

## RATES OF DECAY OF A NONLINEAR MODEL IN THERMOELASTICITY

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### Abstract

We consider a coupled system of equations arising in nonlinear thermoelasticity. They consist of a nonlinear hyperbolic equation with a heat type equation considered in the whole line. We prove the well-posedness of the above problem and analyse the behavior of the total energy as  $t \rightarrow +\infty$ . The main ingredient in the proof is the Fourier splitting method introduced by M.E. Schonbek [8] while analyzing similar problems for the Navier-Stokes equations.

### Resumo

Consideramos um sistema acoplado de equações que aparecem em termoelasticidade não linear. O modelo é descrito por uma equação hiperbólica não linear e uma equação parabólica (tipo calor). Ambas equações são consideradas na reta real e os tempos positivos. Mostramos que o problema de Cauchy é bem posto e analisamos o comportamento da energia total quando  $t \rightarrow +\infty$ . Para a demonstração do resultado central usamos a técnica chamada de “Fourier splitting” introduzida por M.E. Schonbek [8] quando ela fez uma análise similar para problemas tipo Navier-Stokes no  $\mathbb{R}^n$ .

## 1. Introduction

The nonlinear evolution equation

$$u_{tt} + u_{xxxx} - M \left( \int_{\Omega} u_x^2 dx \right) u_{xx} = 0 \quad (1)$$

where  $\Omega \subseteq \mathbb{R}$ ,  $t \geq 0$ , is usually known as Timoshenko’s model and has been recently studied by several authors. It is associated with nonlinear vibration of

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beams. Related models have also been intensively studied in recent years and much of the efforts have been focused in showing local smothing effects of the solutions (see [1], [5], [6] and the references therein). If one adds a term due to rotational inertia, equation (1) is written as

$$u_{tt} - u_{xxtt} + u_{xxxx} - M \left( \int_{\Omega} u_x^2 dx \right) u_{xx} = 0 \quad (2)$$

Let  $\Omega = \mathbb{R}$ . We can easily show that

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left( u_t^2 + u_{xt}^2 + u_{xx}^2 \right) dx + \frac{1}{2} \hat{M} \left( \int_{\mathbb{R}} u_x^2 dx \right) = E(0) ,$$

where  $\hat{M}(t) = \int_0^t M(s) ds$ , for all  $t \geq 0$ , that is, the total energy associated with (2) is conserved.

A more realistic model, from a physical point of view, it is the case with some dissipative effect, say a heat flux is acting on the beam. A natural question then is to ask if we can obtain a uniform rate of decay as  $t \rightarrow +\infty$ . In other words, if the thermal effect is strong enough to stabilize the solutions.

In this paper we will consider the solution-pair  $\{u, \theta\}$  of the following initial value problem

$$\begin{cases} u_{tt} - u_{xxtt} + u_{xxxx} - M \left( \int_{\mathbb{R}} u_x^2 dx \right) u_{xx} + \alpha \theta_{xx} = 0 \\ \theta_t - \theta_{xx} - \alpha u_{xt} = 0 \end{cases} \quad (3)$$

for  $-\infty < x < +\infty$ ,  $t > 0$ , and initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x) \quad (4)$$

The parameter  $\alpha$  is a positive real number and the function  $M(\cdot)$  satisfies

$$M(\cdot) \in C^1(\mathbb{R}^+) \text{ and } M(s) \geq 0 \text{ for any } s \geq 0 \quad (5)$$

The total energy associated to (3)-(4) is given by

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} \left( u_t^2 + u_{xt}^2 + u_{xx}^2 + \theta^2 \right) dx + \hat{M} \left( \int_{\mathbb{R}} u_x^2 dx \right) ,$$

with  $\hat{M}(t) = \int_0^t M(s) ds$ . It is easy to show that

$$\frac{dE(t)}{dt} = - \int_{\mathbb{R}} \theta_x^2 dx$$

As a consequence, the decay of solutions is expected. The problem is to find a uniform rate of decay as  $t \rightarrow +\infty$ . In order to do that we also assume that  $M(\cdot)$  satisfies an additional monotonicity condition. To the best of our knowledge, the rates of decay for such solutions has been studied only in bounded domains. The main purpose of this paper is to investigate the asymptotic behavior of  $E(t)$  as  $t \rightarrow +\infty$ .

Our main results (Theorems 1 and 2) will only be given in the one-dimensional case, but as will become clear during our proofs, most of the ingredients are valid in any dimension.

This paper is organized as follows: In section 2, we prove existence and uniqueness of global (weak) solutions. To obtain this result we consider a weak formulation of (3)-(4) in suitable function spaces and then use semigroup theory. In section 3 we obtain some estimates of the solutions via the Fourier transform together with a choice of a convenient Lyapunov function which has some resemblance to a technique due to M.E. Schonbek [8] who studied similar problems for the Navier-Stokes equations.

The notation we use is standard. The symbol  $(\cdot, \cdot)$  means the inner product in  $L^2(\mathbb{R})$  and the notation  $\langle f, \varphi \rangle$  means the value of  $f \in Z'$  at  $\varphi \in Z$ . By  $H^m$ , we denote the Sobolev space (class of) of functions in  $L^2(\mathbb{R})$  which together with their partial derivatives (in the sence of distributions) up to order  $m$  belong to  $L^2(\mathbb{R})$ . Subscripts denote partial differentiation. The Fourier transform of a function  $f$  is denoted by  $\hat{f}$ .

## 2. Existence and Uniqueness

We assume that

$$M(\cdot) \in C^1(\mathbb{R}^+) \quad \text{and} \quad M(s) \geq 0, \quad \forall s \in \mathbb{R}^+;$$

Consider the Hilbert space  $X = H^2 \times H^1 \times L^2$  with the inner product given by

$$\left( (u, v, \theta), (\tilde{u}, \tilde{v}, \tilde{\theta}) \right)_X = (u, \tilde{u}) + (u_{xx}, \tilde{u}_{xx}) + (v, \tilde{v}) + (v_x, \tilde{v}_x) + (\theta, \tilde{\theta})$$

whenever  $(u, v, \theta)$  and  $(\tilde{u}, \tilde{v}, \tilde{\theta})$  belong to  $X$  where  $(\cdot, \cdot)$  denotes the inner product in  $L^2$ . The norm in  $L^2$  is denoted by  $\| \cdot \|$ .

