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EXTENSIONS OF JETS ON TUBE STRUCTURES

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Abstract

We apply the Ehrenpreis's cohomological extension method to tubes structures. We extend jets which are pointwise solutions of the structure in tubelike closed sets.

Resumo

Aplicamos o método cohomológico de extensão de Ehrenpreis as estruturas Tubo. Extendemos jatos que são soluções pontuais da estrutura definidos em conjuntos fechados tubulares.

1. Introduction

Let $\{u_\alpha\}$ be a subset of functions of $\mathcal{C}(F,\mathbb{C})$ indexed by the set of all nonnegative integer N-uplas $(\alpha_1,...,\alpha_N)$ with $|(\alpha_1,...,\alpha_N)| = \alpha_1 + ... + \alpha_N \in \mathbb{N}$.

The set $\{u_\alpha\}$ defines a smooth jet in a subset F of $\mathbb{R}^N$ if for all bounded subset $K \subset F$

$$u^\alpha(x) = \sum_{|\alpha+\beta|\leq k} \frac{u^{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_{\alpha,k}(x,y) \tag{1.1}$$

and

$$|u^\alpha(x)| \leq M(K)$$

$$|R_{\alpha,k}(x,y)| \leq M|x-y|^{k-|\alpha|} \text{ for all } x,y \in K \subset F\ ,\quad k \in \mathbb{N} \tag{1.2}$$

We denote the set of all jets defined in F by $J(F,\mathbb{C})$.

The Whitney extension theorem asserts that when F is closed smooth jets extend themselves to $\mathbb{R}^N$ as an element of $C^\infty(\mathbb{R}^N,\mathbb{C})$.

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If $F \subset \mathbb{R}^N$ is a closed set and E is a finite vector bundle over $\mathbb{R}^N$, we denote by $J(F, E)$ the sections of E with base in F and coefficients in $J(F, \mathbb{C})$.

Let $\mathcal{L}$ be a subbundle of $\mathbb{C} \otimes T(\mathbb{R}^N)$ generated by n, $2 \leq n \leq N$, linear ly independent smooth complex vector fields expressed by a frame field $L = (L_1, ..., L_n)$. We define $\mathcal{H}(F)$ to be the subalgebra of elements of $J(\mathbb{R}^m \times F, \mathbb{C})$ such that

$$Lu = 0 \ \text{ in } \ \mathbb{R}^m \times F \tag{1.3}$$

The equation (1.3) should be understood as

$$L_j u = 0 \ \text{ in } \ F \quad \text{for} \quad j = 1, ..., n$$

in the classical sense since we can always extend u to $\mathbb{R}^m \times \mathbb{R}^n$.

We say that $\mathcal{L}$ is globally integrable if there is a smooth map

$$Z : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}^m \tag{1.4}$$

such that

$$dZ_1 \wedge, ..., \wedge dZ_m \neq 0 \tag{1.5}$$

in $\mathbb{R}^N$. When we can find global coordinates in $\mathbb{R}^m \times \mathbb{R}^n$ such that

$$Z(x, t)x + i\Phi(t)$$

we say that $\mathcal{L}$ is a Tube structure.

By choosing global coordinates $x_1, ..., x_m, t_1, ..., t_n$ in $\mathbb{R}^m \times \mathbb{R}^n$ we may express the vector fields L_j as

$$L_j = (\partial_t)_j - \sum_{k1}^{m} (\partial_t)_j (Z_k)(\partial_x)_k, \quad j = 1, ..., n \tag{1.6}$$

where $(\partial_x) = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m})$ and

$$L_k Z_j = 0, \quad L_k t_l = \delta_{kl}, \quad (1 \leq j \leq m, \ 1 \leq k, l \leq n) \tag{1.7}$$

Let us denote by $\mathcal{L}^\perp$ the orthogonal of $\mathcal{L}$ with respect to the duality between tangent vector and forms. Associated with $\mathcal{L}$ we have the induced covariant exterior derivative acting in exterior powers of $J(\mathbb{R}^m \times K, E)$, where $E = \mathbb{C} \otimes T^\star/\mathcal{L}^\perp$ and the associated differential complex

$$\bigwedge^{p} J\Big(\mathbb{R}^m \times K, E\Big) \xrightarrow{d_{\mathcal{L}}} \bigwedge^{p+1} J\Big(\mathbb{R}^m \times K, E\Big) \tag{1.8}$$

In this context we will study extensions of jets in $\mathcal{H}(F)$ to jets in $\mathcal{H}(U)$, where U is some open neighborhood of F. We will apply the cohomological extension method as described in [1], that is; for an adequate neighborhood U of F and an extension $\tilde{u} \in C^\infty(\mathbb{R}^m \times U)$ of $u \in \mathcal{H}(K)$ we will solve $d_{\mathcal{L}} v = d_{\mathcal{L}} \tilde{u}$ in some open neighborhood U of K with $v \equiv 0$ in $\mathbb{R}^m \times K$. An extension of u will be $\tilde{u} - v$.

2. Extending Jets

Let us denote by $\mathcal{C}(\nu)$ the family of diadic cubes $\mathcal{Q}_\nu$ with edges of size $2^{-\nu}$ in $\mathbb{R}^n$. Let $\mathcal{Q}_K$ be some fixed diadic cube containing K in its interior and $N(\nu)$ the number of diadic cubes in $\mathcal{C}(\nu)$ contained by $\mathcal{Q}_K$. Let B_R be the ball of radius R centered in the origin in $\mathbb{R}^m$ and $\partial^\alpha = (\partial/\partial_{x_1})^{\alpha_1}...(\partial/\partial_{x_n})^{\alpha_n}$, $\partial_t^\beta = (\partial/\partial_{t_1})^{\beta_1}...(\partial/\partial_{t_n})^{\beta_n}$, $L^\alpha = L_1^{\alpha_1}...L_n^{\alpha_n}$ $x^\alpha = x_1^{\alpha_1}....x_m^{\alpha_m}$ and $t^\beta = t_1^{\beta_1}...t_n^{\beta_n}$. Also let us denote by Γ_t the m-dimensional affine linear manifold

$$\{z \in \mathbb{C}^m : z = y + i\Phi(t)\}$$

and consider

$$G_\epsilon(\xi) = \exp(-\epsilon\ 4^{-1}\xi^2)$$

the Fourier-Laplace transform of $E_\epsilon(z) = (\epsilon\pi)^{-\frac{m}{2}} \exp(-\epsilon^{-1}z^2)$. Let u be in $\mathcal{H}(K)$. We extend u to $\mathbb{R}^m \times \mathbb{R}^n$ as a function $\tilde{u}$ in $C^\infty(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C})$. It follows that $d_{\mathcal{L}}\tilde{u} \in C^\infty\Big(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C} \otimes \wedge^{1,m}\Big)$ and

$$d_{\mathcal{L}}\tilde{u} \equiv 0 \text{ in } \mathbb{R}^m \times F \tag{2.1}$$

