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THE GLOBAL ATTRACTOR OF A WEAKLY DAMPED, FORCED KORTEWEG-DE VRIES EQUATION IN $H^1(\mathbb{R})$

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Dedicated to Professor Roger Temam on the Occasion of his 60^{th} Birthday

Abstract

The aim in this note is to present a proof of the existence of the global attractor of a weakly damped, forced Korteweg-de Vries equation in the phase space $H^1(\mathbb{R})$. Previous results concern the existence of the global attractor in $H^2(\mathbb{R})$ and in spaces of space-periodic functions. It is well-known that the well-posedness of the KdV in $H^s(\mathbb{R})$, for $s \leq 3/2$, requires more than the usual energy-type estimates and depends on subtle regularization properties of dispersive equations. These properties are exploited here both for the well-posedness and for the existence of the global attractor in the presence of weak damping and forcing.

Resumo

O objetivo neste trabalho é provar a existência do atrator global de uma equação do tipo Korteweg-de Vries com dissipação fraca e força externa, no espaço de fase $H^1(\mathbb{R})$. Resultados anteriores demonstraram a existência do atrator global em $H^2(\mathbb{R})$ e em espaços de funções periódicas no espaço. A demonstração de que a equação de KdV é bem-posta em $H^s(\mathbb{R})$, para $s \leq 3/2$, requer mais do que estimativas usuais de energia e depende das propriedades de regularização do termo dispersivo da equação. No caso com dissipação fraca e força externa, essas propriedades são exploradas aqui, tanto para a demonstração de que a equação gera um grupo em $H^1(\mathbb{R})$ quanto para a demonstração da existência do atrator global.

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1. Introduction

Our aim is to prove the existence of the global attractor of a weakly damped, forced Korteweg-de Vries equation in the phase space $H^1(\mathbb{R})$. The equation reads

$$u_t + uu_x + u_{xxx} + \gamma u = f, \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

with the initial condition

$$u|_{t=0} = u_0 \in H^1(\mathbb{R}). \tag{2}$$

It is assumed that f is time-independent and belongs to $H^1(\mathbb{R})$, and that $\gamma > 0$ is a constant. Equation (1) has been derived by E. Ott and R. N. Sudan [16] as a model for ion-sound waves damped by ion-neutral collisions. For $\gamma = 0$ and $f = 0$, equation (1) is the well-known Korteweg-de Vries (KdV) equation [10]. From the mathematical point of view, the extra term with the factor γ accounts for a weak dissipation with no regularization/smoothing property.

We use the function spaces introduced by C. E. Kenig, G. Ponce, and L. Vega [9] for the well-posedness of (1), and the energy equation method of J. Ball [2] for the existence of the global attractor. The well-posedness in $H^1(\mathbb{R})$ of the forced equation without weak dissipation (i.e., with $\gamma = 0$) has been considered by J. Bona and B.-Y. Zhang [3] following the framework of [9]. We follow their steps for the well-posedness when $\gamma \neq 0$.

The existence of the global attractor for hyperbolic equations is usually obtained by splitting the solution operator or by exploiting suitable energy equations (see [18, 11], for instance). For equations on unbounded domains the use of energy equations is particularly suitable since it does not depend on compact imbeddings of function spaces. It does require, however, the weak continuity of the solution operator in the sense that if the initial conditions u_{0n} converge to u_0 weakly, then the corresponding solutions $u_n(t)$ converge weakly to $u(t)$, at all times t. This weak continuity is usually obtained by passing to the limit in the weak formulation of the equation and using the uniqueness of the solutions. The uniqueness is a delicate issue, though, for the KdV (and for (1)) in $H^s(\mathbb{R})$, for $s \leq 3/2$, and regularity, as well, is limited, so that much effort is taken here in proving the weak continuity.

The existence of global attractors for the equation (1) was first considered by J. M. Ghidaglia [5,6], in the spaces $H^2_{\text{per}}(0,L)$ of L-periodic functions in H^2, assuming $f \in H^2_{\text{per}}(0,L)$. This result was extended to $H^k_{\text{per}}(0,L)$, with $k \in \mathbb{N}$, $k \geq 3$, by I. Moise and R. Rosa [14], with $f \in H^k_{\text{per}}(0,L)$, and where the global attractor was proven to be more regular if so is f (with the attractor as regular as f in the scale of Sobolev spaces $H^m_{\text{per}}(0,L)$, $m \in \mathbb{N}$, $m > k \geq 3$). The whole space case in the phase space $H^2(\mathbb{R})$ was first treated independently by P. Laurençot [12], using energy-type equations and weighted spaces, and by I. Moise, R. Rosa, and X. Wang [15], using only energy-type equations. To the best of our knowledge the case $H^1(\mathbb{R})$ is being treated here for the first time. During the completion of this work, it was brought to our attention a recent work by O. Goubet [7] where the space-periodic case is considered in the phase space $L^2(0,L)$, using the framework introduced by J. Bourgain for the well-posedness of the Korteweg-de Vries equation. The method used is the splitting of the solution operator and exploits the compact imbeddings in bounded domains. This is achieved by a suitable regularization property of the linear part of the equation with respect to the nonhomogeneous term, as it appears in the framework of J. Bourgain [4] (see also J. Bona and B.-Y. Zhang [3]). A nice by-product presented in [7] is that further regularity of the global attractor (it belongs to $H^3_{\text{per}}(0,L)$) is obtained assuming only that the forcing term f belongs to $L^2(0,L)$, thus extending and improving the result in [14]. It seems reasonable to conjecture that a similar result holds in the whole space (in $L^2(\mathbb{R})$); this will be considered elsewhere.

2. Function Spaces and Preliminary Estimates

We consider the spaces $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ endowed respectively with the norms

$$\|u\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |u(x)|^2 \, dx\right)^{1/2},$$

and

$$\|u\|_{H^1(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + \|u_x\|^2_{L^2(\mathbb{R})}\right)^{1/2},$$

and the associated inner products $(\!(\cdot,\cdot)\!)_{L^2(\mathbb{R})}$ and $(\!(\cdot,\cdot)\!)_{H^1(\mathbb{R})}$. We set $A = (1-\partial_x^2)$, which is a positive self-adjoint operator in $L^2(\mathbb{R})$. We denote by $H^{-1}(\mathbb{R})$ the dual of $H^1(\mathbb{R})$ and we identify $L^2(\mathbb{R})$ with its dual. The operator A can be extended to an isomorphism from $H^1(\mathbb{R})$ onto $H^{-1}(\mathbb{R})$, and the norm in $H^{-1}(\mathbb{R})$ can be written as

$$\|u\|_{H^{-1}(\mathbb{R})} = \|A^{-1}u\|_{H^1(\mathbb{R})} = \|A^{-1/2}u\|_{L^2(\mathbb{R})}. \tag{3}$$

