,b)$ is compact if and only if b is a successor of a.

$\mathcal{M}(a,b)$ can be given a framing as follows. Let t be any regular value between $f(a)$ and $f(b)$. Notice that the intersection

$$W^u(a) \cap f^{-1}(t)$$

is a sphere of dimension $p-1$ which we denote by $S^u(a)$. By construction we have

$$\mathcal{M}(a,b) \cong W(a,b)^t = W(a,b) \cap f^{-1}(t) = W^s(b) \cap W^u(a) \cap f^{-1}(t) = W^s(b) \cap S^u(a).$$

Thus we have a natural embedding $\mathcal{M}(a,b) \hookrightarrow S^u(a)$ which has codimension r. The normal bundle of this embedding is the pull - back of the normal bundle of the embedding

$$W^s(b) \hookrightarrow M$$

which comes equipped with a unique framing (up to sign) because $W^s(b)$, being diffeomorphic to the disk D^{n-r} is contractible. This induces a framing α on the normal bundle of $\mathcal{M}(a,b) \hookrightarrow S^u(a)$.

The following theorem, proved by Franks in [7], describes the relative attaching maps in the CW - complex $C(f)$ in terms of these framed moduli spaces.

Theorem 1.7 *Let $a > b$ be critical points of a Morse - Smale function $f : M \longrightarrow \mathbf{R}$ having indices p and r respectively. Then in the CW - complex $C(f)$ the relative attaching map*

$$\phi_{a,b} \in \pi_{p-1}(S^r)$$

described in (1.5) is represented via the Thom - Pontryagin construction by the $p-r-1$ - dimensional framed moduli space of flows $(\mathcal{M}(a,b),\alpha)$.

Idea of Proof The attaching map $\phi_{a,b} : S^{p-1} \to S^r$ is the map $S^u(a) \to W^u(b) \cup \infty \cong S^r$ defined by taking a point $x \in S^u(a)$, going along the flow γ_x, composing with the deformation retractions of (1.2) and (1.4) and collapsing onto $W^u(b) \cup \infty$ as in (1.5). By considering these retractions, one sees that the pre - image of $b \in W^u(b) \subset S^u(b) \cup \infty$ consists of those points $x \in S^u(a)$ such that the flow $\gamma_x(t)$ approaches b as $t \to \infty$. This is precisely the space $\mathcal{M}(a,b)$. The induced framing is clearly α as described above.

Notice that the cell structure of $C(f)$ yields the associated cellular - chain complex, known as the *Morse - Smale* chain complex

$$\cdots C_\lambda \xrightarrow{\partial_\lambda} C_{\lambda-1} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \tag{1.8}$$

where C_λ is the free abelian group generated by the cells of $C(f)$ of dimension λ, which is to say the critical points of f of index λ. The boundary homomorphisms are determined by the relative attaching maps of the λ - dimensional disks onto the $\lambda - 1$ - dimensional skeleton of $C(f)$. This can be computed by the above theorem as follows.

Suppose a and b are critical points of relative index one. Say $index(a) = p$ and $index(b) = p - 1$. Then the space of flows, $(\mathcal{M}(a,b), \alpha)$ is a zero dimensional, framed, compact manifold; that is a finite set of points (flow lines) with signs attached to them induced by the framing. Let $n(a,b) \in \mathbf{Z}$ denote the signed number of flow lines:

$$n(a,b) = \sum_{\gamma \in \mathcal{M}(a,b)} \alpha(\gamma)$$

where $\alpha(\gamma) = \pm 1$ is the sign associated to the flow line γ by the framing α.

$$n(a,b) \in \mathbf{Z} = \pi_{p-1} S^{p-1}$$

is therefore the integer given by the degree of the relative attaching map

$$\phi_{a,b} : S^{p-1} \longrightarrow S^{p-1}.$$

Corollary 1.9 *The coefficient of* $[b] \in C_{p-1}$ *of the boundary* $\partial_p[a]$, $\langle \partial_p[a], [b] \rangle$, *is given by the formula*

$$\langle \partial_p[a], [b] \rangle = n(a,b) \in \mathbf{Z}.$$

2. The Classifying Space of a Morse Function

Let $f : M \longrightarrow \mathbf{R}$ be a Morse function on a compact, closed, manifold. In this section we outline the ideas in [4]. In particular we will construct a topological category $\mathcal{C}_f$ whose objects are the critical points of f and where the space of morphisms between two critical points a and b is a compactification, $\bar{\mathcal{M}}(a,b)$ of the moduli space of flows $\mathcal{M}(a,b)$. We will then describe a theorem in [4] saying that for a generic f (i.e one satisfying the Morse - Smale condition) the classifying space $B\mathcal{C}_f$ is homeomorphic to the manifold M. This will give M the structure of an explicit simplicial space and will complete our goal of recovering the topology of the manifold directly and explicitly in terms of the space of flows of the gradient vector field.

We begin by recalling the definition of the *classifying space of a category*. We refer the reader to [15] for details of this construction.

Let $\mathcal{C}$ be a category. Let $Mor(A,B)$ be the set of morphisms between objects A and B. $\mathcal{C}$ is a *topological category* if the sets of morphisms are topologized and the composition pairings

$$Mor(A,B) \times Mor(B,C) \longrightarrow Mor(A,C)$$

are continuous. If no topology is specified, the morphisms are assumed to have the discrete topology. For $\gamma \in Mor(A,B)$ we say that A is the source of γ and B is the target.

For each $n \geq 0$ define the space $\mathcal{N}\mathcal{C}_n$ to be the space of n - tuples of composable morphisms:

$$\mathcal{N}\mathcal{C}_n = \{(\gamma_1, \cdots, \gamma_n) : \text{the target of } \gamma_i = \text{ the source of } \gamma_{i+1}, i = 1, \cdots, n-1\}$$

We define "face maps" $\partial_i : \mathcal{N}\mathcal{C}_n \to \mathcal{N}\mathcal{C}_{n-1}$, $0 \leq i \leq n$, by the formula

$$\partial_i(\gamma_1, \cdots, \gamma_n) = \begin{cases} (\gamma_2, \cdots, \gamma_n) & \text{for } i = 0 \\ (\gamma_1, \cdots, \gamma_i\gamma_{i+1}, \cdots \gamma_n) & \text{for } 1 \leq i \leq n-1 \\ (\gamma_1, \cdots, \gamma_{n-1}) & \text{for } i = n. \end{cases}$$

There are also "degeneracy maps" $s_j : \mathcal{N}\mathcal{C}_n \longrightarrow \mathcal{N}\mathcal{C}_{n+1}$, $0 \leq j \leq n$ given by

$$s_j(\gamma_1, \cdots, \gamma_n) = \begin{cases} (1, \gamma_1, \cdots, \gamma_n) & \text{for } j = 0 \\ (\gamma_1, \cdots \gamma_j, 1, \gamma_{j+1}, \cdots, \gamma_n) & \text{for } j \geq 1. \end{cases}$$

The *classifying space* of the category $\mathcal{C}$ is a space consisting of one of n - simplex for every point in $\mathcal{N}\mathcal{C}_n$. To make this more precise, let Δ^n be the standard n - simplex in $\mathbf{R}^n$;

$$\Delta^n = \{(t_1, \cdots, t_n) \in \mathbf{R}^n : 0 \leq t_j \leq 1, \text{ and } \sum_{i=1}^{n} t_i \leq 1\}.$$

Now consider the following maps between these simplices:

$$\delta_i : \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \quad \sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$$

for $0 \leq i, j \leq n$ defined by the formulae

$$\delta_i(t_1, \cdots, t_{n-1}) = \begin{cases} (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) & \text{for } i \geq 1 \\ (1 - \sum_{q=1}^{n-1} t_q, t_1, \cdots, t_{n-1}) & \text{for } i = 0 \end{cases}$$

and

$$\sigma_j(t_1, \cdots, t_{n+1}) = \begin{cases} (t_1, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}) & \text{for } i \geq 1 \\ (t_2, \cdots, t_{n+1}) & \text{for } i = 0\ . \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i^{th} face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j^{th} face.

We can now define the *classifying space* $B\mathcal{C}$ of the category $\mathcal{C}$ by the rule

$$B\mathcal{C} = \bigcup_{n \geq 0} \Delta^n \times \mathcal{N}\mathcal{C}_n / \sim$$

where if $t \in \Delta^{n-1}$ and $x \in \mathcal{N}\mathcal{C}_n$, then

$$(t, \partial_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{n+1}$ and $x \in X_n$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

Example. Let G be a group $\mathcal{C}_G$ be the category with one object (say $*$) and $Mor(*,*) = G$. The composition law in the category is given by group multiplication. Then the classifying space $B\mathcal{C}_G$ is the "bar construction" model for the classifying space of the group, $B\mathcal{C}_G = BG$. This is the motivation for the terminology "classifying space of a category".

The basic idea in the work of [4] was to study this construction for the category whose objects are the critical points of a Morse function and whose morphisms are the moduli spaces of flow lines. In order to define this category properly we need to study the compactification of the moduli space of flows $\mathcal{M}(a,b)$. Now lemma 1.6 describes what happens when one approaches the end of such a moduli space of flows $\mathcal{M}(a,b)$. Namely one approaches a sequence of flows (which we will refer to as a *piecewise flow)* $\{\gamma_1, \cdots, \gamma_m\}$ where $\gamma_i \in \mathcal{M}(a_{i-1}, a_i)$ where $\{a = a_0, a_1, \cdots a_{m-1}, a_m = b\}$ is a sequence of critical points. This suggests the following compactification theorem.