In fact all derivatives of $d_{\mathcal{L}}\tilde{u}$ vanish at $\mathbb{R}^m \times F$ since the L^j's and ∂_{x_k}'s commute. Denote by $w = x + i\Phi(t)$ and $z = y + i\Phi(s)$ complex vectors in $\mathbb{C}^n$. Consider $0 < R_0 < R_1 < R_2$, and χ in $C^\infty(B_{R_1})$, where $\chi \equiv 1$ in B_{R_0} and $\omega = d_{\mathcal{L}}(\chi\tilde{u})$.

It follows that $\omega \in J\Big(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{C}\otimes\wedge^{1,m}\Big)$ is an uniformly compact supported form in the first variable in B_{R_1} . Define v_ϵ by

$$v_\epsilon(x,t) = 2\pi^{-m} \int\limits_{\Lambda}\int\limits_{\Gamma_s}\int\limits_{\mathbb{R}^m} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\omega(s,\Re z)\wedge d\xi\wedge dz \tag{2.2}$$

Since

$$d_t \int\limits_{\Gamma_t} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)(\chi\tilde{u})(t,\Re z)dz =$$

$$\int\limits_{\Gamma_t} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\omega(t,\Re z)\wedge dz \tag{2.3}$$

This last 1-differential form is exact for every fixed $\xi \in \mathbb{R}^m$ and (2.2) is independent of the arc Λ joining a fixed point $t_0 \in K$ to $t \in \mathbb{R}^n$. The exponential decay guaranteed by G_ϵ allows one to apply Fubini theorem to represent (2.2) as

$$v_\epsilon(x,t) = 2\pi^{-m} \int\limits_{\mathbb{R}^m}\int\limits_{\Lambda}\int\limits_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\omega(s,\Re z)\wedge dz\wedge d\xi \tag{2.4}$$

it follows also that

$$v_\epsilon^{\alpha,\beta}(x,t) = 2\pi^{-m}\partial_x^\alpha\partial_t^\beta \int\limits_{\mathbb{R}^m}\int\limits_{\Lambda}\int\limits_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\omega(s,\Re z)\wedge d\xi\wedge dz =$$

$$2\pi^{-m}\partial_x^\alpha\partial_t^\beta \int\limits_{\mathbb{R}^m}\int\limits_{\Lambda}\int\limits_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\omega(s,\Re z)\wedge dz\wedge d\xi =$$

$$= 2\pi^{-m} \int\limits_{\Gamma_t}\int\limits_{\mathbb{R}^m} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)L^\beta\partial_y^\alpha(\chi u)(t,\Re z)\wedge d\xi\wedge dz$$

$$+2\pi^{-m} \int\limits_{\Lambda}\int\limits_{\Gamma_s}\int\limits_{\mathbb{R}^m} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)L^\beta\partial_y^\alpha\omega(t,\Re z)\wedge d\xi\wedge dz =$$

$$= \int\limits_{\Gamma_t} E_\epsilon(w-z)L^\beta\partial_y^\alpha(\chi u)(t,\Re z)dz$$

$$+2\pi^{-m} \int\limits_{\mathbb{R}^m}\int\limits_{\Lambda}\int\limits_{\Gamma_s} \exp\left(i\,[x-y]\cdot\xi\right) G_\epsilon(\xi)L^\beta\partial_y^\alpha\omega(t,\Re z)\wedge d\xi\wedge dz \tag{2.5}$$

For any $u \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$ and $c \in \mathbb{R}$ denote the *interior* of the set $\{t \in \mathbb{R}^n : \sup_{x\in\mathbb{R}^m} u(x,t) < c\}$ by u_c .

We will introduce now a semilocal condition $(\star)_0$ for a tube structure $\mathcal{L}$ in order to establish a boundedness criteria for the family

$$\{v_\epsilon^{\alpha,\beta},\ \ \epsilon \leq 1,\ \ |\alpha|, |\beta| \leq l\}$$

We say that $\mathcal{L}$ satisfies $(\star)_0$ at a closed connected set $F \subset \mathbb{R}^n$ if for any connected compact subset $K \subset F$ and open neighborhood $Q \supset K$ there exists an open neighborhood U of K such that for all $c \in \mathbb{R}$ and $u \in \mathcal{H}(\mathbb{R}^n)$ any connected component $\mathcal{C}$ of $\Re u_c \cap U$ is contained by a connected component $\mathcal{C}'$ of $\Re = u_c \cap Q$ with $\mathcal{C}' \cap F \neq \emptyset$ or else $\Re u_c \cap U = \emptyset$

If $G \subset \mathbb{R}^n$ is open and $F \subset \overline{G}$ we say that $\mathcal{L}$ satisfies $(\star)_0$ at F relative to G if the statement above holds with Q and U relative open sets in $\overline{G}$.

When K reduces to a point and $F = \mathbb{R}^n$, the condition $(\star)_0$ was originally stated for tubes structures in [6] and it agrees with the solvability condition appearing for locally integrable structures of codimension one ([3]). We will prove an extension theorem, analog to those proved in [2], for a tube structure $\mathcal{L}$ that verifies the condition $\star_0$ at a compact subset of closed connected hypersurface F.