We also consider the space $H^2(\mathbb{R})$, of functions u in $H^1(\mathbb{R})$ with u_x in $H^1(\mathbb{R})$, and its dual space $H^{-2}(\mathbb{R})$, with their norms denoted in a similar way, and the spaces $H_0^1(-r,r)$, of functions in $H^1(\mathbb{R})$ which vanish outside $(-r,r)$, where $r>0$, and their dual $H_0^{-1}(-r,r)$. Note that for all $r>0$,

$$H_0^1(-r,r) \subseteq H^1(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq H^{-1}(\mathbb{R}) \subseteq H_0^{-1}(-r,r), \tag{4}$$

with continuous injections, and

$$H_0^1(-r,r) \underset{c}{\subsetneq} L^2(\mathbb{R}) \underset{c}{\subsetneq} H_0^{-1}(-r,r), \tag{5}$$

with compact injections. Other useful relations involving the operator A are

$$(\!(Au,v)\!)_{L^2(\mathbb{R})} = (\!(u,v)\!)_{H^1(\mathbb{R})}, \qquad \frac{1}{2}\|u\|_{L^2(\mathbb{R})} \leq \|A^{-1}u\|^2_{H^2(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}. \tag{6}$$

Another space we consider is the space $L^\infty(\mathbb{R})$ of essentially bounded functions, with its norm denoted by $\|\cdot\|_{L^\infty(\mathbb{R})} = \operatorname{ess.sup}|\cdot|$. We will often use the Agmon inequality, which reads

$$\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|^{1/2}_{L^2(\mathbb{R})}\|u_x\|^{1/2}_{L^2(\mathbb{R})}, \tag{7}$$

for every u in $H^1(\mathbb{R})$. We also consider spaces of the type $\mathcal{C}_c^\infty$, of infinitely differentiable functions with compact support.

We now consider a space similar to that introduced by C. E. Kenig, G. Ponce, and L. Vega [9]. For each $T > 0$ and each measurable function $u : \mathbb{R} \times [-T, T] \to \mathbb{R}$ we set

$$\lambda_1(T;u) = \text{ess.} \sup_{t\in[-T;T]} \|u(\cdot,t)\|_{H^1(\mathbb{R})}, \tag{8}$$

$$\lambda_2(T;u) = \left(\text{ess.} \sup_{x\in\mathbb{R}} \int_{-T}^{T} |u_{xx}(\cdot,t)|^2 \, dt\right)^{1/2}, \tag{9}$$

$$\lambda_3(T;u) = \left(\int_{-T}^{T} \|u_x(\cdot,t)\|^6_{L^\infty(\mathbb{R})} \, dt\right)^{1/6}, \tag{10}$$

$$\lambda_4(T;u) = \frac{1}{1+T}\left(\int_{\mathbb{R}} \text{ess.} \sup_{t\in[-T,T]} |u(x,t)|^2 \, dx\right)^{1/2}, \tag{11}$$

and

$$\Lambda(T;u) = \max\{\lambda_1(T;u), \lambda_2(T;u), \lambda_3(T;u), \lambda_4(T;u)\}. \tag{12}$$

We then define the space

$$X_T = \{u : \mathbb{R} \times [-T,T] \to \mathbb{R};\ \Lambda(T;u) < \infty\}. \tag{13}$$

The space X_T is a Banach space endowed with the norm $\|\cdot\|_{X_T} = \Lambda(T,\cdot)$. It could also have been defined (except for the norm), in a more explicit, and hopefully self-explanatory, form as the space

$$L^\infty_t(-T,T;H^1_x(\mathbb{R})) \cap W^{2,\infty}_x(\mathbb{R};L^2_t(-T,T)) \\ \cap L^6_t(-T,T;W^{1,\infty}_x(\mathbb{R})) \cap L^2_x(\mathbb{R};L^\infty_t(-T,T)).$$

For a given interval I and a given Banach space E we also consider the spaces $L^p(I;E)$, $1 \le p \le \infty$, of E-valued functions on I whose norm in E to the p-th power is integrable on I (or is essentially bounded if $p = \infty$), and the space $\mathcal{C}_b(I;E)$ of bounded, continuous functions on I with values in E. Their respective norms are denoted $\|\cdot\|_{L^p(I;E)}$ and $\|\cdot\|_{\mathcal{C}_b(I;E)}$. When I is an unbounded interval and $1 \le p < \infty$, it is also useful to consider the functions which are locally in L^p, i.e., the space of functions which belong to $L^p(J;E)$ for every bounded subinterval $J \subseteq I$. These spaces are Fréchet spaces and are denoted

$L^p_{\mathrm{loc}}(I;E)$. Similarly, we denote by $\mathcal{C}(I;E)$ the space of functions which belong to $\mathcal{C}_b(J;E)$ for each bounded subinterval $J \subseteq I$.

We turn our attention now to the linear part of equation (1) and the associated regularization properties which are fundamental for the well-posedness in lower order Sobolev spaces. For each $\gamma \in \mathbb{R}$ we denote by $\{W_\gamma(t)\}_{t\in\mathbb{R}}$ the group associated with the linear equation

$$w_t + w_{xxx} + \gamma w = 0, \qquad w|_{t=0} = w_0, \tag{14}$$

The case $\gamma = 0$ is connected to the KdV equation, and the fundamental estimates on the linear group for the well-posedness of the KdV in $H^1(\mathbb{R})$ were obtained by C. E. Kenig, G. Ponce, and L. Vega [9]. We borrow from them the following estimates:

Lemma 1 [9] *There is a numerical constant c_1, which we can assume greater than 1, such that*

$$\left(\int_{-\infty}^{\infty} \|W_0(t)w_0\|^6_{L^\infty(\mathbb{R})}\,dt\right)^{1/6} \le c_1\|w_0\|_{L^2(\mathbb{R})}, \tag{15}$$

for all $w_0 \in L^2(\mathbb{R})$, and

$$\left(\int_{-\infty}^{\infty} \sup_{t\in[-T,T]} |W_0(t)w_0|^2\,dx\right)^{1/2} \le c_1(1+T)\|w_0\|_{H^1(\mathbb{R})}, \tag{16}$$

for all $w_0 \in H^1(\mathbb{R})$ and all $T > 0$.

Estimate (15) follows from [10, Theorem 2.4 with $\theta = 1$, $\beta = 0$, and $\alpha = 2$], while estimate (16) follows from [10, Corollary 2.9 with $\alpha = 2$, $s = 1$, and $\rho = 1$].

From the estimates in Lemma 1, and using that $W_\gamma(t) = e^{-\gamma t}W_0(t)$, one can deduce the corresponding estimates for $\gamma \neq 0$:

Lemma 2 *Let $\gamma \in \mathbb{R}$ and $T > 0$. Then,*

$$\left(\int_{-T}^{T} \|W_\gamma(t)w_0\|^6_{L^\infty(\mathbb{R})}\,dt\right)^{1/6} \le c_1 e^{|\gamma|T}\|w_0\|_{L^2(\mathbb{R})}, \tag{17}$$

for all w_0 in $L^2(\mathbb{R})$, and

$$\left(\int_{-\infty}^{\infty} \sup_{t\in[-T,T]} |W_\gamma(t)w_0|^2\, dx\right)^{1/2} \leq c_1(1+T)e^{|\gamma|T}\|w_0\|_{H^1(\mathbb{R})}, \tag{18}$$

for all w_0 in $H^1(\mathbb{R})$, where c_1 is the constant in Lemma 1.