With the above considerations, the first step to use semigroup theory is to rewrite (3) as a first order system, letting  $u_t = v$ :

$$\begin{aligned} u_t - v &= 0 \\ v_t - v_{xxt} + u_{xxxx} - M \left( \int_{\mathbb{R}} u_x^2 dx \right) u_{xx} + \alpha \theta_{xx} &= 0 \\ \theta_t - \theta_{xx} - \alpha v_{xx} &= 0 \end{aligned} \quad (6)$$

Now we consider a weak formulation of (6) taking in the first equation the inner product in  $H^2$  with a function  $\varphi(\cdot, t) \in H^2$ . In the second and third equations we take the inner product in  $L^2$  with functions  $\psi(\cdot, t) \in H^1$  and  $\phi(\cdot, t) \in L^2$ , respectively.

Integrating the second term in the second equation of (6) by parts, we arrive to the variational system:

$$\begin{aligned} (u_t, \varphi)_{H^2} - (v, \varphi)_{H^2} &= 0 \\ (v_t, \psi)_{H^1} + (u_{xxxx}, \psi) + \alpha(\theta_{xx}, \psi) &= M \left( \int_{\mathbb{R}} u_x^2 dx \right) (u_{xx}, \psi) \\ (\theta_t, \phi) + (\theta_{xx}, \phi) - \alpha(v_{xx}, \phi) &= 0 \end{aligned} \quad (7)$$

that can be seen as an abstract equation in  $X$ : Let us introduce the following operators

$$\begin{aligned} A : H^2 &\longrightarrow H^{-2} \quad , \quad u \longrightarrow \langle Au, w \rangle = (u, w)_{H^2} \quad , \quad \forall w \in H^2 \\ C : H^1 &\longrightarrow H^{-1} \quad , \quad u \longrightarrow \langle Cu, w \rangle = (u, w)_{H^1} \quad , \quad \forall w \in H^1 \\ N : H^2 &\longrightarrow L^2 \quad , \quad u \longrightarrow \langle N(u), w \rangle = -M \left( \int_{\mathbb{R}} u_x^2 dx \right) (u_{xx}, w) \quad , \quad \forall w \in L^2 \end{aligned}$$

$A$  and  $C$  are isometric isomorphisms (Riesz isomorphism) and  $N$  is a locally Lipschitz function, as we show in lemma 1.

**Lemma 1.** *Let  $M(\cdot)$ , satisfy the assumptions given at beginning of this section. Then  $N$ , is a locally Lipschitz function.*



**Proof:** Let  $u, v \in H^2$ , such that,  $\|u(\cdot)\|_{H^2} \leq c$  and  $\|v(\cdot)\|_{H^2} \leq c$ . Since  $M(\cdot) \in C^1(\mathbb{R}^+)$ , the mean value theorem and triangle inequality give us that

$$\begin{aligned} \|N(u) - N(v)\| &\leq M\left(\int_{\mathbb{R}} u_x^2 dx\right) \|u_{xx} - v_{xx}\| + \left|M\left(\int_{\mathbb{R}} u_x^2 dx\right) - M\left(\int_{\mathbb{R}} v_x^2 dx\right)\right| \|v_{xx}\| \\ &\leq M\left(\int_{\mathbb{R}} u_x^2 dx\right) \|u - v\|_{H^2} + \max_{0 \leq s \leq c} |M'(s)| \left|\int_{\mathbb{R}} (u_x^2 - v_x^2) dx\right| \|v\|_{H^2} \end{aligned} \quad (8)$$

Using Hölder's inequality we get

$$\left|\int_{\mathbb{R}} (u_x^2 - v_x^2) dx\right| \leq \|u - v\|_{H^1} (\|u\|_{H^1} + \|v\|_{H^1})$$

which implies that the right-hand side of (8) is less than or equal to

$$M\left(\int_{\mathbb{R}} u_x^2 dx\right) \|u - v\|_{H^2} + \max_{0 \leq s \leq c} |M'(s)| \|v\|_{H^2} (\|u\|_{H^1} + \|v\|_{H^1}) \|u - v\|_{H^2}$$

Hence, the function  $N$  is locally Lipschitz from  $H^2$  into  $L^2$ . This completes the proof of Lemma 1.

We introduce the operators

$$D = \begin{bmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 0 & A & 0 \\ -\frac{d^4}{dx^4} - I & 0 & -\alpha \frac{d^2}{dx^2} \\ 0 & \alpha \frac{d^2}{dx^2} & \frac{d^2}{dx^2} \end{bmatrix}$$

$$\text{and } \tilde{N}(U) = \begin{bmatrix} 0 \\ N(u) + u \\ 0 \end{bmatrix} \text{ where } U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \text{ and}$$

$$D : H^2 \times H^1 \times L^2 \longrightarrow H^2 \times H^{-1} \times L^2 \quad \tilde{J} : H^3 \times H^2 \times H^2 \longrightarrow H^{-2} \times H^{-1} \times L^2$$

$$\tilde{N} : H^2 \times L^2 \times L^2 \longrightarrow L^2 \times L^2 \times L^2$$

With these considerations, we can rewrite (7) as abstract evolution equation in  $X$ :

$$\frac{d}{dt} DU - \tilde{J}U = \tilde{N}(U) \quad (9)$$

and since  $D$  is an isometric isomorphism (Riesz isomorphism), we can invert  $D$  in (9) obtaining

$$\frac{dU}{dt} - (D^{-1}\tilde{J})U = D^{-1}\tilde{N}(U) \quad (10)$$

Observe that the choice of the domain of  $\tilde{J}$  was such that  $D^{-1}\tilde{J}$  in (10) to make sense.