Recall from section one the partial ordering on critical points defined by setting $a_1 > a_2$ if there exists a flow from a_1 to a_2, that is if $\mathcal{M}(a_1, a_2)$ is nonempty. A sequence $\mathbf{a} = \{a_0, \cdots, a_n\}$ is ordered if $a_i > a_{i+1}$ for all i. Given such a sequence we let $s(\mathbf{a}) = a_0$ and $e(\mathbf{a}) = a_n$. We define the *length* of this sequence, $l(\mathbf{a})$, to be $n-1$. Finally we define

$$\mathcal{M}(\mathbf{a}) = \mathcal{M}(a_0, a_1) \times \cdots \times \mathcal{M}(a_{n-1}, a_n).$$

Theorem 2.1 *A compactification of the moduli space of flows $\mathcal{M}(a,b)$ is given by the following space:*

$$\bar{\mathcal{M}}(a,b) = \mathcal{M}(a,b) \cup \bigcup_{\mathbf{a}} \mathcal{M}(\mathbf{a})$$

where the union is taken over all ordered sequences of critical points $\mathbf{a}$ *with* $s(\mathbf{a}) = a$ *and* $e(\mathbf{a}) = b$.

We will state this compactification theorem more precisely later, and in particular describe the topology of $\bar{\mathcal{M}}(a,b)$. We first describe how it will be used to define the category induced by a Morse function and state how its classifying space is related to the underlying manifold. We will begin by stating the main theorem to be discussed in this section. The details of the proof of this theorem will appear in [4].

Theorem 2.2 *(a). Let $f : M \longrightarrow \mathbf{R}$ be a Morse function on a compact manifold satisfying the Morse - Smale transversality condition. Consider the topological category C_f whose objects are the critical points of f, and whose spaces of morphisms are the compactified moduli spaces,*

$$Mor(a, b) = \bar{\mathcal{M}}(a, b).$$

Composition in this category is given by the inclusion

$$\bar{\mathcal{M}}(a, b) \times \bar{\mathcal{M}}(b, c) \hookrightarrow \bar{\mathcal{M}}(a, c)$$

described in the compactification theorem 2.1. Then there is a natural homeomorphism of the classifying space of this category with the underlying manifold

$$BC_f \xrightarrow{\cong} M.$$

(b). For any Morse function $f : M \longrightarrow \mathbf{R}$ (i.e. not necessarily satisfying the Morse - Smale condition) then there is a homotopy equivalence

$$BC_f \xrightarrow{\simeq} M.$$

Notice that this theorem defines an explicit simplicial space description of the manifold M in terms of the moduli spaces of flow lines of the gradient vector field of the Morse function f. Indeed the k - simplices of this decomposition are parameterized by the space of k - tuples of "composable" flow lines. In order to outline the proof of this theorem we need a more precise version of the compactification theorem. This theorem is proved in [4] using the general gluing results in [2].

Theorem 2.3 *There exists an $\epsilon > 0$ and maps*

$$\mu : (0, \epsilon] \times \mathcal{M}(a, a_1) \times \mathcal{M}(a_1, b) \longrightarrow \mathcal{M}(a, b)$$

which we write as

$$(t, \gamma_1, \gamma_2) \longrightarrow \gamma_1 \circ_t \gamma_2$$

that satisfies the following properties:

(1)*μ is a diffeomorphism onto its image.*

(2)*μ satisfies the following associativity law:*

$$(\gamma_1 \circ_s \gamma_2) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_2 \circ_t \gamma_3)$$

for all $s, t \leq \epsilon$.

(3) *This associativity property defines maps*

$$\mu : (0, \epsilon]^l \times \mathcal{M}(\mathbf{a}) \longrightarrow \mathcal{M}(a, b)$$

where $(\mathbf{a})$ is any ordered sequence of critical points of length l with $s(\mathbf{a}) = a$ and $e(\mathbf{a}) = b$. These maps are also diffeomorphisms onto their images.

(4)*Define $K(a, b) \subset \mathcal{M}(a, b)$ to be*

$$K(a, b) = \mathcal{M}(a, b) - \bigcup \mu \left((0, \epsilon)^l \times \mathcal{M}(\mathbf{a})\right)$$

where the union is taken over all ordered sequences $(\mathbf{a})$ of length ≥ 1 having $s(\mathbf{a}) = a$ and $e(\mathbf{a}) = b$. Then $K(a, b)$ is compact.

(5) *Define the compactification $\bar{\mathcal{M}}(a, b)$ to be the union along μ*

$$\bar{\mathcal{M}}(a, b) = \mathcal{M}(a, b) \cup \bigcup_{\mu} [0, \epsilon]^l \times \mathcal{M}(\mathbf{a}).$$

Then $\mathcal{M}(a, b)$ is homeomorphic to $K(a, b)$.

This theorem says that the ends of the moduli space $\mathcal{M}(a, b)$ consist of spaces of half open cubes parameterized by composable sequences of flow lines. The compact space $K(a, b)$ is formed by removing the associated open cubes. The compactification $\bar{\mathcal{M}}(a, b)$ is formed by formally closing the cubes. It should therefore not be surprising that they are homeomorphic. If $\gamma_1 \in \mathcal{M}(a, a_1)$ and $\gamma_2 \in \mathcal{M}(a_1, b)$, then the parameter $t \in (0, \epsilon]$ in the flow $\gamma_1 \circ_t \gamma_2 \in \mathcal{M}(a, b)$ can be viewed as a measure of how close this flow comes to the critical point a_1. Thus the fact that the pairing μ is a diffeomorphism onto its image allows us to view the space $K(a, b)$ as the space of flows that stay at least ϵ away from all critical points other than a and b (in this undefined measure). On the other hand, the homeomorphic space $\bar{\mathcal{M}}(a, b)$ can be viewed as formally adjoining piecewise flows to $\mathcal{M}(a, b)$. From now on we will rescale the metric if necessary so as to be able to assume $\epsilon = 1$ and so we will drop this from the notation.

We now outline the proof of part (a) of theorem 2.2. We refer the reader to [4] for details.

We begin by describing a filtration of the spaces $\mathcal{M}(a, b)$. We can think of the set $K(a, b)$ of theorem 2.3 as the space of flow lines from a to b which keep distance of at least 1 from any critical points c with $a > c > b$. More generally we can filter the space $\bar{\mathcal{M}}(a, b)$ by saying that a curve in $\mathcal{M}(\bar{a}, b)$ has filtration k if it gets within distance less than 1 of at most k intermediate critical points. Precisely, we define

$$K^{(k)}(a, b) = \bigcup_{l \leq k} \bigcup_{a > a_1 > \ldots > a_l > b} \theta\left([0, 1]^l \times K(a, a_1) \times \ldots K(a_l, b)\right).$$

so that $K^{(0)}(a, b) = K(a, b)$ and

$$K^{(k-1)}(a, b) \subseteq K^{(k)}(a, b).$$

Thus γ is in $K^{(k)}(a, b)$ if and only if γ can be decomposed as

$$\gamma = \gamma_0 \circ_{s_1} \ldots \circ_{s_l} \gamma_l$$

where $\gamma_i \in K(a_{i-1}, a_i)$, $0 \leq s_i \leq 1$ for all i, and $l \leq k$. We have the following obvious properties.

Lemma 2.4

$$K^{(k)}(a,b) \setminus K^{(k-1)}(a,b) \cong \coprod_{a>a_1>\ldots>a_k>b} [0,1)^k \times K(a,a_1) \times \ldots K(a_k,b)$$

Lemma 2.5

$$\bigcup K^{(k)}(a,b) = \mathcal{M}(a,b)$$

Let $\mathbf{a} = (a_0, \cdots, a_{l+1})$ denote an arbitrary decreasing sequence of critcal points with length $l(\mathbf{a}) = l$, starting point $s(\mathbf{a}) = a_0 = a$ and ending point $e(\mathbf{a}) = a_{l+1} = b$. We define

$$K(\mathbf{a}) = K(a_0, a_1) \times \cdots \times K(a_l, a_{l+1}).$$

Using these lemmas we see that the map

$$\coprod_l \coprod_{l(\mathbf{a})=l} [0,1]^l \times K(\mathbf{a}) \longrightarrow \bar{\mathcal{M}}(a,b)$$

defined by

$$(s_1, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \longrightarrow \gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l$$

is onto and therefore $\bar{\mathcal{M}}(a,b)$ can be recovered by imposing an equivalence relation on the above disjoint union. It is straightforward to extract this equivalence relation; it is generated by

$$(s_1, \ldots, s_{i-1}, 1, s_{i+1}, \ldots, s_l; \gamma_1, \ldots, \gamma_l) \simeq$$
$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_l; \gamma_1, \ldots, \gamma_{i-1} \circ_1 \gamma_i, \ldots, \gamma_l).$$

Thus the relations only involve the faces of the cubes which do not contain the point $(0, \ldots, 0)$.

From this argument we draw the following conclusion.