With the notation above we state the Lemma 2.1, whose proof we postpone to Section 3.

Lemma 2.1. *Let $\mathcal{L}$ be a tube structure in $\mathbb{C} \otimes T(\mathbb{R}^m \times \mathbb{R}^n)$. Let $F \subset \mathbb{R}^n$ be a connected closed set, $K \subset F$ compact connected set and $\mathcal{Q}_K$ an open diadic cube containing K. If $\mathcal{L}$ satisfies $(\star)_0$ at F, then for a fixed l the family*

$$\{v_\epsilon^{\alpha,\beta},\ \ \epsilon \leq 1,\ \ |\alpha|, |\beta| \leq l\} \tag{2.6}$$

is bounded in $\{t \in \mathbb{R}^n : |t| \leq k\} \times U$ by a same constant, depending only on K, l and k. Let $G \subset \mathbb{R}^n$ be an open set with regular boundary. If $G \subset \mathbb{R}^n$ is an open subset with regular boundary and $\mathcal{L}$ satisfies $(\star)_0$ at F relatively to $G \subset \mathbb{R}^n$ then the same statement holds for U a relative open set in $\overline{G}$.

The next two definitions deals with maximum of functions in compact sets and will be used in the statement of the main Theorem 2.6.

Definition 2.2. *Let $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ be continuous and a closed set $F \subset \mathbb{R}^N$. We say that u attains a local maximum at a compact subset $K \subset F$ relatively to F if there is a open neighborhood U of K in $\mathbb{R}^N$ such that $u_{|K} = M$ and $u(z) \leq M$ if $z \in F \cap U \setminus K$. If $F = \mathbb{R}^N$ we simply say that u attains a local maximum at a compact subset K.*

We derive from definition 2.2 a new one for Tubes structures

Definition 2.3. *Let $\mathcal{L}$ be a Tube structure defined in $\mathbb{C} \otimes T\Big(\mathbb{R}^m \times \mathbb{R}^n\Big)$. We say that $\mathcal{L}$ satisfies the local maximum principle at $F \subset \mathbb{R}^n$ if for all $K \subset F$ compact $\Re u$ does not have a local maximum at $\{x_0\} \times K$ relatively to $\mathbb{R}^m \times F$ for any $x_0 \in \mathbb{R}^m$ and $u \in \mathcal{H}(\mathbb{R}^n)$, unless u is a constant. If $F = \mathbb{R}^n$ we simply say that $\mathcal{L}$ satisfies the local maximum principle.*

We shall restate for tube structures a theorem of Treves [5].

Theorem 2.4. *Let $u \in \mathcal{H}(\mathbb{R}^n)$ and $V \subset \mathbb{R}^m \times \mathbb{R}^n$,be an open connected set. Then if u vanishes in V, it vanishes in all orbit $\mathcal{O}$ (in the sense of Sussmman [S]) that intercepts V.*

This theorem lead us to the following definition

Definition 2.5. *Let $\mathcal{L}$ be a Tube structure in $\mathbb{C} \otimes T\Big(\mathbb{R}^m \times \mathbb{R}^n\Big)$. We say that a closed set $F \subset \mathbb{R}^n$ has the uniqueness property if for any compact set $K \subset F$ there is an open neighborhood U of K such that any orbit $\mathcal{O}$ of $\mathcal{L}$ which meets $\mathbb{R}^m \times U$ also meets $\mathbb{R}^m \times U \cap F$.*

Now we state the main Theorem.

Theorem 2.6. *Let $\mathcal{L}$ be a Tube structure in $\mathbb{C} \otimes T\Big(\mathbb{R}^m \times \mathbb{R}^n\Big)$ and $F \subset \mathbb{R}^n$ a closed connected set with the uniqueness property. Assume that $\mathcal{L}$ satisfies the local maximum principle at F. Then the following three statements are equivalent:*

i) For any compact connected set $K \subset F$ and for any open neighborhood

$Q \supset K$ there exists an open subset U, $K \subset U \subset Q$ such that any function $u \in \mathcal{H}(F \cap Q)$ admits an unique extension to $\mathbb{R}^m \times U$, as a function in $\mathcal{H}(U)$.

ii) There is no $u \in \mathcal{H}(\mathbb{R}^n)$ and no compact subset $K \subset F$ such that $\Re u$ attains a local maximum at $\{x_0\} \times K$ relatively to $\mathbb{R}^m \times F$, unless u is constant.

iii) There is no $\theta \in S^{m-1}$ and no compact set $K \subset F$ compact such that $\theta \cdot \Phi$ attains local maximum at K relatively to F, unless $\theta \cdot \Phi$ is constant.

iv) The tube structure $\mathcal{L}$ satisfies the condition $(\star)_0$ at F .

Proof: Suppose i) holds and ii) does not. Then one can find a compact set $K \subset F$, $x_0 \in \mathbb{R}^m$, an open neighborhood Q of K, a nonconstant $u \in \mathcal{H}(\mathbb{R}^n)$ such that $\Re u_{|\{x_0\}\times K} = M$ and $\Re u_{|\mathbb{R}^m \times F \cap Q} \leq M$. If $R \in \Im u(\{x_0\} \times K)$, the smooth function w defined in $\mathbb{R}^m \times \Re u_{M+\epsilon}$ as

$$w(x,t) = exp[u(x,t) - (M+\epsilon+iR)]^{-1} \text{ if } (x,t) \in F \cap Q \setminus u^{-1}(\{M+\epsilon+iR\}))$$

and

$$w(x,t) = 0 \text{ if } (x,t) \in \mathbb{R}^m \times F \cap Q \cap u^{-1}(\{M+\epsilon+iR\})$$