**Lemma 2.** *Consider the operators  $D$  and  $\tilde{J}$  defined above. Then,  $D^{-1}\tilde{J}$  and  $(D^{-1}\tilde{J})^*$  are dissipative operators.*

**Proof:**

Let  $U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}$  and  $V = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix}$  be in  $H^3 \times H^2 \times H^2$ . It follows from the definition of  $\tilde{J}$  and  $D$ , that

$$\begin{aligned} ((D^{-1}\tilde{J})U, V) &= \left( \begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} Av \\ -u_{xxxx} - u - \alpha\theta_{xx} \\ \alpha v_{xx} + \theta_{xx} \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} \right)_{H^2 \times H^1 \times L^2} \\ &= \left( \begin{pmatrix} v \\ C^{-1}(-u_{xxxx} - u - \alpha\theta_{xx}) \\ \alpha v_{xx} + \theta_{xx} \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix} \right)_{H^2 \times H^1 \times L^2} \\ &= (v, \tilde{u})_{H^2} - (C^{-1}(u_{xxxx}), \tilde{v})_{H^1} - (C^{-1}u, \tilde{v})_{H^1} - \alpha(C^{-1}\theta_{xx}, \tilde{v})_{H^1} + \\ &+ \alpha(v_{xx}, \tilde{\theta})_{L^2} + (\theta_{xx}, \tilde{\theta})_{L^2} \\ &= (v, \tilde{u})_{H^2} - \langle u_{xxxx}, \tilde{v} \rangle_{H^{-1} \times H^1} - \langle u, \tilde{v} \rangle_{H^{-1} \times H^{-1}} - \alpha \langle \theta_{xx}, \tilde{v} \rangle_{H^{-1} \times H^{-1}} + \alpha(v_{xx}, \tilde{\theta})_{L^2} \\ &+ (\theta_{xx}, \tilde{\theta})_{L^2}, \end{aligned}$$

because  $C : H^1 \rightarrow H^{-1}$  and  $\langle Cu, v \rangle_{H^{-1} \times H^1} = (u, v)_{H^1}$ ,  $\forall u \in H^1$ .

Now, we rewrite  $\langle u, \tilde{v} \rangle_{H^{-1} \times H^1}$  as  $(u, \tilde{v})_{H^1}$ ,  $\langle \theta_{xx}, \tilde{v} \rangle_{H^{-1} \times H^1} = (\theta_{xx}, \tilde{v})_{L^2}$ , to obtain

$$\begin{aligned} ((D^{-1}\tilde{J})U, V) &= (v, \tilde{u})_{H^2} + (u_{xx}, \tilde{v}_x)_{L^2} - (u, \tilde{v})_{H^1} - \alpha(\theta_{xx}, \tilde{v})_{L^2} \\ &+ \alpha(v_{xx}, \tilde{\theta})_{L^2} + (\theta_{xx}, \tilde{\theta})_{L^2} \end{aligned}$$

After integration by parts, we deduce that

$$(v, \tilde{u})_{H^2} + (u_{xxx}, \tilde{v}_x)_{L^2} - (u, \tilde{v})_{H^1} = -[(\tilde{v}, u)_{H^2} - (\tilde{u}, v)_{H^1} - \langle \tilde{u}_{xxxx}, v \rangle_{H^{-1} \times H^1}]$$

and

$$-\alpha(\theta_{xx}, \tilde{v}) + \alpha(v_{xx}, \tilde{\theta}) = -\alpha(\tilde{v}_{xx}, \theta) + \alpha(\tilde{\theta}_{xx}, v)$$

Consequently,

$$\begin{aligned} ((D^{-1}\tilde{J})U, V) &= -[(\tilde{v}, u)_{H^2} - \langle \tilde{u}_{xxxx}, v \rangle_{H^{-1} \times H^1} - (\tilde{u}, v)_{H^1} - \alpha(\tilde{\theta}_{xx}, v) \\ &\quad + \alpha(v_{xx}, \theta)] + (\tilde{\theta}_{xx}, \theta)_{L^2} \\ &= -\left( \begin{pmatrix} \tilde{v} \\ C^{-1}(-\tilde{u}_{xxxx} - \tilde{u} - \alpha\tilde{\theta}_{xx}) \\ \alpha\tilde{v}_{xx} + \tilde{\theta}_{xx} \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right)_{H^2 \times H^1 \times L^2} \\ &= ((D^{-1}L)^*V, U) \end{aligned}$$

The above calculations show that

$$((D^{-1}\tilde{J})U, U) = ((D^{-1}\tilde{J})^*U, U) = -\|\theta_x\|^2 \leq 0$$

which completes the proof of Lemma 2.

**Lemma 3.** *Let  $D$ ,  $\tilde{J}$  and  $\tilde{N}$  as before, where  $M$  and  $\alpha$  satisfy the assumptions given at the beginning of this section. Then, if  $U_0 \in H^3 \times H^2 \times H^2$ , there exist a function  $U(t)$  and  $T_0 > 0$ , such that*

$$U \in L^\infty(0, T_0; H^3 \times H^2 \times H^2) \quad U_t \in L^\infty(0, T_0, X)$$

which solve the problem

$$\frac{dU}{dt} - (D^{-1}\tilde{J})U = \tilde{N}(U) \tag{11}$$

$$U(0) = U_0$$

**Proof:** Lemma 2 implies that  $D^{-1}\tilde{J}$  is the infinitesimal generator of a semigroup of operators  $\{T(t)\}_{t \geq 0}$ . Therefore to prove local existence it is sufficient to

show that the right hand of (11) is a locally Lipschitz function but this is a consequence of Lemma 1 and the fact that  $D^{-1}$  is an isometry.

**Theorem 1.** (Existence and Uniqueness). *Assume that  $M(\cdot)$  and  $\alpha$  satisfy*

a)  $M(\cdot) \in C^1(\mathbb{R})$  and  $M(s) \geq 0, \forall s \in \mathbb{R}^+$ ;

b)  $\alpha$  is a positive constant.