Theorem 2.6

$$\bar{\mathcal{M}}(a,b) = \coprod_l \coprod_{l(\mathbf{a})=l} [0,1]^l \times K(\mathbf{a}) / \simeq$$

The next step is to go from this description of the spaces $\bar{\mathcal{M}}(a,b)$ to one of the manifold M. Let $\gamma \in \mathcal{M}(a,b)$ be a flow line. Then we may compactify γ by adding the points a and b to form the curve $\bar{\gamma}$. This curve is closed in the sense that it contains all of its limit points. The function f is decreasing along flow lines and so it gives a natural diffeomorphism

$$f : \bar{\gamma} \longrightarrow [f(b), f(a)].$$

Now suppose that $\gamma = \gamma_0 \circ_0 \cdots \circ_0 \gamma_l$ is a point of $\bar{\mathcal{M}}(a,b)$ which is not in $\mathcal{M}(a,b)$. Thus $\gamma_0, \cdots, \gamma_l$ is a sequence of flow lines joining critical points $a > a_1 > \cdots > a_l > b$. In this case define

$$\bar{\gamma} = \bar{\gamma}_0 \cup \cdots \cup \bar{\gamma}_l,$$

so $\bar{\gamma}$ is a curve in M joining a to b; in an obvious sense $\bar{\gamma}$ is a *piecewise flow line* joining a to b. The function f defines a diffeomorphism

$$f : \bar{\gamma}_i \longrightarrow [f(a_i), f(a_{i-1})]$$

and these diffeomorphisms piece together to define a homeomorphism

$$f : \bar{\gamma} \longrightarrow [f(b), f(a)].$$

This shows that each element in $\bar{\mathcal{M}}(a,b)$ can be identified with a well defined curve $\gamma : [f(b), f(a)] \longrightarrow M$ parameterized by the inverse of the above homeomorphism. However, with this parameterization, none of these curves satisfy the flow equations. In any case we get a map

$$\phi : [f(b), f(a)] \times \bar{\mathcal{M}}(a,b) \longrightarrow M$$

whose image is the closure of the space $W(a,b) \subset M$ since we have added to $W(a,b)$ all points on the curves $\bar{\gamma} : [f(b), f(a)] \longrightarrow M$ where $\gamma \in \bar{\mathcal{M}}(a,b)$. Therefore the map

$$\coprod_{\mathbf{a}} [f(a_{l+1}), f(a_0)] \times [0,1]^l \times K(\mathbf{a}) \longrightarrow M$$

defined by

$$(t; s_1, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \longrightarrow (\gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l)(t)$$

is onto. The disjoint union is taken over all decreasing sequences of critical points $\mathbf{a} = (a_0, \cdots, a_{l+1})$. Once more it is not difficult to extract the appropriate equivalence relation on the disjoint union. Given a sequence $\mathbf{a}$ of critical points as above with $l(\mathbf{a}) = l$, we define

$$J_{\mathbf{a}} = [f(a_{l+1}), f(a_0)], \quad I^{l(\mathbf{a})} = [0, 1]^l.$$

Now we define

$$\mathcal{R}_f = \coprod_{\mathbf{a}} J_{\mathbf{a}} \times I^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a}) / \sim \tag{2.7}$$

where the relations $\sim$ are given by

$$(t; s_1, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \tag{I}$$

$$\begin{cases} (t; s_1, \cdots, s_{i-1}; \gamma_0, \cdots, \gamma_{i-1}), & \text{if } t \in [f(a_i), f(a_0)] \\ (t; s_{i+1}, \cdots, s_l; \gamma_{i+1}, \cdots, \gamma_l), & \text{if } t \in [f(a_{l+1}), f(a_i)] \end{cases}$$

and

$$(t; s_1, \cdots, s_{i-1}, 1, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \tag{II}$$

$$(t; s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_0, \cdots, \gamma_{i-1} \circ_1 \gamma_i, \cdots, \gamma_l)$$

The map ϕ respects the equivalence relation $\sim$ so it gives a well defined map

$$\mathcal{R}_f \longrightarrow M.$$

Theorem 2.8 *The map*

$$\phi : \mathcal{R}_f \longrightarrow M$$

is a homeomorphism.

Proof The first step is to check that the second set of relations are the only relations which can occur if all the s_i's are non - zero. This follows from the fact that if two flow lines have a point in common then they are equal, together with the previous theorem. If one of the s_i's is zero then we are dealing with a piecewise flow line. If two piecewise flow lines have a point in common then this point must be one of the joining points or else they have a common segment. An elementary analysis leads to the conclusion that the only identifications which can take place if one of the s_i's is zero are consequences of the first set of relations.

Since the spaces $\mathcal{K}(a_{i-1}, a_i)$ are diffeomorphic to the compactified spaces $\bar{\mathcal{M}}(a_{i-1}, a_i)$ in such a way that the composition in the category corresponds to $\circ_1$, this shows us how to recover the manifold M from the category $\mathcal{C}_f$. To prove theorem 2.2 we are therefore reduced to the combinatorial exercise necessary to identify the space $\mathcal{R}_f$ with the classifying space $B\mathcal{C}_f$. Recall from chapter III the definition of the classifying space of the category $\mathcal{C}_f$. Comparing it with definition 2.7 of the space $\mathcal{R}_f$ we see that these spaces are very similar but they are not obviously the same. The essential difference is that $\mathcal{R}_f$ is built up out of cubes whereas the classifying space is built out of simplices. Nonetheless these two spaces are homeomorphic. Verifying this is a combinatorial argument for which we refer the reader to [4] for details.

3. Generalizations

In this section we describe various generalizations of theorem 2.2 including a theorem that applies to rather general infinite dimensional settings. In the following two sections we apply this result to obtain topological information about the moduli spaces of flows in the examples studied by Floer in [5] and [6]. These are the moduli spaces of pseudo - holomorphic spheres in a symplectic manifold, and of asymptotically flat self dual connections on $Y \times \mathbf{R}$, where Y is a closed, 3 - dimensional manifold. These results all represent work in progress by myself, Jones, and Segal. Details of this work will appear in due course.

The first basic generalization is to the setting of *Morse - Bott* functions

$f : M^m \longrightarrow \mathbf{R}$ on compact manifolds [3]. Recall that in such a function the critical points are not isolated, but rather are accumulated into a disjoint union of connected, critical submanifolds. On a critical submanifold $N^n \hookrightarrow M^m$ with normal bundle $\nu(N)$, the Hessian at a point $x \in N^n$ restricts to define a symmetric bilinear form

$$Hess_x^N(f) : \nu_x(N) \times \nu_x(N) \longrightarrow \mathbf{R}$$

The critical submanifold N is said to be *nondegenerate* if $Hess_x^N(f)$ is a nonsingular form at every $x \in N$. $f : M \longrightarrow \mathbf{R}$ is a *Morse - Bott* function if all of its critical points lie in a disjoint union of nondegenerate critical submanifolds. An important class of examples of Morse - Bott functions are equivariant Morse functions. These are smooth functions $f : M \longrightarrow \mathbf{R}$ where M has a smooth action of a group G, and f is invariant under that action (i.e $f(g \cdot x) = f(x)$). In this case the critical submanifolds are orbits under the action.

In his thesis [2], M. Betz defined a category C_f associated to a Morse - Bott function f. As before, the objects of f are the critical points of f, however the topology of the critical submanifolds is taken into account, in that the set of objects, $Obj(C_f)$ is topologized as the disjoint union of the critical submanifolds of $f : M \longrightarrow \mathbf{R}$. As before, for a and b critical points of f, the space of morphisms $Mor(a, b)$ is given by the compactification of the space of flows,

$$Mor(a, b) = \bar{\mathcal{M}}(a, b).$$

However, now in the entire space of morphisms of the category $Mor(C_f)$, which is the space of piecewise gradient flow lines,the topology of critical submanifolds is taken into account in the natural way. In particular, the maps

$$s : Mor(C_f) \longrightarrow Obj(C_f) \quad \text{and} \quad t : Mor(C_f) \longrightarrow Obj(C_f)$$

defined by sending a morphism to its source and target respectively, are continuous maps. There is also a Morse - Smale transversality condition on Morse

- Bott functions defined like above, and it is easy to verify that in this context it is a generic condition as well. In [2] Betz proved the following generalization of (2.3).

Theorem 3.1 *Let $f : M \longrightarrow \mathbf{R}$ be a Morse - Bott function on a closed manifold M, that satisfies the Morse - Smale transversality condition. Then there is a homeomorphism*

$$BC_f \xrightarrow{\cong} M.$$

Furthermore, if f is an equivariant Morse function, then there is a naturally induced action on BC_f and the above homeomorphism is equivariant.

Next we discuss the generalization due to M. Sanders [14] to the case where $f : M \longrightarrow \mathbf{R}$ is a Morse function on a complete, infinite dimensional Hilbert manifold. Recall from [12] the Palais - Smale condition (C):

Condition C *On any subset S on which f is bounded and the gradient ∇f is not bounded away from zero, then f has a critical point in the closure of S.*

Now observe that on an infinite dimensional manifold the gradient flow lines may not be defined on the entire real line. Moreover, flows that are defined on the entire real line may not converge to critical points. This makes studying Morse theory qualitatively different in infinite dimensions than in the compact manifold setting, even under the assumption that the Morse function satisfies condition (C). In his thesis [14] M. Sanders proved the following.