The next dispersive-type regularization that we need does not follow directly from the one in [10, Lemma 2.1 ($c_\alpha = \sqrt{3}/3$ for $\alpha = 2$)] for $\gamma = 0$, but its proof requires only minor modifications, so we limit ourselves to stating the result.

Lemma 3 *Let $\gamma \in \mathbb{R}$, and $T > 0$. Then,*

$$\left(\sup_{x\in\mathbb{R}} \int_{-T}^{T} |\partial_x W_\gamma(t)w_0|^2\, dt\right)^{1/2} \leq \frac{\sqrt{3}}{3} e^{|\gamma|T}\|w_0\|_{L^2(\mathbb{R})}, \tag{19}$$

for all w_0 in $L^2(\mathbb{R})$.

Taking the inner product in $L^2(\mathbb{R})$ of equation (14) with w one finds also the estimate

$$\|W_\gamma(t)w_0\|_{L^2(\mathbb{R})} = e^{\gamma t}\|w_0\|_{L^2(\mathbb{R})}. \tag{20}$$

By linearity, we also have

$$\|W_\gamma(t)w_0\|_{H^1(\mathbb{R})} = e^{\gamma t}\|w_0\|_{H^1(\mathbb{R})}. \tag{21}$$

If the initial condition w_0 belongs to $H^1(\mathbb{R})$, the solution is continuous as a function with values in $H^1(\mathbb{R})$, i.e.,

$$w_0 \in H^1(\mathbb{R}) \Rightarrow t \mapsto W_\gamma(t)w_0 \in \mathcal{C}_b([-T,T]; H^1(\mathbb{R})), \tag{22}$$

for all $T > 0$.

From Lemmas 2 and 3, and relation (21), it is straightforward to prove the following result:

Lemma 4 *Let $\gamma \in \mathbb{R}$, $T > 0$, and $w_0 \in H^1(\mathbb{R})$. Let, $w(\cdot,t) = W_\gamma(t)w_0$. Then, $w \in X_T$ with*

$$\|w\|_{X_T} \leq c_1 e^{|\gamma|T}\|w_0\|_{H^1(\mathbb{R})}, \tag{23}$$

where c_1 is the constant from Lemma 1, which is independent of γ, T, and w_0.

Now, we proceed essentially as in [9] and we derive estimates for the non-homogeneous equation associated with (14):

$$w_t + w_{xxx} + \gamma w = g, \qquad w|_{t=0} = 0, \tag{24}$$

where g is time-dependent and is assumed to belong to $L^1(-T,T;H^1(\mathbb{R}))$.

Lemma 5 *Let $\gamma \in \mathbb{R}$, $T > 0$, and $g \in L^1(-T,T;H^1(\mathbb{R}))$. Let*

$$w(\cdot,t) = \int_0^t W_\gamma(t-\tau)g(\cdot,\tau)\,d\tau, \tag{25}$$

for $t \in [-T,T]$. Then, w belongs to X_T with

$$\|w\|_{X_T} \le c_1 e^{|\gamma|T}\|g\|_{L^1(-T,T;H^1(\mathbb{R}))}, \tag{26}$$

where c_1 is the constant from Lemma 1, which is independent of γ, T, and g.

Proof: We need to estimate $\lambda_i(T;w)$ for each $i = 1,\dots,4$. These estimates are obtained in a very similar way, so we illustrate only the cases $i = 1$ and 2. We let $\chi_t = \chi_t(\tau)$ be the characteristic function of the closed interval with endpoints 0 and t, with t either positive or negative. Hence, we can write w given in (25) as

$$w(\cdot,t) = \int_{-T}^{T} \chi_t(\tau)W_\gamma(t-\tau)g(\cdot,\tau)\,d\tau. \tag{27}$$

Then, for $i = 1$, we use (21):

$$\begin{aligned}
\lambda_1(T;w) &= \operatorname{ess.}\sup_{t\in[-T;T]} \|w(\cdot,t)\|_{H^1(\mathbb{R})}, \\
&\le \int_{-T}^{T} \operatorname{ess.}\sup_{t\in[-T,T]} \|\chi_t(\tau)W_\gamma(t-\tau)g(\cdot,\tau)\|_{H^1(\mathbb{R})}\,d\tau \\
&\le \int_{-T}^{T} \operatorname{ess.}\sup_{s\in[-T,T]} \|W_\gamma(s)g(\cdot,\tau)\|_{H^1(\mathbb{R})}\,d\tau \\
&\le \int_{-T}^{T} e^{|\gamma|T}\|g(\cdot,\tau)\|_{H^1(\mathbb{R})}\,d\tau \\
&\le e^{|\gamma|T}\|g\|_{L^1(-T,T;H^1(\mathbb{R}))}.
\end{aligned}$$

For $i = 2$, we use (19):

$$
\begin{aligned}
\lambda_2(T;w) &\leq \int_{-T}^{T} \lambda_2(T;\chi_t(\tau)W_\gamma(t-\tau)g(\cdot,\tau))\,d\tau \\
&= \int_{-T}^{T} \left(\operatorname*{ess.\,sup}_{x\in\mathbb{R}} \int_{-T}^{T} |\partial_x^2(\chi_t(\tau)W_\gamma(t-\tau)g(\cdot,\tau))|^2\,dt\right)^{1/2} d\tau \\
&\leq \int_{-T}^{T} \left(\operatorname*{ess.\,sup}_{x\in\mathbb{R}} \int_{-T}^{T} |\partial_x(W_\gamma(s)g_x(\cdot,\tau))|^2\,ds\right)^{1/2} d\tau \\
&\leq \frac{\sqrt{3}}{3}\int_{-T}^{T} e^{|\gamma|T}\|g_x(\cdot,\tau)\|_{L^2(\mathbb{R})}\,d\tau \\
&\leq \frac{\sqrt{3}}{3} e^{|\gamma|T}\|g\|_{L^1(-T,T;H^1(\mathbb{R}))}.
\end{aligned}
$$

The proofs for $i = 3$ and 4 follow the same idea and make use of (17) and (18), respectively, introducing the factor $c_1 \geq 1$. Note that for $i = 4$ the estimate (18) introduces a factor of $(1+T)$, but this is absorbed in the definition of $\lambda_4(T;\cdot)$ and does not appear explicitly in estimate (26). □

We now turn our attention to the nonlinear term uu_x in the equation (1). It is a quadratic term which can be written as $(u^2)_x/2$, and which is associated to the bilinear term $(uv)_x/2$. We have

$$
\int_{-T}^{T} \|(uv)_x\|_{H^1(\mathbb{R})}\,dt \leq \sqrt{2}\int_{-T}^{T} (\|(uv)_x\|_{L^2(\mathbb{R})} + \|(uv)_{xx}\|_{L^2(\mathbb{R})})\,dt. \tag{28}
$$