*If  $(u_0, u_1, \theta_0) \in H^3 \times H^2 \times H^1$ , then there exists only one pair  $\{u, \theta\}$  satisfying*

$$\begin{aligned} u &\in L_{loc}^\infty(0, +\infty; H^3), \quad u_t \in L_{loc}^\infty(0, +\infty, H^2), \quad u_{tt} \in L_{loc}^\infty(0, +\infty; H^1) \\ \theta &\in L_{loc}^\infty(0, +\infty; H^2), \quad \theta_t \in L_{loc}^\infty(0, +\infty; L^2) \end{aligned}$$

$$\begin{cases} (u_{tt}, \varphi) + (u_{xtt}, \varphi_x) + (u_{xxx}, \varphi_x) - M\left(\int_{\mathbb{R}} u_x^2 dx\right)(u_{xx}, \varphi) + \alpha(\theta_{xx}, \varphi) = 0 \\ \theta_t - \theta_{xx} - \alpha u_{xxt} = 0 \end{cases} \quad (12)$$

for every  $\varphi(\cdot, t) \in H^1(\mathbb{R})$ , and

$$u(x, 0) = u_0(x) \quad , \quad u_t(x, 0) = u_1(x) \quad , \quad \theta(x, 0) = \theta_0(x)$$

**Proof:** First, observe that if we return to the original model (that is (7)), lemma 3 and integration by parts guarantee that (12) is satisfied for all  $0 \leq t < T_0$ .

By Zorn's lemma we can assume that  $T_0 = T_{\max}$  that is we have local existence in the maximal interval of existence. We want to show that  $T_{\max} = +\infty$ . Suppose (by contradiction) that  $T_{\max} < +\infty$ . Let  $0 < T < T_{\max}$  with  $T$  as close to  $T_{\max}$  as we want. To show global existence it is enough to prove that the solution  $\{u, \theta\}$  of (12) remains bounded in  $0 \leq t \leq T$  by a positive constant  $C$  (which may depend of  $T$ ) in the norm of the space which is the one where  $\{u, u_t, \theta\}$  lies, i.e.  $H^3 \times H^2 \times H^2$ .

We need to find some apriori estimates. The energy method give us that

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_{\mathbb{R}} [u_t^2 + u_{xt}^2 + u_{xx}^2 + \theta^2] dx + \hat{M} \left( \int_{\mathbb{R}} u_x^2 dx \right) \right\} = - \int_{\mathbb{R}} \theta_x^2 dx$$

where  $\hat{M}(t) = \int_0^t M(s)ds$ . Thus

$$\int_{\mathbb{R}} [u_t^2 + u_{xt}^2 + u_{xx}^2 + \theta^2] dx + \hat{M} \left( \int_{\mathbb{R}} u_x^2 dx \right) \leq 2C \quad (13)$$

where  $C \geq 0$  depends only on the initial data.

Hölder's inequality and the above estimate implies that

$$\frac{d}{dt} \int_{\mathbb{R}} u^2 dx \leq 2(2C)^{1/2} \|u\|.$$

Gronwall's inequality gives us that  $\|u\|^2$  is bounded by a constant  $C_1 = C_1(T)$ , which together with (13) that

$$\|u\|_{H^2}^2 + \|u_t\|_{H^1}^2 + \|\theta\|^2 \leq C_1(T) \quad (14)$$

Now, differentiating both equations in (12) with respect to  $x$  and letting  $\varphi = u_{xt}$ , multiplying the second equation by  $\theta_x$  and integrating in  $x \in \mathbb{R}$  we obtain after adding both identities that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [u_{xt}^2 + u_{xxt}^2 + u_{xxx}^2 + \theta_x^2] dx + \int_{\mathbb{R}} \theta_{xx}^2 dx = M \left( \int_{\mathbb{R}} u_x^2 dx \right) \int_{\mathbb{R}} u_{xxx} u_{xt} dx$$

Due to (14) we know that  $\int_{\mathbb{R}} u_x^2 dx \leq C_1(T)$  therefore  $M \left( \int_{\mathbb{R}} u_x^2 dx \right)$  is bounded in  $0 \leq t \leq T$ . Hölder and Gronwall's inequalities together with the estimate (14) give us that

$$\|u\|_{H^3}^2 + \|u_t\|_{H^2}^2 \leq C_2(T) \quad (15)$$

for some positive constant  $C_2(T)$ .

Now, differentiating both equations in (12) with respect to  $t$  and letting  $\varphi = u_{tt}$ , multiplying the second equation by  $\theta_t$  and integrating in  $x \in \mathbb{R}$  we obtain after adding both identities that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [u_{tt}^2 + u_{xtt}^2 + u_{xxt}^2 + \theta_t^2] dx + \int_{\mathbb{R}} \theta_{xt}^2 dx = \int_{\mathbb{R}} u_{xxt} u_{tt} dx M \left( \int_{\mathbb{R}} u_x^2 dx \right) + \\ & + 2M' \left( \int_{\mathbb{R}} u_x^2 dx \right) \int_{\mathbb{R}} u_x u_{xt} dx \int_{\mathbb{R}} u_{xx} u_{tt} dx \end{aligned}$$

Arguing as in (15) and using the above estimates we obtain

$$\|u_{tt}\|_{H^1}^2 + \|\theta_t\|^2 \leq C_3(T) \quad (16)$$

for some positive constant  $C_3(T)$ . Finally we want to estimate  $\|\theta\|_{H^2}$ . Using the second equation in (12) we know that

$$\theta_{xx} = \theta_t - \alpha u_{xxt}$$

Estimates (15) and (16) give us an estimate for  $\|\theta_{xx}\|_{L^2}$  which together with our previous estimates give us that  $\|\theta\|_{H^2} \leq C_4(T)$  for some positive constant  $C_4(T)$ . Thus,  $\{u, u_t, \theta\}$  is bounded in  $H^3 \times H^2 \times H^2$  for any  $0 \leq t \leq T < T_{\max}$ . This completes the proof of global existence.