Theorem 3.2 *Let $f : M \longrightarrow \mathbf{R}$ be a Morse function on a complete Hilbert manifold that satisfies the Palais - Smale condition C. Then there is a category C_f whose objects are the critical points of f, and the morphisms between two critical points a and b is the compactification of the moduli space of flows $\mathcal{M}(a,b)$. Moreover there is a natural homotopy equivalence*

$$BC_f \xrightarrow{\simeq} M.$$

Furthermore if one assumes the following additional properties

(1)f *satisfies the Morse - Smale transversality condition, and*

(2)*every flow line* γ *can be defined on the entire real line and both* $\lim_{t\to -\infty}\gamma(t)$ *and* $\lim_{t\to +\infty}\gamma(t)$ *exist and are critical points*

then there is a homeomorphism

$$BC_f \xrightarrow{\cong} M.$$

We now describe the most basic generalization to the infinite dimensional setting. So assume that

$$f : M \longrightarrow \mathbf{R}$$

is a smooth map on a possibly infinite dimensional manifold that satisfies the following properties.

Properties 3.3

(1) *The set of critical points of* f *is a disjoint union of finite dimensional connected submanifolds of* M. *(Note there may be infinitely many such connected components and the dimensions of the components may be arbitrarily large.)*

(2) *Let* A *and* B *be connected, critical submanifold of* M. *Let* $W(A,B)$ *denote the subspace of* M *consisting of points* $x \in M$ *that lie on gradient flow lines* γ_x *with*

$$\gamma_x(0) = x, \quad \lim_{t\to -\infty}\gamma_x(t) \in A, \quad \lim_{t\to +\infty}\gamma_x(t) \in B.$$

Then each such space $W(A,B)$ *is a finite dimensional manifold.*

(3)*Let* $\mathcal{M}(A,B) = W(A,B)/\mathbf{R}$ *be the moduli space of flow lines starting in* A *and ending in* B. *Then each* $\mathcal{M}(A,B)$ *has a compactification* $\bar{\mathcal{M}}(A,B)$ *with the following structure. Given connected critical submanifolds* A, B, *and* C, *there are embeddings*

$$\begin{aligned} \mu : (0,\epsilon] \times \mathcal{M}(A,B) \times \mathcal{M}(B,C) &\longrightarrow \mathcal{M}(A,C) \\ (t,\gamma_1,\gamma_2) &\longrightarrow \gamma_1 \circ_t \gamma_2 \end{aligned}$$

for sufficiently small $\epsilon > 0$, that extend to maps

$$\mu : [0, \epsilon] \times \bar{\mathcal{M}}(A, B) \times \bar{\mathcal{M}}(B, C) \longrightarrow \bar{\mathcal{M}}(A, C)$$

so that for every $\gamma_1 \in \bar{\mathcal{M}}(A, B)$, $\gamma_2 \in \bar{\mathcal{M}}(B, C)$, then

$$\gamma_1 \circ_0 \gamma_2 \in \bar{\mathcal{M}}(A, C) - \mathcal{M}(A, C).$$

First observe that a Morse - Smale function on a compact, closed manifold clearly satisfies these properties. Moreover these properties are the basic ingredients that the various examples of Floer homology in infinite dimensions have in common. In particular the finite dimensionality of the spaces $W(A, B)$ says that even though a critical point (or critical submanifold) of a function may have infinite indices and co- indices (i.e., the stable and unstable manifolds $W^s(B)$ and $W^u(A)$ may be infinite dimensional) the *relative indices* i.e the dimension of the intersections $W^u(A) \cap W^s(B) = W(A, B)$ are finite. The maps μ in this list of properties should be viewed as *gluing* maps. Gluing of moduli spaces is a construction that is not only central to Floer theory, but is an important tool in all of Gauge theory.

Using the techniques of [2] one can show that given a function $f : M \longrightarrow \mathbf{R}$ satisfying properties 3.3 one can reprarameterize the gluing maps μ if necessary so that they obey the associativity rule (compare 2.3 part (2))

$$(\gamma_1 \circ_s \gamma_2) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_2 \circ_t \gamma_3)$$

for all s, t in the domains of definition of the relevant gluing maps. For a and b critical points this allows us to define the space of "piecewise flow lines"

$$\tilde{\mathcal{M}}(a, b) = \bar{\mathcal{M}}(a, b) \cup \bigcup_{\mu} [0, \epsilon]^l \times \mathcal{M}(\mathrm{a}))$$

where, as in the statement of 2.3, the union is taken over all ordered sequences of critical points $(\mathrm{a}) = (a, a_1, \cdots a_l, b)$. For critical submanifolds A and B the space $\tilde{\mathcal{M}}(A, B)$ is defined to be

$$\tilde{\mathcal{M}}(A, B) = \bigcup_{a \in A \, b \in B} \tilde{\mathcal{M}}(a, b)$$

and is topologized in the natural way.

Notice that $\tilde{\mathcal{M}}(A,B)$ is not necessarily compact but we have that

$$\tilde{\mathcal{M}} \subset \bar{\mathcal{M}}.$$

Certainly these spaces are equal when the manifold is compact. The complement

$$\bar{\mathcal{M}}(A,B) - \tilde{\mathcal{M}}(A,B)$$

is accounted for by "bubbling" phenomena in the ends of certain moduli spaces. We will discuss this point more later.

Using these gluing maps we can construct two categories. Like in previous examples, $\mathcal{C}_f$ will be the category whose objects are the critical points of f topologized with respect to the topology of the critical submanifolods. The morphism $Mor(a,b)$ is the space of piecewise flow lines

$$Mor(a,b) = \tilde{\mathcal{M}}(a,b)$$

and the composition law is given by

$$(\gamma_1, \gamma_2) \longrightarrow \gamma_1 \circ_0 \gamma_2.$$

One can also form what we will call the "compactified category" $\bar{\mathcal{C}}_f$, which has the same objects as $\mathcal{C}_f$ (i.e the space of critical points), but the morphism, which we denote by $\bar{Mor}(a,b)$ is given by the compactified moduli space

$$\bar{Mor}(a,b) = \bar{\mathcal{M}}(a,b).$$

Now given connected, critical submanfolds $A > B$, there are subcategories

$$\mathcal{C}_f^{A,B} \subset \mathcal{C}_f \quad \bar{\mathcal{C}}_f^{A,B} \subset \bar{\mathcal{C}}_f$$

defined to be the full subcategories whose objects are critical points c such that $A > c > B$. (That is, there exist $a \in A$ and $b \in B$ with $\mathcal{M}(a,c)$ and $\mathcal{M}(c,b)$ nonempty.) An argument completely analogous to the proof of part (a) of theorem 2.2 proves the following.

Theorem 3.4 *Let $f : M \longrightarrow \mathbf{R}$ be a smooth function on a Hilbert manifold M that satisfies properties (3.3). Let $A > B$ be any connected, critical submanifolds of M. Then there is a homeomorphism*

$$BC_f^{A,B} \xrightarrow{\cong} \bar{W}(A,B)$$

where

$$\bar{W}(A,B) = \bigcup_{A \geq A' > B' \geq B} W(A', B').$$

Notice that if M is compact, then $\bar{W}(A,B)$ is the closure of $W(A,B)$ in the manifold M.

This theorem also allows us to identify the classifying space of the entire category C_f (assuming it satisfies properties 3.3). To do this, consider the space of "algebraic points" in the manifold:

Definition 3.5 *For $f : M \longrightarrow \mathbf{R}$ a smooth function on a Hilbert manifold let $M_{alg} \subset M$ be the subspace defined by*

$$M_{alg} = \{x \in M : \text{There is a flow line } \gamma_x : \mathbf{R} \longrightarrow M$$
$$\text{with } \gamma_x(0) = x, \text{ and the limits } \lim_{t \to \pm\infty} \gamma(t) \text{ exit and are critical points.}\}$$

Thus the "algebraic points" are the points that lie on flow lines defined for all time and go between critical points. Points that lie on flows that are defined only on a subinterval of the real line or that lie on flows that go off to infinity are not considered algebraic. We will give the motivation for this terminology when we consider the example of the loop space of a symplectic manifold.

A consequence of theorem 3.4 is the following.

Theorem 3.6 *Let $f : M \longrightarrow \mathbf{R}$ be a smooth function on a Hilbert manifold that satisfies properties (4.3). Then there is a homeomorphism between the classifying space of the category C_f and the space of algebraic points,*

$$BC_f \xrightarrow{\cong} M_{alg}.$$

4. Group Completions and the Stability of Moduli Spaces

In this section we apply theorem 3.6 to obtain results about the topology of the moduli space of flows of a smooth function. We will then examine the examples of the symplectic action on the loop space of a compact symplectic manifold, as well as the the Chern - Simons functional on the space of connections on a trivial principal bundle over a closed 3 - dimensional manifold.

First we begin with some generalities. Let $\mathcal{C}$ be a topological category with spaces of objects and morphisms denoted $Obj(\mathcal{C})$ and $Mor(\mathcal{C})$ respectively. Following the notation and terminology used in K - theory, we will call the loop space of the classifying space $\Omega B\mathcal{C}$ the *"group completion"* of the space of morphisms, denoted, following Quillen, by a superscript "+",

$$Mor(\mathcal{C})^+ = \Omega B\mathcal{C}.$$

When the category has one object and the space of morphisms is a topological group G, then of course there is a natural homotopy equivalence

$$G = Mor(\mathcal{C}) \simeq Mor(\mathcal{C})^+ = \Omega BG.$$

In the case when the category $\mathcal{C}$ has one object but the space of morphisms, which forms a monoid under composition, does not form a group (i.e., not all the elements are invertible) then there still is a natural inclusion

$$Mor(\mathcal{C}) \hookrightarrow Mor(\mathcal{C})^+ = \Omega B\mathcal{C}$$

where the loop space is homotopy equivalent to a topological group, which is the classical group completion of Quillen [13]. The notation is inspired from algebraic K - theory, where, given a ring R the space $BGL(R)^+$ is the representing space for the algebraic K -groups of R.