is in $\mathcal{H}(Q \cap F)$. Also $K \subset \partial\overline{(-\Re u)_{-M}}$ because $\mathcal{L}$ satisfies the *local maximum principle* at F. The *uniqueness property* of F implies that $w_\epsilon(x,t) = exp[u(x,t) - (M+\epsilon+iR)]^{-1}$ is the unique possible value for any extension of w to $\mathbb{R}^m \times V \setminus u^{-1}(\{M+\epsilon+iR\})$. Consequently for small ϵ the jet defined in $\mathcal{H}(F \cap Q)$ by $\{w_\epsilon\}$ cannot be extended to the fixed neighborhood $\mathbb{R}^m \times U$ of $\mathbb{R}^m \times K$ granted by i), even as a continuous function. Thus i) implies ii). That ii) implies iii) is trivial. Next assume that iii) does not hold. Then there exists $\theta \in S^{m-1}$, a compact $K \subset F$ and an open neighborhood U of K, such that $\theta \cdot \Phi$ attains a local maximum M at K, in $F \cap U$. Now $K \subset \partial\overline{(-\theta \cdot \Phi)_{-M}}$ and $K \cap (-\theta \cdot \Phi)_{-M} = \emptyset$. Then any $t \in (-\theta \cdot \Phi)_{-M} \cap V$ cannot be connected to K by any continuous path or else $(-\theta \cdot \Phi)_{-M} = \emptyset$, but this last alternative is empty since $\mathcal{L}$ satisfies the *local maximum principle* at F. This contradicts iv). Assume now that iv) holds. Let $K_k \subset \mathcal{Q}_k$ be an exhausting sequence of compact connected subsets for F, $\mathcal{Q}_k$ an open diadic cube containing K_k and $U_k \subset \mathcal{Q}_k$ an open connected neighborhood of K_k for which the condition $(\star)_0$

holds. Denote $U = \bigcup_{k1}^{\infty} U_k$ and B_k the ball of radius k centered in the origin in $\mathbb{R}^m$. Let $\tilde{u}$ be an extension of u to $\mathbb{R}^n \times \mathbb{R}^m$, as in Lemma 2.1. Then the family of functions $\{L^\beta \partial_x^\alpha v_\epsilon,\ |\alpha|, |\beta| \leq l\}$ is bounded in $B_k \times U_k$ by a constant depending only on l, k. By a standard diagonal process one can find a subsequence $\epsilon_k \to 0$ such that $v_{\epsilon_k} \to v$, where $v \in C^\infty(\mathbb{R}^m \times U)$. Then $v_{|F} \equiv 0$ and $L(\tilde{u} - v) = 0$ in $\mathbb{R}^m \times U$. Thus $\tilde{u} - v \in \mathcal{H}(U)$ is one extension of the original jet u. The uniqueness follows from the connectedness of the U_k's. The proof is finished.

Let $F = \varrho^{-1}(0)$ be a connected hypersurface, $\varrho_+ = \{t \in \mathbb{R}^n : \varrho(t) \geq 0\}$ and $\varrho_- = \{t \in \mathbb{R}^n : \varrho(t) \leq 0\}$ all defined by a smooth $\varrho : \mathbb{R}^n \to R$

One can prove the analogue of Theorem 2.6 for an open relative neighborhood of $\varrho^{-1}(0)$ in $\varrho_- = \{t \in \mathbb{R}^n : \varrho(t) \leq 0\}$. The condition $(\star)_0$ is taken relatively to ϱ_-. Recall that $\mathcal{L}$ satisfies $(\star)_0$ at $\varrho^{-1}(0) \subset \varrho_-$ relatively to ϱ_- if for any connected compact set $K \subset \varrho^{-1}(\{0\})$ and any open neighborhood $Q \supset K$ there exist an open relative neighborhood U of K in ϱ_- such that for all $u \in \mathcal{H}(\varrho_-)$ and $c \in \mathbb{R}$ any connected component $\mathcal{C}$ of $\Re u_c \cap U$ is contained by a connected component $\mathcal{C}'$ of $\Re u_c \cap Q$ with $\mathcal{C}' \cap \varrho^{-1}(0) \neq \emptyset$ or else $\Re u_c \cap U = \emptyset$.

Let $\varrho^{-1}(0)$ be a noncharacteristic hypersurface with respect to $\mathcal{L}$, that is $d_{\mathcal{L}}\varrho \neq 0$. If $u \in C^\infty(\mathbb{R}^m \times \varrho^{-1}(0))$, we say that u satisfies the tangential tube structure $\mathcal{L}$ at $\varrho^{-1}(0)$ if $d_{\mathcal{L}}u \wedge d_{\mathcal{L}}\varrho(t) = 0$ for all $t \in \varrho^{-1}(0)$ and in this case it is possible to find an extension $\tilde{u}$, such that $\tilde{u} \in C^\infty(\mathbb{R}^m \times \varrho_-)$ and $\tilde{u}_{|\varrho^{-1}(0)} \in \mathcal{H}(\mathbb{R}^m \times \varrho^{-1}(0))$ (see [2] for details).

We now state and prove a lateral extension theorem :

Theorem 2.7. *Let $\varrho \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\varrho^{-1}(0)$ is connected and $d_{\mathcal{L}}\varrho \neq 0$ in $\varrho^{-1}(0)$. If $\mathcal{L}$ is a Tube structure in $\mathbb{C} \otimes T\Big(\mathbb{R}^m \times \mathbb{R}^n\Big)$ which satisfies the local maximum principle at $\varrho^{-1}(0)$, then the following four assertions are equivalent:*

i) for any compact connected set $K \subset \varrho^{-1}(0)$ and for any open neighborhood $Q \supset K$ there exists an open subset U, $K \subset U \subset Q$ such that any $u \in \mathcal{H}(Q \cap \varrho^{-1}(0))$ extends to $\mathbb{R}^m \times U \cap \varrho_-$ as a jet in $\mathcal{H}(U \cap \varrho_-)$.

ii) There is no $u \in \mathcal{H}(\varrho_+)$, $K \subset \varrho^{-1}(0)$ compact and $x_0 \in \mathbb{R}^m$ such that $\Re u$ attains a local maximum at $\{x_0\} \times K$ relatively to $\mathbb{R}^m \times \varrho_+$, unless u is a *constant.*

iii)There is no $\theta \in S^{m-1}$ *such that* $\theta \cdot \Phi_{|\varrho_+}$ *attains a local maximum at* K *relatively to* ϱ_+ *unless* $\theta \cdot \Phi$ *is a constant.*

iv)The tube structure $\mathcal{L}$ *satisfies the condition* $(\star)_0$ *at* $\varrho^{-1}(0)$ *relative to* ϱ_-.