Remains to show uniqueness. To do that we suppose that (2.7) has two solutions, say  $\{u, u_t, \theta\}$  and  $\{\tilde{u}, \tilde{u}_t, \tilde{\theta}\}$  with the same initial data at time  $t = 0$ . The difference  $w = u - \tilde{u}$  and  $z = \theta - \tilde{\theta}$  will satisfy

$$\begin{aligned} (w_{tt}, \varphi)_{H^1} &+ (w_{xx}, \varphi_{xx}) + \alpha(z_{xx}, \varphi) + (w, \varphi) = \\ &= \left( M \left( \int_{\mathbb{R}} u_x^2 dx \right) u_{xx} - M \left( \int_{\mathbb{R}} \tilde{u}_x^2 dx \right) \tilde{u}_{xx}, \varphi \right) + (w, \varphi) \\ (z_t, \psi) &+ (z_x, \psi_x) - \alpha(w_{xxt}, \psi) = 0 \end{aligned}$$

for any  $\varphi(\cdot, t) \in H^2$  and  $\psi(\cdot, t) \in V$ . Notice that just for convenience we added to both sides of the first identity the term  $(w, \varphi)$ .

Letting  $\varphi = u_t - \tilde{u}_t$ ,  $\psi = \theta - \tilde{\theta}$  and adding the identities we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \|u_t - \tilde{u}_t\|_{H^1}^2 + \|u_{xx} - \tilde{u}_{xx}\|^2 + \|\theta - \tilde{\theta}\|^2 + \|u - \tilde{u}\|^2 \right\} + \|\theta_x - \tilde{\theta}_x\|^2 = \\ &= \left( M \left( \int_{\mathbb{R}} u_x^2 dx \right) u_{xx} - M \left( \int_{\mathbb{R}} \tilde{u}_x^2 dx \right) \tilde{u}_{xx}, u_t - \tilde{u}_t \right) + (u - \tilde{u}, u_t - \tilde{u}_t) \equiv A \end{aligned} \quad (17)$$

We want to obtain a bound for  $A$ . Adding and subtracting  $M \left( \int_{\mathbb{R}} u_x^2 dx \right) \tilde{u}_{xx}$  and using Hölder's inequality we obtain that

$$\begin{aligned} A \leq &\left\{ M \left( \int_{\mathbb{R}} u_x^2 dx \right) \|u_{xx} - \tilde{u}_{xx}\| + \left| M \left( \int_{\mathbb{R}} u_x^2 dx \right) - M \left( \int_{\mathbb{R}} \tilde{u}_x^2 dx \right) \right| \|\tilde{u}_{xx}\| \right\} \times \\ &\times \|u_t - \tilde{u}_t\| + \|u - \tilde{u}\| \|u_t - \tilde{u}_t\| \end{aligned} \quad (18)$$

Recall that for any  $T > 0$  we know (see (14)) that  $\|u_x\| \leq C_1(T)$  and  $\|\tilde{u}_x\| \leq C_1(T)$  for  $0 \leq t \leq T$ . Therefore  $M \left( \int_{\mathbb{R}} u_x^2 dx \right) \leq C_2(T)$  for some positive constant  $C_2(T)$ .

Using the mean value theorem and Hölder's inequality we also obtain that

$$\begin{aligned}
 \left| M \left( \int_{\mathbb{R}} u_x^2 dx \right) - M \left( \int_{\mathbb{R}} \tilde{u}_x^2 dx \right) \right| &\leq \left[ \max_{0 \leq s \leq C_2(T)} |M'(s)| \right] \left| \int_{\mathbb{R}} (u_x^2 - \tilde{u}_x^2) dx \right| \leq \\
 &\leq \max_{0 \leq s \leq C_2(T)} |M'(s)| \|u_x - \tilde{u}_x\| (\|u_x\| + \|\tilde{u}_x\|) \\
 &\leq C_3(T) \|u_x - \tilde{u}_x\|
 \end{aligned} \tag{19}$$

Using again (14) and Gagliardo-Nirenberg's interpolation theorem we obtain that the right hand side of (19) can be bounded by  $C\|u - \tilde{u}\| \|u_{xx} - \tilde{u}_{xx}\| \leq CC_1(T)\|u_{xx} - \tilde{u}_{xx}\|$ . Consequently, returning to (18) we deduce that

$$A \leq C_4(T) \|u_{xx} - \tilde{u}_{xx}\| \|u_t - \tilde{u}_t\| + \|u - \tilde{u}\| \|u_t - \tilde{u}_t\|$$

for some positive constant  $C_4(T)$ . Going back to (17) we conclude that

$$\begin{aligned}
 &\frac{d}{dt} \left\{ \|u_t - \tilde{u}_t\|_{H^1}^2 + \|u_{xx} - \tilde{u}_{xx}\|^2 + \|\theta - \tilde{\theta}\|^2 + \|u - \tilde{u}\|^2 \right\} \leq \\
 &\leq C_5(T) \left\{ \|u_t - \tilde{u}_t\|_{H^1}^2 + \|u_{xx} - \tilde{u}_{xx}\|^2 + \|\theta - \tilde{\theta}\|^2 + \|u - \tilde{u}\|^2 \right\}
 \end{aligned}$$

for some positive constant  $C_5(T)$ . Gronwall's inequality implies that  $u \equiv \tilde{u}$  and  $\theta \equiv \tilde{\theta}$  for all  $0 \leq t \leq T$  because they have the same initial data at  $t = 0$ .