Now let $f : M \longrightarrow \mathbf{R}$ be a smooth map on a Hilbert manifold satisfying properties (3.3). Let

$$\mathcal{M}(f) = \coprod \tilde{\mathcal{M}}(A, B)$$

where the union is taken over all connected critical submanifolds of M. So $\mathcal{M}(f)$ is the full moduli space of piecewise flows of f and by definition is the space of morphisms

$$\mathcal{M}(f) = Mor(\mathcal{C}_f).$$

Thus an immediate corollary of (3.6) is the following.

Corollary 4.1 *Let $f : M \longrightarrow \mathbf{R}$ be as above. Then there is a homeomorphism*

$$\mathcal{M}(f)^+ \cong \Omega M_{alg}.$$

The value of this result comes from the fact that K - theory techniques of Quillen [13] allow one to relate the homology or homotopy types of the moduli space $\mathcal{M}(f)$ with its group completion $\mathcal{M}(f)^+$. Thus one obtains information about $\mathcal{M}(f)$ in terms of homotopy theoretic information about the loop space ΩM_{alg}. Of course this is of little value unless one knows something about the homotopy type of the space of algebraic points, M_{alg}. However many interesting examples of functions on infinite dimensional manifolds satisfy the following variational property:

Property "Crit" *The inclusion of the algebraic points in the entire manifold*

$$M_{alg} \hookrightarrow M$$

is a homotopy equivalence.

This property is so named because it says that up to homotopy, one need only consider those points that lie on flow lines connecting critical points (i.e the algebraic points). Now many maps do not satisfy this property, even those that satisfy property 3.3. However it is not difficult to see that if f satisfies the Palais - Smale Condition C then it also satisfies Property Crit. Moreover as we shall see, this is a much more general property than Condition C. That is, many interesting smooth functionals do not satisfy Condition C but do satisfy Property Crit. In any case we can summarize these considerations as follows.

Theorem 4.2 *Let $f : M \longrightarrow \mathbf{R}$ be a smooth map on a Hilbert manifold that satisfies Properties (3.3) as well as Property Crit. Then there is a homotopy equivalence from the group completion of the moduli space of flow lines to the loop space of the manifold,*

$$\mathcal{M}(f)^+ \simeq \Omega M.$$

We now apply this result to two classes of examples. These are the examples considered by Floer in [5] and [6]. First we consider a compact, closed, simply connected symplectic manifold (M^{2n}, ω). Here ω is the symplectic form; that is a closed two - form with $\omega^n \neq 0$. Although it is not a necessary assumption, for the purposes of this exposition we will make the simplifying assumption that the second homotopy group is infinite cyclic,

$$\pi_2(M) \cong \mathbf{Z}.$$

Let LM be the space of smooth, unbased loops on M. LM is a smooth infinite dimensional Hilbert manifold with fundamental group

$$\pi_1(LM) \cong \pi_2(M) \cong \mathbf{Z}.$$

The symplectic two - form ω on M transgresses to give a one-form α on LM. α has the property that if $\sigma : [0,1] \longrightarrow LM$ is any smooth one - simplex in LM, and hence defines a smooth map from the cylinder $\tilde{\sigma} : S^1 \times [0,1] \longrightarrow M$, then

$$\int_\sigma \alpha = \int_{\tilde{\sigma}} \omega.$$

Now α pulls back to a one form (which by abuse of notation we also call α) on the universal cover $\widetilde{LM}$, which, since $\pi_1(LM) \cong \mathbf{Z}$, is a cyclic covering space. Since $\widetilde{LM}$ is simply connected, α, being a one form is exact and so there is a functional

$$\phi : \widetilde{LM} \longrightarrow \mathbf{R}$$

with $d\phi = \alpha$. In fact ϕ determines a map of covering spaces

$$\begin{array}{ccc} \widetilde{LM} & \xrightarrow{\phi} & \mathbf{R} \\ \downarrow & & \downarrow \\ LM & \xrightarrow{\bar{\phi}} & \mathbf{R}/\mathbf{Z} \cong S^1. \end{array}$$

By construction it is clear that the map $\bar{\phi} : LM \longrightarrow S^1$ represents an integral cohomology class lifting the real cohomology class represented by the closed one form α.

In the most basic, but perhaps most important special case, when $M = S^2$, then $\bar{\phi} : LS^2 \longrightarrow \mathbf{R}/\mathbf{Z}$ can be viewed as the area functional. More specifically, if $\beta : S^1 \longrightarrow S^2$ is an embedding, then $\bar{\phi}(\beta)$ is (up to a constant multiple) the area of the surface bounded by β. Now, since β bounds two surfaces, this area is only well defined modulo the surface area of the sphere. That is why $\bar{\phi}$ takes values in $\mathbf{R}/\mathbf{Z}$ (the actual area functional on the unit sphere would take values in $\mathbf{R}/4\pi\mathbf{Z}$).

It is not difficult to prove that the critical points of $\bar{\phi} : LM \to S^1$ are the constant loops which are topologized as the manifold M. Similarly, the space of critical points of $\phi : \widetilde{LM} \to \mathbf{R}$ is given by $\mathbf{Z} \times M$. Notice furthermore that a flow line between critical points x and $y \in M \subset LM$ is a curve

$$\gamma : \mathbf{R} \longrightarrow LM \quad \text{or equivalently} \quad \gamma : S^1 \times \mathbf{R} \longrightarrow M$$

which asymptotically approaches the constant loops at x and y. That is, one can view γ as a map

$$\gamma : S^2 = \mathbf{C} \cup \infty \longrightarrow M$$

with $\gamma(0) = x$ and $\gamma(\infty) = y$. Viewed in this way, the flow equations are essentially the Cauchy Riemann equations. That is, $\gamma : S^2 \longrightarrow M$ is a flow line if and only it is a "psuedo - holomorophic" map; that is, the map of tangent bundles

$$d\gamma : TS^2 \longrightarrow TM$$

preserves the almost complex structure. (Recall that a symplectic structure on a Riemannian manifold determines an almost complex structure on its tangent bundle.) We refer the reader to [5] for details of these facts.

Now on the level of the universal cover $\widetilde{LM}$, we consider the space of flows between two critical points $(x, n+k)$, $(y, n) \in M \times \mathbf{Z}$. Such a flow is given by a map

$$\gamma : S^2 \longrightarrow M$$

satisfying

(1) γ is pseudo - holomorphic,

(2) $\gamma(0) = x$ and $\gamma(\infty) = y$

(3) γ has degree k; that is, the homotopy class represented by γ in $\pi_2(M) \cong \mathbf{Z}$ is k.

We denote the corresponding moduli space by $Hol^k_{x,y}(S^2, M)$. Notice that the moduli space of piecewise flow lines form $(x, n+k)$ to (y, n) consists of smooth, degree k maps $\gamma : S^2 \longrightarrow M$ with $\gamma(0) = x$ and $\gamma(\infty) = y$ having the property that there are a finite number of circles parallel to the equator in S^2 on which γ is constant, and γ is pseudo holomorphic in the complement of these circles. We denote the corresponding moduli space by $\tilde{Hol}^k_{x,y}(S^2, M)$. The full moduli space of piecewise flow lines we denote $\tilde{Hol}(S^2, M)$. The following is an application of theorem (4.2).

Corollary 4.3 *Suppose that $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ satisfies Properties (3.3) and Property Crit. Then there is a homotopy equivalence*

$$\tilde{Hol}(S^2, M)^+ \simeq \Omega \widetilde{LM}.$$

In order to understand the meaning of this result one can use the techniques of Quillen [13] as follows. Let

$$Hol^k_*(S^2, M)$$

denote the space of based psuedo - holomorphic maps of degree k. By "based" I mean that the value at $0 \in \mathbf{C} \cup \infty = S^2$ is a fixed basepoint, but there is no restriction on the value at ∞. Assuming that $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ satisfies Properties (3.3) the gluing maps in $\tilde{Hol}(S^2, M)$ allow one to define pairings

$$Hol^k_*(S^2, M) \times Hol^r_*(S^2, M) \longrightarrow Hol^{k+r}_*(S^2, M)$$

which makes $\coprod_k Hol^k_*(S^2, M)$ into a (homotopy) monoid. Indeed it is a submonoid (up to homotopy) of the full space of smooth maps $\Omega^2 M$. One can use the above corollary and the techniques of [13] to prove the following, which says that the group completion of the space of holomorphic maps is homotopy equivalent to the space of all smooth maps.

Theorem 4.4 *Suppose that $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ satisfies Properties (3.3) and Property Crit. Then there is a homotopy equivalence*

$$\left(\coprod_k Hol^k_*(S^2, M)\right)^+ \simeq \Omega^2 M.$$

Now in the setting of homotopy monoids, Quillen's theory allows us to identify the group completion. Gluing with a canonical element in $Hol^1_*(S^2, M)$ defines the inclusion maps

$$j : Hol^k_*(S^2, M) \longrightarrow Hol^{k+1}_*(S^2, M).$$

Let $Hol^\infty_*(S^2, M)$ be the (homotopy) limit of these inclusions

$$Hol^\infty_*(S^2, M) = \lim_{\substack{\rightarrow \\ k}} Hol^k_*(S^2, M).$$

Then one can prove the following.