Proof: Now the proof follows the same arguments as in Theorem 2.6 except for the detail that ϱ_- is a smooth manifold with boundary $\varrho^{-1}(0)$, so one may use the mean value theorem and Lemma 2.1 to ensure uniform continuity of the family $\{L^\beta \partial_x^\alpha v_\epsilon : \ |\alpha|, |\beta| \leq k, \ 0 < \epsilon \leq 1\}$ in any compact subset of $\mathbb{R}^m \times U \cap \varrho_-$ where $U = \cup_{k=1}^\infty U_k \subset \varrho_-$.

As a consequence of Theorem 2.7 we have the following corollary, that implies Theorem 2.4 in [2].

Corollary 2.8. $\mathcal{L}$ *is a Tube structure in* $\mathbb{C} \otimes T\Big(\mathbb{R}^m \times \mathbb{R}^n\Big)$ *which satisfies the local maximum principle at* ϱ_- *and suppose that* $\varrho^{-1}(0)$ *is compact. Then* $u \in \mathcal{H}(\varrho^{-1}(0))$ *extends itself to* ϱ_- *as a jet in* $\mathcal{H}(\varrho_-)$ *if and only there is no* $\theta \in S^{m-1}$ *such that the set such that* $\theta \cdot \Phi_{|(c+\varrho)_+}$ *attains a local maximum at* K *relatively to* $(c + \varrho)_+$ *for all* $c \in [0, +\infty)$, *unless* $\theta \cdot \Phi$ *is a constant.*

3. Proof of Lemma 2.1

We begin recalling that in Section 2 we had choose $0 < R_0 < R_1 < R_2$, and χ in $C^\infty(B_{R_1})$, where $\chi \equiv 1$ in B_{R_0} and $\omega d_{\mathcal{L}}(\chi \tilde{u})$. Let $diam(K)$ denote the diameter of K. Throughout the proof we shall assume without loss of generality that $(3 + n)\mathtt{d} = R_2 - R_1$, where

$$\mathtt{d} = diam(K) \sup_{t \in \mathcal{Q}_K} |\nabla \Phi(t)|$$

For each $\theta\xi/|\xi|, \ \xi \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$ denote by $(\Phi \cdot \theta)_{\Phi(t)\cdot\theta}$ the open set $\{s \in \mathbb{R}^n : \Phi(s) \cdot \theta < \Phi(t) \cdot \theta\}$. Let t_0 be fixed in K, $t \in U$ and $t' \in U \cap (\Phi \cdot \theta)_{\Phi(t)\cdot\theta}$. By the hypothesis $(\star_0)$ one can find $q_\theta \in \mathcal{Q}_K \cap F \cap (\Phi \cdot \theta)_{\Phi(t)\cdot\theta}$ in the same connected component of $\mathcal{Q}_K \cap (\Phi \cdot \theta)_{\Phi(t)\cdot\theta}$ to which t' belongs. Since $t \in \mathcal{Q}_K \cap \overline{(\Phi \cdot \theta)_{\Phi(t)\cdot\theta}}$ one can take t' arbitrarily close to t.

Taking advantage of the exactness in (2.3) we will choose a path Λ_ξ for each $\xi \in \mathbb{R}^m$ as a sum of two piecewise linear paths, $\Lambda_\xi = \Lambda_{\xi,0} + \Lambda_{\xi,1}$, where the path indexed by 0 links the fixed point t_0 to q_θ and the other one links q_θ to t and lies nearby $\overline{(\Phi \cdot \theta)_{\Phi(t)\cdot\theta}}$ inside $\mathcal{Q}_K$. By 2.5

$$v_\epsilon^{\alpha,\beta}(x,t) = 2\pi^{-m} \int_\Lambda \int_{\Gamma_s} \int_{\mathbb{R}^m} \exp\left(i\,[w-z]\cdot\xi\right) G(\epsilon\xi)\partial_y^\alpha L^\beta \omega(s,\Re z) \wedge d\xi \wedge dz$$

$$+ \int_{\Gamma_t} E_\epsilon\,(w-z)\,L^\beta \partial_y^\alpha (\chi u)(t,y) \wedge dy A + B \tag{3.1}$$

where $B0$ when $\beta 0$.

We will first estimate the partial sum A in 3.1. Let us write A as

$$A = 2\pi^{-m} \int_{\mathbb{R}^m} \left(\int_{\Lambda_{\xi,0}} + \int_{\Lambda_{\xi,1}} \right) \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\partial_x^\alpha L^\beta \omega(s,\Re z) \wedge d\xi \wedge dz$$

$$= \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,0}} \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\partial_x^\alpha L^\beta \Big(\chi d_{\mathcal{L}} u\Big)(z) \wedge d\xi \wedge dz +$$

$$2\pi^{-m} \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,0}} \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\partial_x^\alpha L^\beta \Big(u d_{\mathcal{L}} \chi\Big)(z) \wedge d\xi \wedge dz +$$

$$2\pi^{-m} \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,1}} \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\partial_x^\alpha L^\beta \Big(\chi d_{\mathcal{L}} u\Big)(z) \wedge d\xi \wedge dz +$$

$$2\pi^{-m} \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,1}} \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi)\partial_x^\alpha L^\beta \Big(u d_{\mathcal{L}} \chi\Big)(z) \wedge d\xi \wedge dz \tag{3.2}$$

Denote the first two integrals in (3.2) by the roman numerals I, II respectively and by III the sum of the last two. Then

$$I = 2\pi^{-m} \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,0}} \int_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G_\epsilon(\xi) L^\beta \partial_x^\alpha \Big(\chi d_{\mathcal{L}} u\Big)(z) \wedge d\xi \wedge dz \tag{3.3}$$

The set K is connected. For each $\xi \in \mathbb{R}^n$ fixed one can find a piecewise linear path $\Lambda_{\xi,0}$ as follows; let $\nu \in N$ be such that

$$2^{-1}(\exp\Big(-2\mathtt{d}\;\; A(|\xi|)\Big)) \le 2^{-\nu} \le (\exp\Big(-2\mathtt{d}\;\; A(|\xi|)\Big)) \le \exp(-2\mathtt{d}\frac{A}{\sqrt{m}}) \tag{3.4}$$

where $A(\rho)\frac{A}{\sqrt{m}} > 0$ if $\rho \leq A$ and $A(\rho)\frac{\rho}{\sqrt{m}}$ if $\rho > A$.