### 3. Asymptotic behavior as $t \rightarrow +\infty$

Let us consider system (3), (4) in  $\mathbb{R} \times \mathbb{R}^+$ . Taking the Fourier transformation in  $x$  we obtain

$$\begin{cases} (1 + y^2)v_{tt} + y^2(y^2 + a(t))v - \alpha y^2\varphi = 0 \\ \varphi_t + y^2\varphi + \alpha y^2v_t = 0 \end{cases} \tag{20}$$

where  $a(t) = M(\int_{\mathbb{R}} u_x^2 dx) = M(\int_{\mathbb{R}} y^2|v|^2 dy)$  due to Parseval's identity. Here  $v = \hat{u}$  and  $\varphi = \hat{\theta}$  denote the Fourier transforms of  $u$  and  $\theta$  respectively. Besides the assumptions on  $M(\cdot)$  given in theorem 1 we will also assume that  $M(s)$  is an increasing function and  $M(0) \neq 0$ .

**Lemma 4.** *Let  $D = \frac{1}{2} \{(1 + y^2)|v_t|^2 + y^2(y^2 + a(t))|v|^2 + |\varphi|^2\}$ . Then, we can find  $r > 0$  such that*

$$D = D(y, t) \leq CD(y, 0)\exp(-wt)$$

holds whenever  $|y| \geq r$  where  $C$  and  $w = w(r)$  are positive constants.

**Proof:** We consider the Lyapunov function  $\mathcal{L} = \mathcal{L}(y, t)$  given by

$$\mathcal{L}(y, t) = Ny^2 D(y, t) + J(y, t)$$

where  $N$  is a suitable positive constant to be chosen later and

$$J(y, t) = \operatorname{Re}\{(1 + y^2)\varphi\bar{v}_t + \frac{\alpha}{4} y^2(1 + y^2)v_t\bar{v}\}$$

Clearly,  $|J(y, t)| \leq \left(\frac{\alpha}{4} + 1\right) (1 + y^2)D(y, t)$ . Consequently,

$$\left[\left(N - \left(\frac{\alpha}{4} + 1\right)\right)y^2 - \left(\frac{\alpha}{4} + 1\right)\right] D \leq \mathcal{L} \leq \left[\left(N + \left(\frac{\alpha}{4} + 1\right)\right)y^2 + \frac{\alpha}{4} + 1\right] D \quad (21)$$

We claim that the following estimate

$$\frac{d\mathcal{L}}{dt} \leq -w(r)\mathcal{L} \quad (22)$$

holds whenever  $|y| > r$  for some  $r > 0$  where  $w(r) > 0$ . Clearly, from (21) and (22) we can conclude the proof of Lemma 4 choosing  $N$  large enough. In order to prove the claim we proceed as follows: Let us multiply equation (20)<sub>1</sub> by  $\bar{v}_t$  and (20)<sub>2</sub> by  $\bar{\varphi}$ . Adding the corresponding identities we obtain

$$\frac{d}{dt}D(y, t) = -y^2|\varphi|^2 + y^2|v|^2 \frac{da}{dt} \quad (23)$$

Next, we multiply equation (20)<sub>2</sub> by  $(1 + y^2)\bar{v}_t$  and replace the term  $(1 + y^2)v_{tt}$  given in (20)<sub>1</sub> to obtain

$$\begin{aligned} (1 + y^2)\frac{d}{dt}\{\varphi\bar{v}_t\} &= -y^2(1 + y^2)\operatorname{Re}\{\varphi\bar{v}_t\} - \alpha y^2(1 + y^2)|v_t|^2 - y^4\operatorname{Re}\{\varphi\bar{v}\} \\ &+ \alpha y^2|\varphi|^2 - a(t)y^2\operatorname{Re}\{\varphi\bar{v}\} \end{aligned} \quad (24)$$

Let us multiply equation (20)<sub>1</sub> by  $y^2\bar{v}$  to obtain

$$\begin{aligned} \frac{d}{dt}\operatorname{Re} y^2(1 + y^2)\bar{v}v_t &= y^2(1 + y^2)|v_t|^2 - y^2|v|^2 + \\ &+ \alpha y^4\operatorname{Re}\{\varphi\bar{v}\} - a(t)y^4|v|^2 \end{aligned} \quad (25)$$



We use identities (24) and (25) together with Young's inequality with  $p = q = 2$  to obtain

$$\begin{aligned} \frac{dJ}{dt} &\leq -\frac{\alpha}{4}y^2(1+y^2)|v_t|^2 + \left(\alpha + \frac{1}{2\alpha}\right)y^2(1+y^2)|\varphi|^2 + \\ &+ \left(\frac{\alpha^2}{4} - 1\right)y^4\operatorname{Re}\{\varphi\bar{v}\} - \frac{\alpha}{4}y^6|v|^2 - \frac{\alpha}{4}a(t)y^4|v|^2 - a(t)y^2\operatorname{Re}\{\varphi\bar{v}\} \end{aligned} \quad (26)$$

Again, Young's inequality give us that for any  $\delta > 0$  we have

$$\left(\frac{\alpha^2}{4} - 1\right)y^2\operatorname{Re}\{\varphi\bar{v}\} \leq \left(\frac{\alpha^2}{4} - 1\right)^2 \frac{1}{2\delta}y^2|\varphi|^2 + \frac{\delta}{2}y^6|v|^2 \quad (27)$$

and

$$-a(t)y^2\operatorname{Re}\{\varphi\bar{v}\} \leq \frac{a(t)}{2\delta}y^2|\varphi|^2 + \frac{\delta}{2}a(t)y^2|v|^2$$