First, we estimate, using Agmon's inequality,

$$
\begin{aligned}
\int_{-T}^{T} \|(uv)_x\|_{L^2(\mathbb{R})}\,dt &\leq \int_{-T}^{T} (\|u_x v\|_{L^2(\mathbb{R})} + \|uv_x\|_{L^2(\mathbb{R})})\,dt \\
&\leq \int_{-T}^{T} (\|u_x\|_{L^2(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R})}\|v_x\|_{L^2(\mathbb{R})})\,dx \\
&\leq 2\int_{-T}^{T} \|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})}\,dt \\
&\leq 2T\|u\|_{L^\infty(-T,T;H^1(\mathbb{R}))}\|v\|_{L^\infty(-T,T;H^1(\mathbb{R}))} \\
&= 2T\lambda_1(T;u)\lambda_1(T;v) \\
&\leq 2T\|u\|_{X_T}\|v\|_{X_T}.
\end{aligned}
$$

For the second term in the right hand side of (28), we use the "full power" of the space X_T, i.e., all the terms that make up the norm of X_T,

$$\begin{aligned}
\int_{-T}^{T} \|(uv)_{xx}\|_{L^2(\mathbb{R})}\, dt &\leq \int_{-T}^{T} (\|u_{xx}v\|_{L^2(\mathbb{R})} + 2\|u_x v_x\|_{L^2(\mathbb{R})} + \|uv_{xx}\|_{L^2(\mathbb{R})})\, dt \\
\leq\ & \int_{-T}^{T} \left(\int_{-\infty}^{\infty} |u_{xx}v|^2\, dx\right)^{1/2} dt + 2\int_{-T}^{T} \|u_x\|_{L^\infty(\mathbb{R})}\|v_x\|_{L^2(\mathbb{R})}\, dt \\
& \quad + \int_{-T}^{T} \left(\int_{-\infty}^{\infty} |uv_{xx}|^2\, dx\right)^{1/2} dt \\
\leq\ & \sqrt{2T} \left(\int_{-T}^{T}\int_{-\infty}^{\infty} |u_{xx}v|^2\, dx\, dt\right)^{1/2} \\
& \quad +2 \text{ ess.} \sup_{t\in[-T,T]} \|v_x\|_{L^2(\mathbb{R})} \int_{-T}^{T} \|u_x\|_{L^\infty(\mathbb{R})}\, dt \\
& \quad +\sqrt{2T} \left(\int_{-T}^{T}\int_{-\infty}^{\infty} |uv_{xx}|^2\, dx\, dt\right)^{1/2} \\
\leq\ & \sqrt{2T} \left(\int_{-\infty}^{\infty}\int_{-T}^{T} |u_{xx}|^2|v|^2\, dt\, dx\right)^{1/2} \\
& \quad +2T^{5/6} \text{ess.} \sup_{t\in[-T,T]} \|v_x\|_{L^2(\mathbb{R})} \left(\int_{-T}^{T} \|u_x\|_{L^\infty(\mathbb{R})}^6\, dt\right)^{1/6} \\
& \quad +\sqrt{2T} \left(\int_{-\infty}^{\infty}\int_{-T}^{T} |u|^2|v_{xx}|^2\, dt\, dx\right)^{1/2} \\
\leq\ & \sqrt{2T} \left(\text{ess.} \sup_{x\in\mathbb{R}} \int_{-T}^{T} |u_{xx}|^2\, dt\right)^{1/2} \left(\int_{-\infty}^{\infty} \text{ess.} \sup_{t\in[-T,T]} |v|^2\, dx\right)^{1/2} \\
& \quad +2T^{5/6}\lambda_1(T;v)\lambda_3(T;u) \\
& \quad +\sqrt{2T} \left(\text{ess.} \sup_{x\in\mathbb{R}} \int_{-T}^{T} |v_{xx}|^2\, dt\right)^{1/2} \left(\int_{-\infty}^{\infty} \text{ess.} \sup_{t\in[-T,T]} |u|^2\, dx\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \sqrt{2T}(1+T)\lambda_2(T;u)\lambda_4(T;v) + 2T^{5/6}\lambda_1(T;u)\lambda_3(T;v) \\
& \quad +\sqrt{2T}(1+T)\lambda_2(T;v)\lambda_4(T;u) \\
\leq\ & (2\sqrt{2T}(1+T) + 2T^{5/6})\|u\|_{X_T}\|v\|_{X_T}.
\end{aligned}$$

Taking the last two estimates together we prove from (28) the following result:

Lemma 6 *Let $T > 0$ and let $u, v \in X_T$. Then,*

$$\|(uv)_x\|_{L^1(-T,T;H^1(\mathbb{R}))} \leq 4(1+\sqrt{2})T^{1/2}(1+T)\|u\|_{X_T}\|v\|_{X_T}. \tag{29}$$

From Lemmas 5 and 6 it is straightforward to deduce the last result of this section:

Lemma 7 *Let $\gamma \in \mathbb{R}$, $T > 0$, and $u, v \in X_T$. Let*

$$w(\cdot, t) = \int_0^t W_\gamma(t-\tau)(uv)_x \, d\tau, \tag{30}$$

for $t \in [-T, T]$. Then,

$$\|w\|_{X_T} \leq c_2 T^{1/2}(1+T)e^{|\gamma|T}\|u\|_{X_T}\|v\|_{X_T}, \tag{31}$$

where $c_2 = 4(1+\sqrt{2})c_1$, and c_1 is the constant from Lemma 1, which is independent of γ, T, u, and v.

3. Local Well-Posedness

Consider $\gamma \in \mathbb{R}$, $f \in L^1_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$, and $u_0 \in H^1(\mathbb{R})$, throughout this section. We look for the solution of (1) in the *mild sense* [8], i.e., as the unique fixed point of the map $\Sigma : X_T \to X_T$ defined by

$$\Sigma(u)(t) = W_\gamma(t)u_0 + \int_0^t W_\gamma(t-\tau)\left(f(\tau) - \frac{1}{2}(u(\tau)^2)_x\right) d\tau, \tag{32}$$

for $t \in [-T, T]$. From Lemmas 4, 5, and 7, it follows that

$$\begin{aligned} \|\Sigma(u)\|_{X_T} \;\leq\; & c_1 e^{|\gamma|T}\|u_0\|_{H^1(\mathbb{R})} + c_1 e^{|\gamma|T}\|f\|_{L^1(-T,T;H^1(\mathbb{R}))} \\ & \frac{c_2}{2}T^{1/2}(1+T)e^{|\gamma|T}\|u\|^2_{X_T}. \end{aligned} \tag{33}$$

Similarly, by writing $u^2 - v^2 = (u+v)(u-v)$, we find

$$\|\Sigma(u) - \Sigma(v)\|_{X_T} \leq \frac{c_2}{2}T^{1/2}(1+T)e^{|\gamma|T}\|u+v\|_{X_T}\|u-v\|_{X_T}. \tag{34}$$

Let

$$R = 2c_1 e^{|\gamma|}(\|u_0\|_{H^1(\mathbb{R})} + \|f\|_{L^1(-T,T;H^1(\mathbb{R}))}). \tag{35}$$