Let ϵ be a positive number and $N(K,\nu)$ the number of closed diadic cubes $\mathcal{Q}_\nu$ in of $\mathcal{C}(\nu)$ whose union is a covering of K. There exist $A > 0$ such that

$$\sum_{k=1}^{N(K,\nu)} s_{\mathcal{Q}_k}^{n+1} \leq 2^{-(n+1)\nu} N(\nu) \leq \epsilon \int\limits_{\mathcal{Q}_K} dx \tag{3.5}$$

where the last inequality is a consequence from the fact that the upper Minkowski dimension of $\mathcal{Q}_K$ is n. Consider a vertex t_1 in

$$\partial \cup_{k=1}^{N(K,\nu)} \mathcal{Q}_k \tag{3.6}$$

which is at a minimum distance of t_0. Suppose that the vertex $t_{k-1} \in \mathcal{Q}_{k-1}$ is defined then select the next vertex $t_k \in \partial \cup_{k=1}^{N} \mathcal{Q}_k$ nearby t_{k-1} (in another cube intersecting the former one), allowing the maximum projection in the direction of the vector $\overrightarrow{t_0 q_\theta}$. In doing so we find after a finite number of steps, vertices $\{t_1, \ldots, t_N\}$ of the cubes $\mathcal{Q}_k$, with $N \leq N(K,\nu)$, such that t_{k-1} and t_k lies in the same cube $\mathcal{Q}_k$, t_N is in the same cube as q_θ and t_1 is in the same cube as t_0. Denote the polygonal line defined by $\Lambda_{\xi,0}$ these vertices. It follows from the hypothesis on the flatness of $d_{\mathcal{L}}u$ that for any $l \in N$

$$\|L^\beta \partial_x^\alpha \big(\chi d_{\mathcal{L}} u\big)\|(y,s) \leq C|s - t^*|^{l-1-|\alpha|-|\beta|} \leq C 2^{-\nu(l-[1+|\alpha|+|\beta|])} \tag{3.7}$$

for all $s \in \mathcal{Q}_k$, $t^* \in \mathcal{Q}_k \cap K \neq \emptyset$ and uniformly in $y \in B_{R_2}$. If $l - (|\alpha| + |\beta| + 1) \geq n + 1$ then by (3.4),(3.5) and (3.7) we obtain

$$|I| \leq \sum_{k=1}^{N(K,\nu)} \int_{\mathbb{R}^m} \int_{\Lambda_{\xi,0} \cap \mathcal{Q}_k} \int_{\Gamma_s} \exp\Big[\sup_{s \in \mathcal{Q}_K} |\nabla\Phi(s)||t_0 - s||\xi|\Big] \|L^\beta \partial_x^\alpha \big(\chi u\big)\|(z,s) dz ds d\xi =$$

$$\sum_{k=1}^{N(K,\nu)} \sup_{s \in \mathcal{Q}_K} \int_{\mathbb{R}^m} \exp\Big[\sup_{s \in \mathcal{Q}_K} |\nabla\Phi(s)|t_0 - s||\xi|\Big] |t_{k-1} - t_k| \int_{\Gamma_s} \|L^\beta \partial_x^\alpha \big(\chi d_{\mathcal{L}} u\big)\|(z) dz d\xi \leq$$

$$\sup_{s \in \mathcal{Q}_K} \int_{\mathbb{R}^m} \exp\Big[(|\nabla\Phi(s)|p - s| - 2\mathtt{d})|\xi|\Big] d\xi \int_{\Gamma_s \cap \operatorname{supp}\chi} dz \sqrt{n} N(K,\nu) 2^{\nu(l-(|\alpha|+|\beta|+1))} \leq$$

$$\int_{\mathbb{R}^m} \exp(-\mathtt{d}|\xi|) d\xi \sup_{s \in \mathcal{Q}_K} \int_{\Gamma_s \cap \operatorname{supp}\chi} dz \sqrt{n} N(K,\nu) 2^{\nu(l-(|\alpha|+|\beta|+1))} d\xi \sqrt{n} \leq N(\nu) 2^{n+1}$$

$$\leq C\epsilon \tag{3.8}$$

The path $\Lambda = \Lambda_{\xi,0} + \Lambda_{\xi,1}$ is fixed for each $\xi \in \mathbb{R}^m$. Apply Fubini theorem and the fact $\exp(i[x + i\Phi(t)] \cdot \xi) = \exp(ix \cdot \xi)\exp(-\Phi(t) \cdot \xi)$, to write

$$II = 2\pi^{-m} \int\limits_{\mathbb{R}^m} \int\limits_{\Lambda_{\xi,0}} \int\limits_{\Gamma_s} \exp\left(i\left[w - z\right] \cdot \xi\right) G_\epsilon(\xi) L^\beta \partial_x^\alpha \Big(ud_{\mathcal{L}}\chi\Big)(s, \Re z) \wedge dz \wedge d\xi =$$

$$2\pi^{-m} \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \int\limits_{\Lambda_{\xi,0}} \exp\left(i\left[x + i\Phi(t) - y - \Phi(s)\right] \cdot \xi\right) G_\epsilon(\xi) L^\beta \partial_x^\alpha \Big(ud_{\mathcal{L}}\chi\Big)(s, y) \wedge d\xi \wedge dy \tag{3.9}$$