Theorem 4.5 *Suppose that $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ satisfies Properties (3.3) and Property Crit. Then there is a homotopy equivalence*

$$\mathbf{Z} \times Hol^\infty_*(S^2, M) \simeq \Omega^2 M.$$

This is an example of a "stability theorem" for these moduli spaces. That is, it identifies the limiting homotopy type of the moduli spaces under gluing with a canonical class of degree one.

Now one can prove that in many cases $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ does not in fact satisfy Property Crit. However in the examples of $M = \mathbf{CP}(n)$, or more generally, a Grassmannian or even a more general flag manifold, then Properties (3.3) and Property Crit are satisfied. Then theorems 4.4 and 4.5 apply and we recover results of Segal, Guest, and Kirwan [16], [8], [10] concerning how, as the degrees of the maps get large, spaces of holomorphic maps from S^2 to these manifolds approximates the homotopy type of the spaces of all continuous maps.

We now discuss the example of the Chern - Simons functional on the space of connections on a 3 - manifold Y, and show how theorem 4.2 leads to a stability theorem about the moduli space of self - dual connections on $Y \times \mathbf{R}$.

Let G be a simply connected, compact Lie group, and consider the space $\mathcal{A}_G(Y)$ of connections on the trivial principal bundle $Y \times G \longrightarrow Y$ over a closed 3 - manifold Y. $\mathcal{A}_G(Y)$ can be identified with the space of Lie algebra valued one forms

$$\mathcal{A}_G(Y) \cong \Omega^1(Y; \mathfrak{g}).$$

In particular there is an natural isomorphism between the tangent space and this vector space,

$$T_A \mathcal{A}_G(Y) \cong \Omega^1(Y; \mathfrak{g}).$$

Now give Y a Riemannian metric. Since Y is a 3 - manifold, the space of 2 - forms $\Omega^2(Y; \mathfrak{g})$ acts on the space of one - forms $\Omega^1(Y; \mathfrak{g})$ by the rule

$$\alpha(\beta) = \int_Y trace\,(\alpha \wedge \beta) d\,vol.$$

Thus we may think of Lie - algebra valued 2 forms as cotangent vectors:

$$\Omega^2(Y; \mathfrak{g}) \hookrightarrow (\Omega^1(Y; \mathfrak{g}))^* \cong T_A^*(\mathcal{A}_G(Y)).$$

In particular the curvature form $F_A \in T_A^*(\mathcal{A}_G(Y))$ and so the mapping

$$A \longrightarrow F_A \in T_A^*(\mathcal{A}_G(Y))$$

is a section of the cotangent bundle $T^*(\mathcal{A}_G(Y))$ and hence is a one form

$$F \in \Omega^1(\mathcal{A}_G(Y)).$$

Notice that using the metric and the induced Hodge star operator we have that the vector field dual to the one form F is given by

$$A \longrightarrow *F_A \in \Omega^1(Y;\mathfrak{g}) \cong T_A(\mathcal{A}_G(Y)).$$

Clearly the zeros of this vector field are given by the flat connections on $Y \times G$.

It is not difficult to see that the curvature F is a closed 1 - form on $\mathcal{A}_G(Y)$ and since $\mathcal{A}_G(Y)$ is contractible, it must be the differential of a function. Equivalently, the vector field $*F_A$ is the gradient vector field of a function on $\mathcal{A}_G(Y)$. This function is the Chern - Simons functional

$$\psi : \mathcal{A}_G(Y) \longrightarrow \mathbf{R}$$

defined by

$$\psi(A) = \int_Y trace(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A)$$

where in this formula A is viewed as a one form.

Now let $\mathcal{G}$ be the *gauge group* of the trivial bundle $Y \times G$. This is the group of bundle automorphisms and hence is given by

$$\mathcal{G} = Map(Y, G).$$

Notice that the set of path components

$$\pi_0(\mathcal{G}) = [Y, G] \cong \pi_3 G \cong \mathbf{Z}$$

and we can think of the path component that an element $g \in \mathcal{G}$ lies in as its degree. The gauge group acts on $\mathcal{A}_G(Y) \cong \Omega^1(Y;\mathfrak{g})$ by the rule

$$g(A) = g^{-1}Ag + g^{-1}dg \in \Omega^1(Y;\mathfrak{g}).$$

The behavior of the Chern - Simons functional under a gauge transformation is given by

$$\psi(g(A)) = \psi(A) + \tau_G \cdot deg(g)$$

where τ_G is a constant depending only on the Lie group G. For example, $\tau_{SU(2)} = \pi$.

Thus, although ψ is not fully gauge invariant, it is invariant under the action of $\mathcal{G}^0 \subset \mathcal{G}$, the subgroup of gauge transformations of degree zero. Since $\mathcal{G}/\mathcal{G}^0 \cong \mathbf{Z}$ (given by degree), ψ induces a map of infinite cyclic universal covering spaces

$$\begin{array}{ccccc} \tilde{\mathcal{B}} & = & \mathcal{A}_G(Y)/\mathcal{G}^0 & \xrightarrow{\psi} & \mathbf{R} \\ & & \downarrow & & \downarrow \\ \mathcal{B} & = & \mathcal{A}_G(Y)/\mathcal{G} & \xrightarrow{\psi} & \mathbf{R}/\tau_G\mathbf{Z} \cong S^1. \end{array}$$

An important issue dealt with in [5] is the fact that the underlying space $\mathcal{B} = \mathcal{A}(Y)/\mathcal{G}$ is not a manifold. The reason for this is because the gauge group $\mathcal{G}$ does not act freely on $\mathcal{A}(Y)$. Actually by the above formula for the action it is apparent that the center of G always acts trivially on any connection. However the singularities $\mathcal{B}$ arise from *reducible* connections, that is those connections A whose isotropy subgroup

$$G_A = \{g \in \mathcal{G} : g(A) = A\}$$

is larger than the center $C(G)$.

One way of dealing with this problem is to restrict to the subgroup of based gauge equivalences

$$\mathcal{G}_* \subset \mathcal{G} \quad \text{and} \quad (\mathcal{G}^0)_* \subset \mathcal{G}^0$$

given by those bundle automorphisms which are the identity on the fiber over a fixed basepoint $y_0 \in Y$. Equivalently, $\mathcal{G}_*$ is given by the basepoint preserving mapping space

$$\mathcal{G}_* = Map_*(Y, G),$$

and similarly $(\mathcal{G}^0)_* = Map_*^0(Y, G)$, the maps of degree zero.

Now $\mathcal{G}_*$ does act freely on $\mathcal{A}_G(Y)$ and so the quotient spaces $\mathcal{B}_* = \mathcal{A}_G(Y)/\mathcal{G}_*$ and $\tilde{\mathcal{B}}_* = \mathcal{A}_G(Y)/(\mathcal{G}^0)_*$ are infinite dimensional manifolds. Thus we can restrict the Chern - Simons functional to these manifolds and view it as a $\mathcal{G}/\mathcal{G}_* \cong G/c(G)$ - *equivariant* functional, where $c(G)$ is the center of G.

Let a and b be flat connections and so represent critical points of ψ. A curve $\gamma : \mathbf{R} \longrightarrow \mathcal{B}_*$ is a gradient flow between a and b if

$$\lim_{t \to -\infty} \gamma(t) = a \quad \text{and} \quad \lim_{t \to \infty} \gamma(t) = b$$

and

$$\frac{d\gamma}{dt} = -\nabla_{\gamma(t)}(\psi) = - * F_{\gamma(t)}.$$

Now one can view a curve of gauge equivalence classes of connections $\gamma : \mathbf{R} \longrightarrow \mathcal{B}$ going between flat connections a and b as a gauge equivalence class of connection $\bar{\gamma}$ on the trivial bundle over $Y \times \mathbf{R}$ which, when viewed as a one-form, is trivial in the $\mathbf{R}$ - direction and satisfies the asymptotic conditions that as $t \to \pm\infty$, $\bar{\gamma}$ approaches the flat connections b and a respectively. A direct calculation, comparing the curvatures of of the connections $\gamma(t)$ on Y at each t with the curvature of the connection $\bar{\gamma}$ on the four manifold $Y \times \mathbf{R}$, yields the following. (See [5] for details.)

Theorem 4.6 *A curve $\gamma : \mathbf{R} \longrightarrow \mathcal{B}(Y)$ going between the flat connections a and b satisfies the flow equation*

$$\frac{d\gamma}{dt} = - * F_{\gamma(t)}$$

if and only if the connection $\bar{\gamma}$ on the 4 - manifold $Y \times \mathbf{R}$ satisfies the anti-self duality equation

$$F_{\bar{\gamma}} = - * F_{\bar{\gamma}}.$$

Any connection on the trivial bundle $Y \times \mathbf{R} \times SU(2)$ is gauge equivalent to one that is trivial in the $\mathbf{R}$ - direction. Hence, we have the following.