For $0 < T \leq 1$ and $\|u\|_{X_T}, \|v\|_{X_T} \leq R$, we have

$$\|\Sigma(u)\|_{X_T} \leq \frac{R}{2} + c_2 T^{1/2} e^{|\gamma|} R^2, \tag{36}$$

and

$$\|\Sigma(u) - \Sigma(v)\|_{X_T} \leq 2c_2 T^{1/2} e^{|\gamma|} \|u - v\|_{X_T}. \tag{37}$$

Take T_R, $0 < T_R \leq 1$, sufficiently small so that

$$\frac{R}{2} + c_2 T_R^{1/2} e^{|\gamma|} R^2 \leq R \quad \text{and} \quad 2c_2 T_R^{1/2} e^{|\gamma|} R \leq \frac{1}{2}, \tag{38}$$

which is given by

$$T_R = \min\left\{1, \frac{1}{16 c_2^2 e^{2|\gamma|} R^2}\right\}. \tag{39}$$

With this choice of T_R, it follows that

$$\|\Sigma(u)\|_{X_{T_R}} \leq R, \qquad \|\Sigma(u) - \Sigma(v)\|_{X_{T_R}} \leq \frac{1}{2}\|u - v\|_{X_{T_R}}, \tag{40}$$

for every $u, v \in X_{T_R}$ with $\|u\|_{X_{T_R}}, \|v\|_{X_{T_R}} \leq R$. Thus, Σ is a strict contraction when restricted to the ball in X_{T_R} of radius R and centered at the origin. By the Banach Fixed Point Theorem there exists a unique u in this ball which is the fixed point of Σ and, hence, a solution of (1) in the mild sense. One can check that there is enough regularity to deduce, by taking the duality product of the equation for u in the mild sense with a test function, that u is also a solution in the weak sense, and that, vice-versa, a weak solution which belongs to X_T is a fixed point of Σ and, hence, is the unique mild solution. By the Uniform Contraction Principle [8], it follows also that for any T' with $0 < T' < T_R$, the fixed point u is continuous in $X_{T'}$ with respect to γ in $\mathbb{R}$, u_0 in $H^1(\mathbb{R})$, and f in $L^1(-T', T'; H^1(\mathbb{R}))$. Working with some $T'' << T_R$, if necessary, and by continuation, one can check that the fixed point u is unique within all X_{T_R}, not only within the ball of radius R and centered at the origin. Finally, using, in particular, (22) and the expression for the solution u as a fixed point of Σ, we find that the solution u is continuous as a function with values in $H^1(\mathbb{R})$, i.e.,

$$t \mapsto u(t) \in \mathcal{C}_b([-T_R, T_R]; H^1(\mathbb{R})),$$

and consequently, the dependence on the data is also continuous in this function space. Hence, we have proven the following result:

Theorem 1 *Let $\gamma \in \mathbb{R}$, $f \in L^1_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$, and $u_0 \in H^1(\mathbb{R})$. Let R and T_R be given by (35) and (39), respectively, where c_1 is the constant given in Lemma 1. Then, there exists a unique solution u in X_{T_R} of equation (1). Moreover, $t \mapsto u(t)$ belongs to $\mathcal{C}_b([-T_R, T_R]; H^1(\mathbb{R}))$ and the map which associates (γ, f, u_0) to the corresponding unique solution is continuous from $\mathbb{R} \times L^1(-T', T'; H^1(\mathbb{R})) \times H^1(\mathbb{R})$ into $X_{T'} \cap \mathcal{C}([-T', T']; H^1(\mathbb{R}))$, for appropriate $T' > 0$.*

4. Global Solutions and Energy-Type Equations

Let $\gamma \in \mathbb{R}$, $f \in H^1(\mathbb{R})$, and $u_0 \in H^1(\mathbb{R})$ be given throughout this section. We want to establish the global existence of the solutions obtained in the previous section. This is achieved with the help of two of the invariants of the KdV, namely,

$$I_0(u) = \int_{-\infty}^{\infty} u(x)^2\, dx, \tag{41}$$

$$I_1(u) = \int_{-\infty}^{\infty} \left(u_x(x)^2 - \frac{u(x)^3}{3} \right) dx. \tag{42}$$

These are two of the countable number of invariants for the KdV equation (see R. M. Miura, C. S. Gardner, and M. D. Kruskal [13], for instance). Upon introducing dissipation and external forcing these integrals are no longer invariants, but lead to energy-type equations which are crucial for proving global existence of the solutions. For a smooth initial condition $\tilde{u}_0 \in \mathcal{C}_c^\infty(\mathbb{R})$, and a smooth forcing term $\tilde{f} \in \mathcal{C}_c^\infty(\mathbb{R})$, the local solution $\tilde{u} \in X_T$ given by Theorem 1, for some $T > 0$, coincides with the classical solution, which exists globally and belongs to $\mathcal{C}^\infty(\mathbb{R} \times \mathbb{R})$. Multiplying equation (1) by $2\tilde{u}$ and $-2\tilde{u}_{xx} - \tilde{u}^2$, respectively, we see that $\tilde{u}$ satisfies the energy-type equations

$$\frac{d}{dt} I_j(\tilde{u}(t)) + 2\gamma I_j(\tilde{u}(t)) = \tilde{K}_j(\tilde{u}(t)), \tag{43}$$

for all $t \in \mathbb{R}$ and for $j = 0, 1$, where

$$\tilde{K}_0(\tilde{u}) = \int_{-\infty}^{\infty} \tilde{f}(x)\tilde{u}(x)\, dx, \tag{44}$$

and

$$\tilde{K}_1(\tilde{u}) = \int_{-\infty}^{\infty} \left(2\tilde{f}_x(x)\tilde{u}_x(x) - \tilde{f}(x)\tilde{u}(x)^2 + \frac{\gamma}{3}\tilde{u}(x)^2\right)\, dx. \tag{45}$$

We integrate (43) to find

$$I_j(\tilde{u}(t)) + 2\gamma \int_0^t I_j(\tilde{u}(\tau))\, d\tau = I_j(\tilde{u}_0) + \int_0^t \tilde{K}_j(\tilde{u}(\tau))\, d\tau, \tag{46}$$

for $t \in [-T, T]$ and $j = 0, 1$. Now, let $u_0 \in H^1(\mathbb{R})$ and $f \in H^1(\mathbb{R})$ and consider approximations of them by smooth functions $\tilde{u}_0$ and $\tilde{f}$ converging to u_0 and f in $H^1(\mathbb{R})$, respectively. By the continuity with respect to the data of the local solution given by Theorem 1, we have that the solutions $\tilde{u}$ with initial condition $\tilde{u}(0) = \tilde{u}_0$ and forcing term $\tilde{f}$ converge in X_T, for some appropriate $T > 0$, to the solution $u \in X_T$ with initial condition $u(0) = u_0$ and forcing term f. Taking the limit in (46) and using the continuity of the solution with respect to the data, in particular using that