The integrand in (3.9) is analytic in the ξ-variable. The exponential decay of G_ϵ allows change of integration domain from $\xi \in \mathbb{R}^m$ to $\zeta \in \mathbb{C}_A^m$, where

$$\mathbb{C}_A^m = \{\zeta \in \mathbb{C}^m \ : \ \zeta_j = \xi_j + iA(|\xi|)sgn(x_j - y_j)\} \text{ for } j = 1, ..., m \tag{3.10}$$

Then

$$|\exp\left(i\left[w - z\right] \cdot \zeta\right)| \exp\left(-\left[\Phi(s) - \Phi(t)\right] \cdot \xi - (x - y) \cdot A(|\xi|)sgn(x - y)\right) \leq$$

$$\exp\left(|\Phi(s) - \Phi(t)|\,|\xi| - A(|\xi|)|x - y|\right) \leq \exp\Big(|\xi|\Big[|\Phi(s) - \Phi(t)| - (3 + n)\mathtt{d}\Big]\Big) \leq$$

$$\exp\Big(\sup_{s \in Q_K} |\nabla\Phi(s)| t_0 - s||\xi| - (3 + n)\mathtt{d}A(|\xi|)\Big) \leq \exp\Big(-(n + 2)\mathtt{d}A(|\xi|)\Big) \tag{3.11}$$

As w is confined to $\overline{B_{R_0}} \times K$, while z is in $\overline{B_{R_2}} \setminus \overline{B_{R_1}} \times K$, we get

$$|II| \leq 2\pi^{-m} \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \int\limits_{\Lambda_{\Re\zeta,0}} \exp\Big(-(n+2)\mathtt{d}A(|\Re\zeta|)\Big) G_\epsilon(\zeta) \, \|L^\beta \partial_x^\alpha \Big(ud_{\mathcal{L}}\chi\Big)\|(s, y) \wedge d\zeta \wedge dy \leq$$

$$2\pi^{-m} \sup_{s \in Q_K} \int\limits_{B_{R_2}} \|L^\beta \partial_x^\alpha \Big(ud_{\mathcal{L}}\chi\Big)\|(s, y) dy \int\limits_{\mathbb{R}^m} \exp\Big(-\mathtt{d}A(|\Re\zeta|) d\zeta \, 2\sqrt{n} N(K, \nu)(2^\nu)^{n+1} \leq$$

$$C 2\sqrt{n} N(\nu)(2^\nu)^{n+1} \leq C'\epsilon \tag{3.12}$$

by (3.11), (3.4) and the fact that G_ϵ is bounded in C_A^m by 1 for $|\Re\zeta| > A$. If $|\Re\zeta| \leq A$ we must observe the constrains imposed by (3.4) and (3.5). We choose ϵ such that $2^{-1}\exp\Big(-2\mathtt{d}\ A\Big) \leq \epsilon \leq 4\mathtt{d}\sqrt{m}A^{-1}$, to obtain $-\mathtt{d}A + 4^{-1}\epsilon m^{-1}A^2 \leq 0$.

This shows in view of (3.12) and (3.8) for $w_0 = x + i\Phi(t_0)$ and $w = x + i\Phi(t)$, that

$$\lim_{\epsilon \to 0} v_\epsilon^{\alpha,\beta}(w) - v_\epsilon^{\alpha,\beta}(w_0) =$$

$$lim_{\epsilon\to 0}2\pi^{-m}\Big(\int_{\Gamma_t}-\int_{\Gamma_{t_0}}\Big)\int_{\mathbb{R}^m}\exp\left(i\left[w-z\right]\cdot\xi\right)G_\epsilon(\zeta)L^\beta\partial_x^\alpha\Big(\chi u(z)\Big)d\xi\wedge dz$$

$$= lim_{\epsilon\to 0}\Big(\int_{\Gamma_t}-\int_{\Gamma_{t_0}}\Big)E_\epsilon(w-z))L^\beta\partial_x^\alpha\Big(\chi u(z)\Big)dz = 0 \tag{3.13}$$

in $\overline{B_{R_0}}+iK$. Thus

$$v_\epsilon^{\alpha,\beta}(w) = (\pi\epsilon)^{-m/2}\int_{\Gamma_{t_0}}E_\epsilon(w-z)L^\beta\partial_x^\alpha\Big(\chi u(z)\Big)dz + O(\epsilon)$$

if $n+1\leq l-(|\alpha|+|\beta|+1)$, uniformly in $\overline{B_{R_0}}\times K$ and it is well known that

$$\int_{\Gamma_t}E_\epsilon(w-z)L^\beta\ \partial_x^\alpha\Big(\chi u(z)\Big)dz$$

converges uniformly to $L^\beta\partial^\alpha u$ in $\overline{B_{R_0}}\times\{t\}$ as $\epsilon\to 0$.

This last step provide a semilocal version of the Baouendi-Treves approximation theorem [5] for jets in $\mathcal{H}(K)$ where $K\subset\mathbb{R}^n$ is a compact set having the following property: *any two points of K can be linked by a rectifiable curve within K of length bounded by a fixed constant.*(In this context the Baouendi-Treves approximation theorem asserts that any $u\in\mathcal{H}(K)$ can be approximated by a sequence of polynomials $P_k(Z(x,t))$ in $\mathcal{H}(K)$).

We may conclude that $v_\epsilon^{\alpha,\beta}$ converges uniformly to $L^\beta\partial^\alpha u$ in $\overline{B_{R_0}}\times K$, as long as $|\alpha|+|\beta|\leq l-(n+2)$.

Now we estimate the last partial sum. We select a piecewise linear $\Lambda_{\xi,1}$ path quite in the same way as we did before. We know by hypothesis that we can find t' and q_θ in a same component of $\mathcal{Q}_K\cap(\Phi\cdot\theta)_{\Phi(t)\cdot\theta}$ with $q_\theta\in F$ and t' arbitrarily close to t.