From (26) and (27) we deduce that

$$\begin{aligned} \frac{dJ}{dt} &\leq -\frac{\alpha}{4}y^2(1+y^2)|v_t|^2 - \left(\frac{\alpha}{4} - \frac{\delta}{2}\right)y^6|v|^2 + \\ &+ c_1y^2(1+y^2)|\varphi|^2 - \left(\frac{\alpha y^2}{4} - \frac{\delta}{2}\right)a(t)y^2|v|^2 + \frac{a(t)}{2\delta}y^2|\varphi|^2 \end{aligned} \quad (28)$$

where  $c_1 = \frac{1}{2\delta} \left(\frac{\alpha}{4} - 1\right)^2 + \alpha + \frac{1}{2\alpha} > 0$ . We choose  $0 < \delta < \frac{\alpha}{2}$ . Thus, for any  $|y| \geq \sqrt{\frac{4\delta}{\alpha}}$  we have that  $\frac{\alpha y^2}{4} - \frac{\delta}{2} \geq \frac{\alpha y^2}{8}$  and from (28) we obtain

$$\begin{aligned} \frac{dJ}{dt} &\leq -\frac{\alpha}{4}y^2(1+y^2)|v_t|^2 - \left(\frac{\alpha}{4} - \frac{\delta}{2}\right)y^6|v|^2 \\ &- \frac{\alpha}{8}y^4a(t)|v|^2 + c_1y^2(1+y^2)|\varphi|^2 + c_2y^2|\varphi|^2 \end{aligned} \quad (29)$$

where  $c_2$  is a positive constant (depending only on the initial energy) such that  $a(t) = M(\int_{\mathbb{R}} u_x^2 dx) \leq c_2$  which is possible due to (13) and the fact that  $M$  is increasing in  $\mathbb{R}$ . Now, we estimate  $\frac{d\mathcal{L}}{dt}$  using (23) and (29):

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= Ny^2 \frac{dD}{dt} + \frac{dJ}{dt} \leq -Ny^4|\varphi|^2 + Ny^4|v|^2 \frac{da(t)}{dt} \\ &- \frac{\alpha}{4}y^2(1+y^2)|v_t|^2 - \left(\frac{\alpha}{4} - \frac{\delta}{2}\right)y^6|v|^2 \\ &- \frac{\alpha}{8}y^4a(t)|v|^2 + c_1y^2(1+y^2)|v|^2 + c_2y^2|v|^2 \end{aligned} \quad (30)$$

We choose  $N$  such that  $N > c_1$ . The inequality

$$-Ny^4 + c_1y^2(1 + y^2) + c_2y^2 < -y^2$$

will be true whenever  $|y| > \left(\frac{c_1 + c_2 + 1}{N - c_1}\right)^{1/2}$ . With this choice

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &\leq -y^2|\varphi|^2 + Ny^4|v|^2 \frac{da}{dt} - \frac{\alpha}{4}y^2(1 + y^2)|v_t|^2 - \\ &- \left(\frac{\alpha}{4} - \frac{\delta}{2}\right)y^6|v|^2 - \frac{\alpha}{8}y^4a(t)|v|^2 \end{aligned} \quad (31)$$

for all  $|y| > \text{Max} \left\{ \left(\frac{c_1 + c_2 + 1}{N - c_1}\right)^{1/2}, \left(\frac{4\delta}{\alpha}\right)^{1/2} \right\} \equiv \rho > 0$  and  $0 < \delta < \frac{\alpha}{2}$ . Observe that

$$\begin{aligned} \left| \frac{da}{dt} \right| &= \left| M' \left( \int_{\mathbb{R}} u_x^2 dx \right) 2 \int_{\mathbb{R}} u_x u_{xt} dx \right| = \left| -2 \int_{\mathbb{R}} u_{xx} u_t dx M' \left( \int_{\mathbb{R}} u_x^2 dx \right) \right| \\ &\leq E(0) \text{Max}_{0 \leq s \leq c} |M'(s)| < +\infty \end{aligned}$$

because (23) implies that  $M(0) \int_{\mathbb{R}} u_x^2 dx \leq 2E(0)$  for any  $t$ . Thus, we take  $c = 2E(0)/M(0)$ . From (31) it follows that

$$\frac{d\mathcal{L}}{dt} \leq -c_3y^2D(y, t) + NE(0)D(y, t) \quad (32)$$

for some positive constant  $c_3 = c_3(\rho, \delta, \alpha)$  and all  $|y| \geq \rho$ . To complete the proof of (22) we choose

$$r \equiv \text{Max} \left\{ \rho, \left( \frac{2NE(0)}{c_3} \right)^{1/2} \right\}$$

to obtain from (32)

$$\frac{d\mathcal{L}}{dt} \leq -\frac{c_3}{2}y^2D(y, t) \quad \forall |y| > r$$

Using (21) we can find  $w = w(r) > 0$  such that

$$\frac{d\mathcal{L}}{dt} \leq -w(r)\mathcal{L}(y, t)$$

**Lemma 5.** *Let  $\{v, \varphi\}$  the solution of (20), and  $D$  as in Lemma 4. Then, same conclusion of Lemma 4 holds whenever  $0 < \delta_1^2 \leq |\xi|^2 \leq r_1^2$  for any  $\delta_1^2 > 0$  and some  $r_1 > 0$ .*

**Proof:** We consider the Lyapunov function  $\mathcal{L}_1 = \mathcal{L}_1(y, t)$  given by

$$\mathcal{L}_1(y, t) = N_1 D(y, t) + J(y, t)$$

where  $N_1$  is a suitable positive constant and  $J(y, t)$  is as in Lemma 1. Clearly  $|J(y, t)| \leq \left(\frac{\alpha}{4} + 1\right) (1 + y^2) D(y, t)$  therefore

$$\left[ N_1 - \left(\frac{\alpha}{4} + 1\right) (1 + y^2) \right] D \leq \mathcal{L}_1 \leq \left[ N_1 + \left(\frac{\alpha}{4} + 1\right) (1 + y^2) \right] D \quad (33)$$

we will proceed as in the proof of Lemma 4 to estimate  $\frac{d\mathcal{L}_1}{dt}$ . Thus, the inequality corresponding to (30) will be

$$\begin{aligned} \frac{d\mathcal{L}_1}{dt} &\leq -N_1 y^2 |\varphi|^2 + N_1 y^2 |v|^2 \frac{da}{dt} - \frac{\alpha}{4} y^2 (1 + y^2) |v_t|^2 \\ &\quad - \left(\frac{\alpha}{4} - \frac{\delta}{2}\right) y^6 |v|^2 - \frac{\alpha}{8} y^4 a(t) |v|^2 + c_1 y^2 (1 + y^2) |\varphi|^2 + c_2 y^2 |\varphi|^2 \end{aligned} \quad (34)$$