$$I_j(u(t)) = \lim I_j(\tilde{u}(t)), \quad \text{and} \quad K_j(u(t)) = \lim \tilde{K}(\tilde{u}(t)),$$

for all $t \in [-T, T]$, and $j = 0, 1$, where

$$K_0(u) = \int_{-\infty}^{\infty} f(x)u(x)\, dx, \tag{47}$$

$$K_1(u) = \int_{-\infty}^{\infty} \left(2f_x(x)u_x(x) - f(x)u(x)^2 + \frac{\gamma}{3}u(x)^3\right)\, dx, \tag{48}$$

we find that

$$I_j(u(t)) + 2\gamma \int_0^t I_j(u(\tau))\, d\tau = I_j(u_0) + \int_0^t K_j(u(\tau))\, d\tau, \tag{49}$$

for all $t \in [-T, T]$, and $j = 0, 1$. From the energy-type equations (49) one can extend the solution u indefinitely and obtain a global solution $u = u(t)$, $t \in \mathbb{R}$, with $u \in X_T \cap \mathcal{C}_b([-T, T], H^1(\mathbb{R}))$ for all $T > 0$. One can also check that for each $T > 0$ and each initial condition $u_0 \in H^1(\mathbb{R})$, there exists a constant $C = C(\|u_0\|_{H^1(\mathbb{R})}, T)$ such that

$$\|u\|_{X_T} \leq C(\|u_0\|, T). \tag{50}$$

This can be obtained by dividing each interval $[-T, T]$ into small enough subintervals as required by the proof of local existence and using the estimate provided by the energy-type equations (49) for the norm of the solution in $H^1(\mathbb{R})$ at each instant of time. We omit the details since this is straightforward and classical. The energy-type equations (49) also hold for all time and the continuity of the solutions with respect to the data can be extended to all large times, as well. Hence, we have the following result:

Theorem 2 *Let $\gamma \in \mathbb{R}$, $f \in H^1(\mathbb{R})$, and $u_0 \in H^1(\mathbb{R})$. Then, there exists a solution $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}))$ of equation (1) which is the unique solution which belongs to X_T for all $T > 0$. Moreover, the solution $t \mapsto u(t)$ satisfies the energy equations*

$$\frac{d}{dt} I_j(u(t)) + 2\gamma I_j(u(t)) = K_j(u(t)), \tag{51}$$

for all $t \in \mathbb{R}$, and for $j = 0, 1$, where I_j and K_j are given in (41), (42), (47), and (48). Furthermore, the map which associates the data (γ, f, u_0) to the corresponding unique solution u is continuous from $\mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ into $X_T \cap \mathcal{C}([-T, T]; H^1(\mathbb{R}))$ for all $T > 0$, with, in particular,

$$\|u\|_{X_T} \leq C(\gamma, \|f\|_{H^1(\mathbb{R})}, \|u_0\|_{H^1(\mathbb{R})}, T), \tag{52}$$

for some constant C depending monotonically on the data.

Thanks to Theorem 2 we can define a group associated with equation (1):

Definition 1 *For $\gamma \in \mathbb{R}$ and $f \in H^1(\mathbb{R})$ fixed, we denote by $\{S(t)\}_{t \in \mathbb{R}}$ the group in $H^1(\mathbb{R})$ defined by $S(t)u_0 = u(t)$ where $u = u(t)$ is the unique solution of (1) which belongs to X_T for all $T > 0$.*

5. Bounded Absorbing Sets

From this section on we are interested in the long time behavior of equation (1) taking the dissipation into account. Therefore, we assume that $\gamma > 0$. We also assume that the forcing term f belongs to $H^1(\mathbb{R})$. We want first to obtain the existence of bounded aborbing sets for the solution operator $\{S(t)\}_{t\in\mathbb{R}}$. This is achieved with the help of the energy-type equations proved earlier. We first obtain an absorbing ball in $L^2(\mathbb{R})$, then we find an absorbing ball in $H^1(\mathbb{R})$. Since this is a standard procedure, we just outline it here.

Estimating the term K_0 given in (47) using Young's inequality it is straightforward to deduce from the energy-type equation (51) for $j = 0$ the following asymptotic behavior:

$$\limsup_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R})} \leq \rho_0 \equiv \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R})}, \tag{53}$$

uniformly for u_0 bounded in $L^2(\mathbb{R})$.

For the absorbing ball in $H^1(\mathbb{R})$, we need to estimate K_1, which is given in (48). We first estimate the following term using Agmon's inequality (7):

$$\begin{aligned}\int_{\infty}^{\infty} |u(x)|^3\, dx &\leq \|u\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})} \leq \|u\|^{5/2}_{L^2(\mathbb{R})}\|u_x\|^{1/2}_{L^2(\mathbb{R})} \\ &\leq \frac{3}{2^{8/3}}\|u\|^{10/3}_{L^2(\mathbb{R})} + \|u_x\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{54}$$

Therefore,

$$\frac{2}{3}\|u_x\|^2_{L^2(\mathbb{R})} - \frac{1}{2^{8/3}}\|u\|^{10/3}_{L^2(\mathbb{R})} \leq I_1(u) \leq \frac{4}{3}\|u_x\|^2_{L^2(\mathbb{R})} + \frac{1}{2^{8/3}}\|u\|^{10/3}_{L^2(\mathbb{R})}, \tag{55}$$

which holds for all u in $H^1(\mathbb{R})$.

Now, we use (55) and Agmon's and Young's inequalities to obtain the following estimate for K_1:

$$|K_1(u)| \leq \gamma I_1(u) + \frac{3}{\gamma}\|f_x\|^2_{L^2(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2^{5/3}}\|u\|^{10/3}_{L^2(\mathbb{R})}.$$

Therefore, from the energy-type equation (51) for $j = 1$,

$$\frac{d}{dt}I_1(u) + \gamma I_1(u) \leq \frac{3}{\gamma}\|f_x\|^2_{L^2(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2^{5/3}}\|u\|^{10/3}_{L^2(\mathbb{R})}. \tag{56}$$

Applying the Gronwall Lemma and using (53) we find

$$\begin{aligned}\limsup_{t\to\infty} I_1(u(t)) &\leq \frac{3}{\gamma^2}\|f_x\|^2_{L^2(\mathbb{R})} + \frac{1}{\gamma^3}\|f\|_{L^\infty(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})} \\ &\quad + \frac{1}{2^{5/3}\gamma^{10/3}}\|f\|^{10/3}_{L^2(\mathbb{R})}.\end{aligned}$$

Finally, using (55) we obtain

$$\limsup_{t\to\infty} \|u_x(t)\|^2_{L^2(\mathbb{R})} \le \rho_0^2 \equiv \frac{9}{2\gamma^2}\|f_x\|^2_{L^2(\mathbb{R})} + \frac{3}{2\gamma^3}\|f\|_{L^\infty(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})} + \frac{9}{2^{11/3}\gamma^{10/3}}\|f\|^{10/3}_{L^2(\mathbb{R})}. \qquad (57)$$

Thus, we have proven the following result:

Proposition 1 *Let $\gamma > 0$ and $f \in H^1(\mathbb{R})$. Then, the solution operator associated with equation (1) possesses a bounded absorbing set in $H^1(\mathbb{R})$, with the radius of absorbing ball given according to (53) and (57).*

6. Asymptotic Compactness and the Global Attractor

As in the previous section, we are interested in the long time behavior of equation (1) taking the dissipation into account, so that we assume $\gamma > 0$. We also assume that the forcing term f belongs to $H^1(\mathbb{R})$. Let $\{S(t)\}_{t\in\mathbb{R}}$ be the solution operator associated with equation (1). From Proposition 1 we know that there exists a bounded set in $H^1(\mathbb{R})$ which is absorbing for $\{S(t)\}_{t\in\mathbb{R}}$. Therefore, for the existence of the global attractor it suffices to prove the following property known as the *asymptotic compactness property:* If $\{u_{0n}\}_n$ is a sequence bounded in $H^1(\mathbb{R})$ and $\{t_n\}_n$ is a sequence such that $t_n \to \infty$, then $\{S(t_n)u_{0n}\}_n$ is precompact in $H^1(\mathbb{R})$. See, for instance, O. Ladyzhenskaya [11], F. Abergel [1], R. Rosa [17], and R. Temam [18].

Our aim in this section is to show that the asymptotic compactness property follows from the energy-type equations (51). The idea of proving the existence of the global attractor by exploiting energy-type equations is due to J. Ball [2], and it was later used by several authors. The present case fits the abstract framework given in I. Moise, R. Rosa, and X. Wang [15]. The only delicate point is the weak continuity of the solution operator. We need enough regularity to pass to the weak limit in the equation and we need uniqueness to obtain the weak convergence to the right solution. These are two delicate issues for the KdV in lower order Sobolev spaces.

Lemma 8 *The solution operator $\{S(t)\}_{t\in\mathbb{R}}$ is weakly continuous in $H^1(\mathbb{R})$ in the sense that if u_{0n} converges weakly in $H^1(\mathbb{R})$ to some u_0, as $n\to\infty$, then $S(t)u_{0n}$ converges to $S(t)u_0$ weakly in $H^1(\mathbb{R})$ for all $t\in\mathbb{R}$.*

Proof: Let $u_{0n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R})$. We fix $T>0$ and consider $u_n(t) = S(t)u_{0n}$, for $-T\le t\le T$. Since u_{0n} is bounded in $H^1(\mathbb{R})$, it follows from Theorem 2 that

$$\{u_n\}_n \text{ is bounded in } X_T. \tag{58}$$

In particular,

$$\{u_n\}_n \text{ is bounded in } \mathcal{C}_b([-T,T];H^1(\mathbb{R})). \tag{59}$$

From the equation (1), we see then that

$$\{u_n'\}_n \text{ is bounded in } \mathcal{C}_b(-T,T;H^{-2}(\mathbb{R})), \tag{60}$$

where u_n' denotes the time-derivative of u_n.

From (58), we have

$$u_{n'} \overset{*}{\rightharpoonup} u \text{ weakly star in } X_T \tag{61}$$

for some element u in X_T, for some subsequence $\{n'\}$.

On the other hand, from (60) we find that for every v in $H^2(\mathbb{R})$ and every t and $t+\tau$ in $[-T,T]$,

$$\begin{aligned}(u_n(t+\tau)-u_n(\tau),v)_{L^2(\mathbb{R})} &= \int_t^{t+\tau}(u_n'(s),v)_{L^2(\mathbb{R})}ds\\ &\le \|u_n'\|_{L^\infty(-T,T;H^{-2}(\mathbb{R}))}\|v\|_{H^2(\mathbb{R})}\\ &\le c\tau\|v\|_{H^2(\mathbb{R})},\end{aligned} \tag{62}$$

where c is a constant independent of n. Taking

$$v = A^{-1}(u_n(t+\tau)-u_n(\tau)),$$

for fixed t and τ, where $A=(1-\partial_x^2)$, we obtain, using (3) and (6),

$$\begin{aligned}\|u_n(t+\tau)-u_n(t)\|_{H^{-1}(\mathbb{R})}^2 &\le c\tau\|u_n(t+\tau)-u_n(t)\|_{L^2(\mathbb{R})}\\ &\le 2c\tau\|u_n\|_{\mathcal{C}_b([-T,T];L^2(\mathbb{R}))}\le c\tau,\end{aligned} \tag{63}$$

for a possibly larger constant c.

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\varphi(s) = 1$, if $|s| \leq 1$, and $\varphi(s) = 0$, if $|s| \geq 2$. For each $r > 0$, let $\varphi_r(s) = \varphi(s/r)$. Then, $\varphi_r u_n$ belongs to $H_0^1(-2r, 2r)$ and, from (59), (63), the compact injections (5), and the continuous injections (4), we see that $\{\varphi_r u_n\}_n$ is equibounded and equicontinuous in $\mathcal{C}_b([-T,T]; H_0^{-1}(-2r, 2r))$ for every $r > 0$, with $\{\varphi_r u_n(t)\}_n$ precompact in $H_0^{-1}(-2r, 2r)$. Therefore, by the Arzela-Ascoli Theorem, for each $r > 0$, the sequence $\{\varphi_r u_n\}_n$ is precompact in the space $\mathcal{C}_b([-T,T]; H_0^{-1}(-2r, 2r))$. By a diagonalization process, and passing to a further subsequence still denoted $\{n'\}$, we find

$$u_{n'} \to u \text{ strongly in } \mathcal{C}_b([-T,T]; H_0^{-1}(-r, r)), \ \forall r > 0. \tag{64}$$

From the weak-star convergence (61) and the strong convergence (64) one can check that we may pass to the limit in either the weak or the mild formulation of the equation (1) to deduce that the limit function u is a solution of (1). For this passage to the limit, we do not need the weak-star convergence (61) in X_T. This weak-star convergence in X_T is only needed to assure that the limit u belongs to X_T, in which case u must be the unique solution provided by Theorem 2. Hence, $u(t) = S(t)u_0$. By contradiction, one can check that in fact the whole sequence u_n converges to u in the sense of (64) and (61).

It still remains to show that $u_n(t)$ converges weakly in $H^1(\mathbb{R})$ to $u(t)$ for every $t \in [-T,T]$. We know that the convergence is strong in $H_0^{-1}(-r,r)$, for each $r > 0$. Thus, taking $v \in \mathcal{C}_c^\infty(\mathbb{R})$, we find that for r large enough, Av belongs to $H_0^1(-r,r)$, so that

$$\begin{aligned} ((u_n(t), v))_{H^1(\mathbb{R})} &= ((u_n(t), Av))_{L^2(\mathbb{R})} \\ &\to ((u(t), Av))_{L^2(\mathbb{R})} \\ &= ((u(t), v))_{H^1(\mathbb{R})}. \end{aligned} \tag{65}$$

Then, from (59) and the density of $\mathcal{C}_c^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$ we find that

$$((u_n(t), v))_{H^1(\mathbb{R})} \to ((u(t), v))_{H^1(\mathbb{R})}, \tag{66}$$

for every v in $H^1(\mathbb{R})$, which proves the desired weak continuity in $H^1(\mathbb{R})$.