Corollary 4.7 *Let a and b be flat connections on $Y \times G$ and so represent critical points of the Chern - Simons functional $\psi : \mathcal{B}_* \longrightarrow S^1$. Then the "moduli* *space of flows" $\mathcal{M}(a,b)$ is equal to the moduli space of based gauge equivalence classes of anti - self - dual connections (instantons) on $Y \times \mathbf{R} \times G$ which in the sense described above, asymptotically approach the flat connections a and b.*

We now turn our attention to the functional $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$ on the universal cover. We first make the following observation. Let $\tilde{\mathcal{B}}_*(a,b)$ denote the space of $(\mathcal{G}^0)_*$ - equivalence classes of connections on $(Y \times \mathbf{R}) \times G$ which in the sense described above, asymptotically approach the flat connections a and b. The following observation is proved using a rather standard homotopy theoretic argument.

Lemma 4.8 *There is a natural homotopy equivalence*

$$\tilde{\mathcal{B}}_*(a,b) \simeq Map_*(Y \times S^1/y_o \times S^1, BG)$$

and hence the set of path components is given by

$$\pi_0(\tilde{\mathcal{B}}_*(a,b)) \cong \mathbf{Z}.$$

We refer to the integer representing the path component of the connection as its *degree*. We then have the following description of the dynamics of $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$.

Corollary 4.9 *The critical points of $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$ are given by pairs (n,a) where $n \in \mathbf{Z}$ and a is a $(\mathcal{G}^0)_*$ - equivalence class of flat connections on $Y \times G$. The moduli space of flows $\mathcal{M}((n+k,a);(n,b))$ is equal to the moduli space of $(\mathcal{G}^0)_*$ - equivalence classes of degree k anti - self dual connections on $Y \times \mathbf{R} \times G$ which asymptotically approach the flat connections a and b. We denote this moduli space by $\mathcal{M}_k(a,b)$.*

Let $\mathcal{M}_k = \bigcup_{a,b} \mathcal{M}_k(a,b)$ where the union is taken over all $(\mathcal{G}^0)_*$ - equivalence classes of flat connections. We then have the following application of theorem 4.2. (Compare with (4.3).)

Corollary 4.10 *Suppose that $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$ satisfies Properties (3.3) and Property Crit. Then there is a homotopy equivalence*

$$\left(\coprod_k \mathcal{M}_k\right)^+ \simeq \Omega \tilde{\mathcal{B}}_*.$$

In order to understand the meaning of this group completion theorem we impose further basepoint conditions. Let $\tilde{\mathcal{B}}^0_*$ denote the space of $(\mathcal{G}^0)_*$ - gauge equivalence classes of connections γ on $(Y \times \mathbf{R}) \times G$ which satisfy the following asymptotic conditions.

(1) $\lim_{t \to -\infty} \gamma_t$ is the trivial connection on $Y \times G$, and

(2) $\lim_{t \to +\infty} \gamma_t$ is any flat connection on $Y \times G$.

The following is another easy homotopy theoretic exercise.

Lemma 4.11 *There is a natural homotopy equivalence*

$$\tilde{\mathcal{B}}^0_* \simeq Map_*(\Sigma Y; BG) \simeq Map_*(Y, G)$$

where ΣY denotes the suspension of Y.

Let $\mathcal{M}^0_k = \mathcal{M}_k \cap \tilde{\mathcal{B}}^0_*$ be the space of "based" anti - self dual connections. Then Taubes "gluing of instantons" defines pairings

$$\mathcal{M}^0_k \times \mathcal{M}^0_r \longrightarrow \mathcal{M}^0_{k+r}$$

which makes $\coprod_k \mathcal{M}^0_k$ into a (homotopy) monoid. Like we saw above in the holomorphic mapping setting, Quillen's theory [13] implies that the group completion is given by

$$\left(\coprod_k \mathcal{M}^0_k\right)^+ \simeq \mathbf{Z} \times \mathcal{M}^0_\infty$$

where $\mathcal{M}^0_\infty = \lim_{k \to \infty} \mathcal{M}^0_k$. Like above, the limit is taken over the process of gluing with a fixed instanton of degree one. We then get the following "stability theorem".

Theorem 4.12 *Suppose that $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$ satisfies Properties (3.3) and Property Crit. Then there is a homotopy equivalence*

$$\mathbf{Z} \times \mathcal{M}^0_\infty \simeq \tilde{\mathcal{B}}^0_* \simeq Map_*(Y, G).$$

This theorem is the analogue of theorem 4.5. It says that in the limit (over degree), the space of instantons on $Y \times \mathbf{R}$ approximates the homotopy type of the space of all connections. Theorem 4.5 says that the limiting homotopy type of holomorphic maps is all continuous maps. Recall that the analogous statement for instantons on a closed four - manifold (rather than on a four manifold of the form $Y \times \mathbf{R}$) was proved in complete generality by Taubes in [19]. Indeed work in progress by T. Mrowka suggests that the basic analytic setup in [19] can be used to show that $\psi : \tilde{\mathcal{B}}_* \longrightarrow \mathbf{R}$ satisfies Property Crit. for any closed 3 - manifold Y and any compact simply connected Lie group G. We therefore make the following conjecture.

Conjecture 4.13 *Theorem 4.12 holds for any closed three - manifold Y and any compact, simply connected Lie group G.*

5. Floer homotopy type

In [5] and [6] Floer defined homological invariants to the functionals $\phi : \widetilde{LM} \longrightarrow \mathbf{R}$ and $\psi : \tilde{\mathcal{B}} \longrightarrow \mathbf{R}$ for certain examples. (Actually Floer did not work on the level of universal covers but instead dealt with $\mathbf{R}/\bar{Z}$ - valued functionals). These invariants are the homology groups of Morse - Smale type chain complexes generated by the critical points, and whose boundary homomorphisms are computed by counting the number of flow lines between critical points of relative index one. In this section we describe work in progess, in which Jones, Segal, and I are attempting to describe the underlying homotopy theory in these constructions in terms of the categorical constructions studied above. In order to motivate we go back to the compact, finite dimensional setting.

Given a Morse - Smale function $f : M^n \longrightarrow \mathbf{R}$ on a closed n - dimensional manifold, for each $0 \leq k \leq n$ let $C_f^k \subset C_f$ be the full subcategory whose objects are critical points of index $\leq k$. Notice that by applying classifying spaces we obtain a filtration of M,

$$BC_f^0 \hookrightarrow \cdots \hookrightarrow BC_f^{k-1} \hookrightarrow BC_f^k \hookrightarrow \cdots \hookrightarrow BC_f^n = BC_f \cong M.$$

The following is an immediate corollary of theorem 3.4.

Corollary 5.1 *(a). The strata of this filtration are given by the unstable manifolds*

$$BC_f^k - BC_f^{k-1} \cong \bigcup_{a \in Crit_k} W^u(a)$$

where $Crit_k$ is the set of all critical points of index k. Since each such $W^u(a)$ is diffeomorphic to a disk D^k, we also have

(b). The subquotients of this filtration are given by a wedge of spheres,

$$BC_f^k / BC_f^{k-1} \cong \bigvee_{a \in Crit_k} S^k$$

An immediate outcome of this result is that the spectral sequence converging to the homology $H_*(M)$ coming from this filtration has as its E_1 - term the Morse - Smale chain complex and hence the spectral sequence collapses at the E_2 - level. In infinite dimensions, when one can define this filtration, (that is, when there is a well defined notion of index) one would not expect the spectral sequence to collapse and indeed the differentials should contain some interesting geometric information.

In general in infinite dimensions one does have the notion of index, (that is the dimension of the stable and unstable manifolds are infinite) and so the Morse - Smale complex, determined by the homotopy of pairs of the form (BC_f^k, BC_f^{k-1}) must be replaced by a more general construction. When $f : M \longrightarrow \mathbf{R}$ is a smooth map on a Hilbert manifold satisfying Properties (3.3) one simply studies the homotopy type of pairs $(BC_f^{A,B}, BC_f^{A,B'})$ for any triple of connected critical submanifolds, $A > B > B'$. The "Floer homotopy type" is the homotopy type of this collection of pairs.

In a wide class of examples this data can be studied as follows. For example, when Properties 3.3 are satisfied, theorem 3.4 says that $BC_f^{A,B} \cong \bar{W}(A,B)$, and so we are really studying the homotopy types of pairs of the form $(\bar{W}(A,B'), \bar{W}(A,B))$. Now consider the normal bundle $\nu^A_{B,B'}$ of the embedding $W(A,B) \hookrightarrow W(A,B')$. In many cases this extends canonically to a bundle (which we still call $\nu^A_{A,B'}$) over $\bar{W}(A,B) \cong BC_f^{A,B}$. In this case we study the Thom space this bundle $T(\nu^A_{B,B'})$ which by definition is the corresponding unit disk bundle modulo the boundary unit sphere bundle. Moreover, it is usually the case that when one has connected critical submanifolds $A' > A > B > B'$, then the bundle $\nu^A_{B,B'}$ over $BC_f^{A,B}$ is the restriction of the bundle $\nu^{A'}_{B,B'}$ over $BC_f^{A',B}$. Hence there is a finite dimensional bundle $\nu_{B,B'}$ over $BC_f^{-,B}$, where $C_f^{-,B} \subset C_f$ is the full subcategory whose objects are the points of the critical submanifolds A with $A > B$. In this setting "Floer homotopy type" is given by the homotopy types of the Thom spaces $T(\nu_{B,B'})$.