Let $N(\xi,t)$ be the minimum number of closed cubes $\mathcal{Q}_\nu\in\mathcal{C}(\nu)$ whose union covers $\mathcal{Q}_K\cap\overline{(\Phi\cdot\theta)_{\Phi(t)\cdot\theta}}$ and such that $\mathcal{Q}_\nu\cap\mathcal{Q}_K\cap(\Phi\cdot\theta)_{\Phi(t)\cdot\theta}\neq\emptyset$. Moreover we choose ν such that

$$2(1+|\xi|^2)^{-\frac{1}{2}}\geq 2^\nu\geq(1+|\xi|^2)^{-\frac{1}{2}} \tag{3.14}$$

We always have

$$\sum_{k=1}^{N(\xi,t)}2^{\nu n}\leq N(\nu)2^\nu = \int_{\mathcal{Q}_K}dx \tag{3.15}$$

We will select the vertices of $\Lambda_{\xi,1}$ in $\cup_{k=1}^{N(\xi,t)} \partial \mathcal{Q}_k$. Consider a vertex t_1 in the boundary of one of $N(\xi,t)$ cubes in the covering of

$$\mathcal{Q}_K \cap (\Phi \cdot \theta)_{\Phi(t)\cdot\theta} \tag{3.16}$$

which contains the point q_θ. Suppose that the vertex $t_{k-1} \in \mathcal{Q}_{k-1}$ is defined. Then select the next vertex t_k in the boundary of another cube $\mathcal{Q}_k$ in the covering of $\mathcal{Q}_K \cap (\Phi \cdot \theta)_{\Phi(t)\cdot\theta}$ distinct of $\mathcal{Q}_{k-1}$, with $\mathcal{Q}_k \cap \mathcal{Q}_{k-1} \neq \emptyset$, allowing the maximum projection in the direction of the vector $\overrightarrow{q_\theta t}$. As before in the construction of $\Lambda_{\xi,0}$ we find after at most $N(\xi,t)$ number of steps, vertices $\{t_1, \ldots, t_N\}$ of the cubes $\mathcal{Q}_k$ such that t_{k-1} and t_k lies in a same cube $\mathcal{Q}_k$. The point t_1 is in the same cube as q_θ and the point t_N will be in the same cube as t. The process is fullfilled because there is a cube which contains t and in its interior points t' of $\mathcal{Q}_K \cap (\Phi\cdot\theta)_{\Phi(t)\cdot\theta}$ which are in the same connected component of q_θ. Denote the polygonal line defined by these vertices by $\Lambda_{\xi,1}$. It follows from (3.15) that

$$(1+|\xi|^2)^{-\frac{1}{2}(n-1)} N 2^{-\nu} \leq N(\xi,t) 2^{-n\nu} \leq N(\nu) 2^{-n\nu} = \int\limits_{\mathcal{Q}_K} dx \tag{3.17}$$

Integrating III by parts with respect to $P(\partial_x)$, where $P(X) = 1 - \sum_{j=1}^{m} X_j^2$, at least $\mathtt{k} = \left[\frac{m+n}{2}\right]$ times we get

$$III = 2\pi^{-m} \int\limits_{\mathbb{R}^m} \int\limits_{\Lambda_{\xi,1}} \int\limits_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G(\epsilon\xi) P^{-\mathbf{k}}(-i\xi) P^{\mathbf{k}}(\partial_x) L^\beta \partial_x^\alpha \omega(z) \wedge d\xi \wedge dz \tag{3.18}$$

and each point of $\Lambda_{\xi,1}$ is within $\sqrt{n}2^{\nu}$ of some point of $\mathcal{Q}_K \cap (\Phi\cdot\theta)_{\Phi(t)\cdot\theta}$. Then we can bound the exponential integrand in III by

$$2\pi^{-m} \left| \int\limits_{\mathbb{R}^m} \int\limits_{\Lambda_{\xi,1}} \int\limits_{\Gamma_s} \exp\left(i\,[w-z]\cdot\xi\right) G(\epsilon\xi) P^{-\mathbf{k}}(-i\xi) P^{\mathbf{k}}(\partial_x) L^\beta \partial_x^\alpha \omega(z) \wedge d\xi \wedge dz \right| \leq$$

$$C(K,l) \sum_{k=1}^{N(\xi,t)} \int_{\mathbb{R}^m} \exp\Big(\sup_{s\in\mathcal{Q}_K} |\nabla\Phi(s)|\sqrt{n}2^{-\nu}|\xi|\Big) n^{\frac{n}{2}} 2^{-n\nu} P^{-\mathbf{k}+\frac{n}{2}}(-i\xi) d\xi \tag{3.19}$$

where $C(K,l) = 2\pi^{-m} \sup_{s\in\mathcal{Q}_K} \int_{\Gamma_s} \|P^{\mathbf{k}}(\partial_x)L^\beta\partial_x^\alpha\omega(s,\Re z)\|dz$. Now from 3.18 we can estimate 3.19 and get

$$n^{\frac{n}{2}}C(K,l)\exp\Big(\sqrt{n}C\sup_{s\in\mathcal{Q}_K}|\nabla\Phi(s)|\Big)N(\xi,t)2^{-n\nu}\int_{\mathbb{R}^m}P^{-\mathbf{k}+\frac{n}{2}}(-i\xi)d\xi \leq$$

$$n^{\frac{n}{2}}C(K)\exp\Big(\sqrt{n}C\sup_{s\in\mathcal{Q}_K}|\nabla\Phi(s)|\Big)N(\nu)2^{-n\nu}\int_{\mathbb{R}^m}P^{-\mathbf{k}+\frac{n}{2}}(-i\xi)d\xi \leq C(K,\mathtt{k},l) \tag{3.20}$$

Finally the partial sum B in 3.1 is given by

$$v_\epsilon^{\alpha,\beta}(x,t) = \int\limits_{\Gamma_t} E_\epsilon(x-y)L^\beta\partial_y^\alpha(\chi u)(t,z)\wedge dz \tag{3.21}$$

which converges uniformly to $L^\beta\partial_y^\alpha(\chi u)(\cdot,t)$ in $\mathbb{R}^m\times\mathcal{Q}_K$. This proves the first part of the Lemma. We omit the rest of the proof as it uses essentially the same ideas; we only remind that in this case we must construct paths Λ_ξ reaching a point in $F\cap\partial G$ laying inside G and this can be done since ∂G is a regular manifold. The proof is finished.

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