□

With the previous lemma in mind we can proceed as in I. Moise, R. Rosa, and X. Wang [15]. We let u_{0n} be bounded in $H^1(\mathbb{R})$ and let $t_n \to \infty$. From the existence of an absorbing ball in $H^1(\mathbb{R})$ (Proposition 1), if follows that $\{S(t_n)u_{0n}\}_n$ is weakly precompact in $H^1(\mathbb{R})$, with

$$S(t_{n'})u_{0n'} \rightharpoonup w, \text{ weakly in } H^1(\mathbb{R}), \tag{67}$$

for some subsequence $n' \to \infty$ and some $w \in H^1(\mathbb{R})$ with

$$\|w\|_{L^2(\mathbb{R})} \leq \rho_0, \qquad \|w_x\|_{L^2(\mathbb{R})} \leq \rho_1. \tag{68}$$

From Lemma 8, if follows that for every $t \in \mathbb{R}$,

$$S(t_{n'} - t)u_{0n'} \rightharpoonup S(-t)w, \text{ weakly in } H^1(\mathbb{R}), \tag{69}$$

as $n' \to \infty$, with

$$\|S(-t)w\|_{L^2(\mathbb{R})} \leq \rho_0, \qquad \|(S(-t)w)_x\|_{L^2(\mathbb{R})} \leq \rho_1. \tag{70}$$

Applying the variation of constant formula to the energy-type equation (51) with $j = 0$ from $t'_n - T$ to t'_n, for $T > 0$, to the solution $S(\cdot)u_{0n'}$ we find

$$\begin{aligned} \|S(t_{n'})u_{0n'}\|^2_{L^2(\mathbb{R})} &= \|S(t_{n'} - T)u_{0n'}\|^2_{L^2(\mathbb{R})}e^{-2\gamma T} \\ &\quad +2\int_0^T e^{-2\gamma(T-\tau)}((f, S(\tau)S(t_{n'} - T)u_{0n'}))_{L^2(\mathbb{R})}\, d\tau. \end{aligned} \tag{71}$$

Similarly, for the solution $S(\cdot)w$, we find

$$\begin{aligned} \|w\|^2_{L^2(\mathbb{R})} &= \|S(-T)w\|^2_{L^2(\mathbb{R})}e^{-2\gamma T} \\ &\quad +2\int_0^T e^{-2\gamma(T-\tau)}((f, S(\tau)S(-T)w))_{L^2(\mathbb{R})}\, d\tau. \end{aligned} \tag{72}$$

Subtracting (72) from (71), taking the limit as n' goes to infinity, and using (69) and (70) we obtain

$$\limsup_{n' \to \infty} \|S(t_{n'})u_{0n'}\|^2_{L^2(\mathbb{R})} \leq \|w\|^2_{L^2(\mathbb{R})} + 2\rho_0^2 e^{-2\gamma T}, \tag{73}$$

where T is arbitrary. Letting T go to infinity we find that

$$\limsup_{n'\to\infty} \|S(t_{n'})u_{0n'}\|^2_{L^2(\mathbb{R})} \leq \|w\|^2_{L^2(\mathbb{R})}, \tag{74}$$

which, together with the weak convergence (67), implies, since $L^2(\mathbb{R})$ is a Hilbert space, the strong convergence

$$S(t_{n'})u_{0n'} \to w, \text{ strongly in } L^2(\mathbb{R}), \tag{75}$$

as $n' \to \infty$.

Repeating this argument with $t_{n'}$ replaced by $t_{n'} - t$, for each fixed $t \in \mathbb{R}$, one obtains also

$$S(t_{n'} - t)u_{0n'} \to S(-t)w, \text{ strongly in } L^2(\mathbb{R}), \tag{76}$$

as $n' \to \infty$, for all $t \in \mathbb{R}$.

Now, we repeat the argument above for $I_1(\cdot)$. Applying the variation of constant formula to the energy-type equation (51) with $j = 1$ from $t'_n - T$ to t'_n, for $T > 0$, to the solution $S(\cdot)u_{0n'}$ we find

$$\begin{aligned} I_1(S(t_{n'})u_{0n'}) &= I_1(S(t_{n'} - T)u_{0n'})e^{-2\gamma T} \\ &\quad + \int_0^T e^{-2\gamma(T-\tau)} K_1(S(\tau)S(t_{n'} - T)u_{0n'})\, d\tau. \end{aligned} \tag{77}$$

Similarly for the solution $S(\cdot)w$:

$$I_1(w) = I_1(S(-T)w)e^{-2\gamma T} + \int_0^T e^{-2\gamma(T-\tau)} K_1(S(\tau)S(-T)w)\, d\tau. \tag{78}$$

Subtracting (78) from (77), taking the limit as n' goes to infinity, and using (69), (76), and (70), we obtain

$$\limsup_{n'\to\infty} I_1(S(t_{n'})u_{0n'}) \leq I_1(w) + ce^{-2\gamma T}, \tag{79}$$

where T is arbitrary and c is a constant independent of T. Notice that in order to pass to the limit in the term involving $K_1(\cdot)$ we used not only the weak convergence (69), but also the strong convergence (76). This strong convergence was the main reason for going through the first step with the $L^2(\mathbb{R})$ energy equation.

Letting T go to infinity in (79) we find that

$$\limsup_{n'\to\infty} I_1(S(t_{n'})u_{0n'}) \leq I_1(w). \tag{80}$$

Using the strong convergence (75) and the boundedness in $H^1(\mathbb{R})$ of the sequence $S(t_{n'})u_{0n'}$ (see (69)), we deduce from (80) that

$$\limsup_{n'\to\infty} \|(S(t_{n'})u_{0n'})_x\|^2_{L^2(\mathbb{R})} \leq \|w_x\|^2_{L^2(\mathbb{R})}. \tag{81}$$

This, together with the weak convergence (67) and the strong convergence (75), implies, since $H^1(\mathbb{R})$ is a Hilbert space, the strong convergence

$$S(t_{n'})u_{0n'} \to w, \text{ strongly in } H^1(\mathbb{R}), \tag{82}$$

as $n' \to \infty$. This proves the asymptotic compactness property of the solution operator and, hence, the existence of the global attractor. We summarize the result with the following theorem:

Theorem 3 *Let $\gamma > 0$ and $f \in H^1(\mathbb{R})$. Then, the solution operator $\{S(t)\}_{t\in\mathbb{R}}$ in $H^1(\mathbb{R})$ associated with the equation (1) possesses a connected global attractor in $H^1(\mathbb{R})$, i.e., a compact (in $H^1(\mathbb{R})$), connected, invariant set which attracts all the orbits (in the $H^1(\mathbb{R})$-metric) of the system, uniformly on bounded sets of initial conditions, and is maximal with respect to the inclusion relation among the bounded invariant sets and minimal (idem) among the globally attracting sets.*

The connectedness of the attractor follows from the continuity of the individual solutions (See Theorem 2).

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