The homotopy type of these Thom spaces can be studied using techniques of stable homotopy theory in the following manner. Let $\mathbf{B}$ denote a decreasing sequence of critical submanifolds

$$B > B_1 > B_2 \cdots > B_i > \cdots$$

and let $\nu_i = \nu_{B,B_i}$ denote the associated normal bundles over $BC_f^{-,B}$. For $A > B$, let ν_i^A denote the restriction of ν_i to $C_f^{A,B}$. Let $-\nu_i^A$ denote a bundle over $BC_f^{A,B}$ which has the property that $\nu_i^A \oplus -\nu_i^A$ is trivial. By the finiteness properties in (3.3) such a bundle exists. Now the stable homotopy type of the Thom space $T(-\nu_i^A)$ is well defined (up to suspension), so we let $T(-\nu_i^A)$ denote the *spectrum* associated to this Thom space. In the category of spectra one can formally suspend and desuspend, which has the effect of shifting the dimensions of the homology groups. The convention we use is the following. If ν_i^A is a k dimensional bundle, then the Thom isomorphism theorem gives an isomorphism (with $\tilde{Z}_2$ - coefficients if ν_i^A is not orientable)

$$H^q(BC_f^{A,B}) \cong H^{q+k}(T\nu_i^A)$$

and so we formally suspend the spectrum $T(-\nu_i^A)$ so that the Thom isomorphism is in dimensions

$$H^q(BC_f^{A,B}) \cong H^{q-k}(T(-\nu_i^A)).$$

Now the Thom - Pontryagin construction gives an inverse system of these spectra

$$BC_f^{A,B} \leftarrow T(-\nu_1^A) \leftarrow \cdots \leftarrow T(-\nu_i^A) \leftarrow \cdots \tag{5.2}$$

which fit together (over choices of $A > B$) to give an inverse system of spectra

$$BC_f^{-,B} \leftarrow T(-\nu_1) \leftarrow \cdots \leftarrow T(-\nu_i) \leftarrow \cdots \tag{5.3}$$

We will not go into the detail of the Thom - Pontryagin constructions in these contexts, but we will simply recall that given an embedding of manifolds $N \hookrightarrow M$ with normal bundle ν, then by identifying the total space of the disk bundle of ν with a tubular neighborhood of N in M there is a Thom - Pontryagin map $\tau : M \longrightarrow T(\nu)$ which collapses everything outside the tubular neighborhood to a point. Given a bundle $\varsigma \longrightarrow M$, there is a more general Thom - Pontryagin map $\tau : T(\varsigma) \longrightarrow T(\varsigma_{|_N} \oplus \nu)$. The maps in the inverse systems (5.2) and (5.3) are of this type.

The *Floer homotopy type* of $f : M \longrightarrow \mathbf{R}$ with respect to the sequence $\mathbf{B}$ is the homotopy type of the inverse system of Thom - spectra (5.3). Moreover we define the *Floer homology* with respect to $\mathbf{B}$ by the rule

$$HF_*(M;\mathbf{B}) = \lim_{i\to\infty} H^*(T(-\nu_i)). \tag{5.4}$$

Notice that this definition of Floer homology measures the homological data of the pairs $(BC_f^{-,B_i}, BC_f^{-,B_{i+1}})$. However this is only an example of the homotopy theoretic data that the inverse system (5.3) contains. We also define the *compactified Floer homotopy type* to be the homotopy type of the inverse system of Thom spectra as in (5.3) of bundles over the compactified categories $B\bar{\mathcal{C}}_f^{-,B}$ as defined earlier. Of course to do this one needs to know that the normal bundles ν_i extend over the classifying spaces of the compactified categories. When one is in this situation, one may also define the compactified Floer homology as the limit of the cohomology groups of the Thom spectra as in (5.4), but now these bundles live over $B\bar{\mathcal{C}}_f^{-,B}$. We end by describing an example where these constructions may be studied in a particularly clean way.

Let $\phi : \widetilde{LS^2} \longrightarrow \mathbf{R}$ be the map induced by the canonical symplectic form on $S^2 = \mathbf{CP}(1)$ as defined above. In this case one may verify that ϕ satisfies Properties (3.3) and so by theorem 3.6 one has

$$B\mathcal{C}_\phi \cong \widetilde{LS^2}_{alg}.$$

We begin by identifying this space. By viewing S^2 as $\mathbf{CP}(1)$ we have that

$$LS^2 = L(\mathbf{C}^2 - (0,0))/L(\mathbf{C}^*)$$

where $\mathbf{C}^* = \mathbf{C} - 0$ is the group of units. Letting $L_0(\mathbf{C}^*)$ be the space of loops in $\mathbf{C}^*$ of winding number zero, one sees that

$$\widetilde{LS^2} = L(\mathbf{C}^2 - (0,0))/L_0(\mathbf{C}^*).$$

Now an element of $L(\mathbf{C}^2 - (0,0))$ is a pair of smooth, complex valued functions f and g on the circle S^1 with no zeros in common. By the description of the flow lines of ϕ as holomorphic maps as explained above, it is easy to see that such a pair (f,g) represents an algebraic point of $\widetilde{LS^2}$ if and only if f and g are both polynomials. That is,

$$B\mathcal{C}_\phi \cong \widetilde{LS^2}_{alg} = \{(p,q) \text{ polynomials with no zeros in common on } S^1\}/\mathbf{C}^*.$$

Now as described above, the critical points are given by $\tilde{Z} \times S^2$. To ease notation we write

$$\mathcal{C}^{n+k,n} = \mathcal{C}_\phi^{(n+k)\times S^2, n\times S^2}.$$

It is then rather straightforward to see that

$BC^{n+k,n} \cong \{(p,q)$ polynomials of degree $\leq k$ with no zeroes in common on $S^1\}/\mathbf{C}^{\cdot}$.

Now let P_k be the (complex) vector space of all polynomials of degree $\leq k$. Then with this description of $BC^{n+k,n}$ we see that it is diffeomorphic to an open dense subspace of

$$\left(P_k \times P_k - (0,0)\right)/\mathbf{C}^{\cdot} \cong \mathbf{CP}(2k+1).$$

It is not difficult to see that the difference between $BC^{n+k,n}$ and this model of $\mathbf{CP}(2k+1)$ comes from the fact that the moduli spaces of flows, which consist of piecewise holomorphic maps from S^2 to itself, are not compact. There is bubbling phenomena that occurs when a root of a holomorphic map of a fixed degree approaches a pole. The upshot is that when one compactifies these moduli spaces to form the category $\bar{C}_\phi$ one obtains the following result.

Theorem 5.5 *There is a natural homeomorphism from the classifying space of the compactified category to the complex projective space*

$$B\bar{C}^{n+k,n} \cong \mathbf{CP}(2k+1).$$

Furthermore the normal bundle of the embedding

$$B\bar{C}^{n+k,n} \hookrightarrow B\bar{C}^{n+k,n-1}$$

is isomorphic to the normal bundle of the natural embedding

$$\mathbf{CP}(2k+1) \hookrightarrow \mathbf{CP}(2k+3)$$

which is given by 2ς, where $\varsigma \longrightarrow \mathbf{CP}(2k+1)$ is the canonical (complex) line bundle.

Now the Thom space of k - times the line bundle

$$k\varsigma \longrightarrow \mathbf{CP}(m)$$

is homeomorphic to the stunted projective space $\mathbf{CP}(m+k)/\mathbf{CP}(k-1)$ which we denote by $\mathbf{CP}_k^{m+k}$. This notation makes sense in the stable category of spectra even when k is negative. Moreover if one considers the line bundle over the infinite projective space,

$$\varsigma \longrightarrow \mathbf{CP}(\infty),$$

then we write $T(k\varsigma) = \mathbf{CP}_k^\infty$ for all $k \in \bar{Z}$,

$$H^q(T(k\varsigma)) \cong \begin{cases} \mathbf{Z}, & \text{for } q = 2r \text{ where } r > k \\ 0 & \text{for } q \text{ odd.} \end{cases}$$

Now consider the decreasing sequence of critical submanfolds

$$\{0\} \times S^2 > \{-1\} \times S^2 > \cdots > \{-i\} \times S^2 > \cdots$$

The corresponding sequence of classifying spaces

$$B\bar{C}^{-,0} \hookrightarrow B\bar{C}^{-,-1} \hookrightarrow \cdots B\bar{C}^{-,-i} \hookrightarrow \cdots$$

is homeomorphic to a sequence of embeddings

$$\mathbf{CP}(\infty) \hookrightarrow \mathbf{CP}(\infty) \hookrightarrow \cdots \mathbf{CP}(\infty) \hookrightarrow \cdots$$

where the successive normal bundles are given by 2ς. Hence we have the following.

Theorem 5.6 *The compactified Floer homotopy type of $\phi : \widetilde{LS^2} \longrightarrow \mathbf{R}$ is given by the inverse system of spectra*

$$\mathbf{CP}(\infty) = \mathbf{CP}_0^\infty \leftarrow \mathbf{CP}_{-2}^\infty \leftarrow \cdots \leftarrow \mathbf{CP}_{-2i}^\infty \leftarrow \cdots$$

The compactified Floer homology is given by

$$HF_q(\widetilde{LS^2}) \cong \begin{cases} \bar{Z}, & \text{for } q \text{ any even integer (positive or negative)} \\ 0 & \text{for } q \text{ odd.} \end{cases}$$

We end by noting that if Floer's calculations in [5] are adapted to this context, he would have that

$$HF_*(\widetilde{LS^2}) \cong \bigoplus_{Z} \tilde{H}_*(S^2)$$

which is isomorphic to our result after regrading.

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