for any  $|y| \geq \left(\frac{4\delta}{\alpha}\right)^{1/2}$  and  $0 < \delta < \frac{\alpha}{2}$ . Furthermore, if we choose  $N_1$  such that  $N_1 > c_1 + c_2 + 1$  then, in the interval

$$\left(\frac{N_1 E(0)}{\frac{\alpha}{4} - \frac{3\delta}{2}}\right)^{1/4} < |y| < \left(\frac{N_1 - c_1 - c_2 - 1}{c_1}\right)^{1/2} \quad (35)$$

with  $0 < \delta < \frac{\alpha}{6}$ , we obtain from (34) that

$$\frac{d\mathcal{L}_1}{dt} \leq -y^2 |\varphi|^2 - \frac{\alpha}{4} y^2 (1 + y^2) |v_t|^2 - \frac{\alpha}{4} y^4 a(t) |v|^2 - y^6 |v|^2 \quad (36)$$

for any “ $y$ ” satisfying (35) and  $0 < \delta < \frac{\alpha}{6}$ . From (36) and (33) we get

$$\frac{d\mathcal{L}_1}{dt} \leq -c_4 y^2 D(y, t) \leq -c_5 y^2 \frac{1}{N_1} \mathcal{L}_1(y, t)$$

for some positive constant  $c_5$  which together with (33) give us

$$\mathcal{L}_1(y, t) \leq \mathcal{L}_1(0, t) \exp(-c_6 y^2 t)$$

for some positive  $c_6$  and  $0 < \delta_1^2 \leq y^2 \leq r_1^2$  where  $r_1 > \text{Max} \left\{ r, \left(\frac{N_1 - c_1 - c_2 - 1}{c_1}\right)^{1/2}, \left\{\frac{4\delta}{\alpha}\right\}^{1/2} \right\}$ ,  $r$  being as in Lemma 4 and  $N_1 = c_1 \delta_1^2 + c_1 + c_2 + 1$  with  $\delta_1 > 0$  arbitrary.

**Theorem 2.** *Let us consider the solution-pair  $\{u, \theta\}$  of system (20) obtained in Theorem 1 with  $(u_0, u_1, \theta_0) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R})$ . We assume additionally that  $(u_0, u_1, \theta_0) \in [L^1(\mathbb{R})]^3$  and  $M(\cdot)$  is an increasing function with  $M(0) \neq 0$ . Then, the total energy*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} (u_t^2 + u_{xx}^2 + u_{xt}^2 + \theta^2) dx + \hat{M} \left( \int_{\mathbb{R}} u_x^2 dx \right)$$

satisfies

$$E(t) \leq cE(0) \left\{ \exp(-wt) + \frac{1}{\sqrt{t}} \right\} \quad \forall t > 1$$

where  $c$  and  $w$  are positive constants and  $\hat{M}(s) = \int_0^s M(\tau) d\tau$ .

**Proof:** Via the Fourier transform we can write  $E(t)$  as

$$E(t) = \int_{\mathbb{R}} D(y, t) dy$$

where  $D(y, t)$  is given in Lemma 4. Let  $\delta > 0$  and write

$$\int_{\mathbb{R}} Ddy = \int_{|y| \leq \delta} Ddy + \int_{\delta \leq |y| \leq r_1} Ddy + \int_{r_1 \leq |y|} Ddy$$

Due to Lemmas 4 and 5 we know that

$$\int_{\delta \leq |y| \leq r_1} Ddy + \int_{r_1 \leq |y|} Ddy \leq cE(0) \exp(-wt)$$

for some positive constants  $w$  and  $c$ . Remains to get an estimate for the term  $\int_{|y| \leq \delta} Ddy$ .

In fact, as we observe from (30) for any  $0 < \delta < \frac{\alpha}{2}$  we can obtain the bound

$$\frac{d\mathcal{L}}{dt} \leq cy^2 D(y, t) \leq \tilde{c}y^2 \mathcal{L}(y, t) \quad (37)$$

some positive constans  $c$  and  $\tilde{c}$ . We are only interested in (37) when  $|y| \leq \delta$ .

Clearly from our above discussion we can obtain from (37) that

$$\mathcal{L}(y, t) \leq c_1 \exp(cy^2 t) \mathcal{L}(y, 0)$$

or

$$D(y, t) \leq c_2 \exp(cy^2 t) D(y, 0) \quad (38)$$

for some positive constants  $c_1, c_2$ . Integration of (38) in  $|y| \leq \delta$  give us

$$\int_{|y| \leq \delta} D(y, t) dy \leq c_2 \exp(c\delta^2 t) \int_{|y| \leq \delta} D(y, 0) dy \quad (39)$$

we claim that  $\sup_{|y| \leq \delta} D(y, 0) \leq F(0) < +\infty$  for some constant  $F(0)$  which (independent of  $\delta$ ) depends on the initial data  $(u_0, u_1, \theta_0)$ . In fact, since we are interested only with  $\delta$  small let us take  $0 < \delta < 1$ . Using the fact that

$$\sup_{y \in \mathbb{R}} \{ |\hat{u}_1|^2 + |\hat{u}_0|^2 + |\hat{\theta}_0|^2 \} \leq \frac{1}{2\pi} \{ \|u_1\|_{L^1}^2 + \|u_0\|_{L^1}^2 + \|\theta_0\|_{L^1}^2 \} \equiv F(0)$$

it follows from (39) that

$$\int_{|y| \leq \delta} D(y, t) dy \leq c_2 \exp(c\delta^2 t) F(0) \delta \quad (40)$$

We take  $\delta = \frac{E(0)}{\sqrt{t}}$  in (40) and conclude that

$$\int_{|y| \leq \delta} D(y, t) dy \leq c_2 E(0) \exp(cE^2(0)) \frac{F(0)}{\sqrt{t}}, \quad \forall t > 1.$$

which concludes the proof of theorem 2.

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