

HYPERBOLIC-PARABOLIC PROBLEM WITH DEGENERATE SECOND-ORDER BOUNDARY CONDITIONS *

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Abstract

The initial boundary value problem for a hyperbolic-parabolic equation with nonlinear second-order degenerate boundary condition is considered. Existence and uniqueness of a global generalized solution are proved.

Resumo

Neste trabalho é considerado o problema de valores iniciais e de fronteira para uma equação hiperbólica-parabólica com uma condição de fronteira não linear degenerada envolvendo derivadas de segunda ordem no tempo. É provado um teorema de existência e unicidade de solução global no sentido generalizado.

1. Introduction

In [1], J.L. Lions considers nonlinear problems on manifolds in which the unknown function satisfies the Laplace equation in a cylinder Q and a nonlinear evolution equation of the second order on the lateral boundary Σ of Q . This problem models water waves with free boundaries [2, 3]. The dissipative first-order boundary condition of the type considered in [1] arises when one studies flows of a gas in channels with porous walls [4, 5]. The presence of the second derivative with respect to t in the boundary condition is due to internal forces acting on particles of the medium.

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A similar problem for the wave equation has been considered in [6]. The coefficient of the principal second-order term of the boundary condition is a strictly positive function. From the physical point of view, it means that vacuum is forbidden. Our purpose here is to omit the condition of strict positivity and to consider the degenerate case when the coefficient is only nonnegative.

We study in the present paper the hyperbolic-parabolic equation

$$Pv_{tt} - \Delta v + \alpha v_t = f, \quad \text{in } Q \quad (1.1)$$

with nonlinear boundary condition

$$\frac{\partial v}{\partial \nu} + K(v)v_{tt} + g(v_t) = 0, \quad \text{on } \Sigma \quad (1.2)$$

and with initial data

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x). \quad (1.3)$$

The type of equation (1.1) depends on the sign of the function $P \geq 0$. This is a hyperbolic equation when $P > 0$ and parabolic when $P = 0$.

The term $K(v)v_{tt}$ with $K(v) \geq 0$ models internal forces when the density of the medium depends on the displacement. When $K(v) = 0$, vacuum occurs.

Notice that the boundary dissipation $g(v_t)$ in (1.2) is more general than the corresponding dissipative term in [6].

In this article we use the ideas from [6] and [7] to prove the existence of global generalized solutions to the problem (1.1)-(1.3). We exploit the Faedo-Galerkin method, a priori estimates and compactness arguments. Uniqueness is proved in the one-dimensional case.

2. The Main Result

For $T > 0$, let Ω be a bounded open set of R^n with sufficiently smooth boundary Γ and $Q = \Omega \times (0, T)$. We consider the following hyperbolic-parabolic problem,

$$P(x, t)v_{tt} - \Delta v + \alpha v_t = f(x, t), \quad (x, t) \in Q; \quad (2.1)$$

$$\left(\frac{\partial v}{\partial \nu} + K(v)v_{tt} + g(v_t) \right) \Big|_{\Sigma} = 0; \quad (2.2)$$

$$v(x, 0) = v_0(x); \quad v_t(x, 0) = v_1(x), \quad x \in \Omega. \quad (2.3)$$

Here $P(x, t)$ and $K(u)$ are continuously differentiable non-negative functions of their arguments; α is a strictly positive constant; ν is the outward unity normal vector on Γ ; $\Sigma = \Gamma \times (0, T)$.

We use the usual notations,

$$(u, v)(t) = \int_{\Omega} u(x, t)v(x, t) dx, \quad (u, v)_{\Gamma}(t) = \int_{\Gamma} u(x, t)v(x, t) d\Gamma,$$

$$\|u\|^2(t) = (u, u)(t), \quad \Delta u = \sum_{i=1}^n \partial^2 u / \partial x_i^2$$

and impose the following compatibility conditions,

$$-\Delta v_0(x) + \alpha v_1(x) = f(x, 0), \quad x \in \Omega, \quad (2.4)$$

$$\frac{\partial v_0}{\partial \nu} + g(v_1) = 0, \quad x \in \Gamma. \quad (2.5)$$

We consider functions $K(v)$ satisfying the assumptions

$$0 \leq K(v) \leq C(1 + |v|^{\rho}), \quad (2.6)$$

$$|K'(v)|^{\frac{\rho}{\rho-1}} \leq \eta + C(\eta)K(v), \quad (2.7)$$

where $\rho \in (1, \infty)$ and η is a sufficiently small positive number.

These conditions mean that the density of the medium can not increase “too rapidly” as a function of displacement. The condition (2.7) appears quite naturally because functions with polynomial growth, such as $K(v) = |v|^s$ with $1 < s \leq \rho$, satisfy it. Besides, the inequality $K(v) \geq 0$ means that vacuum is not forbidden.

Function $g(\xi)$ satisfies the following conditions,

$$g(\xi)\xi \geq \alpha_1|\xi|^{\rho+2} + \alpha_2|\xi|^2, \quad (2.8)$$

$$g'(\xi) \geq \alpha_3|\xi|^{\rho} + \alpha_4, \quad (2.9)$$

$$|g(\xi)| \leq C(1 + |\xi|^{\rho+1}) \quad (2.10)$$

with $\alpha_i > 0$, $i = 1, \dots, 4$.

Finally, coefficients $P(x, t)$ and α satisfy the hypothesis,

$$P \geq 0 \quad \text{and} \quad 2\alpha - |P_t| \geq \delta, \quad \text{in } \overline{Q}, \quad (2.11)$$

where δ is some positive number.

Definition. A function $v(x, t)$ satisfying conditions

$$\begin{aligned} v &\in L^\infty(0, T; H^1(\Omega)), \\ v_t &\in L^\infty(0, T; H^1(\Omega)) \cap L^{\rho+2}(\Sigma), \\ \sqrt{P}v_{tt} &\in L^\infty(0, T; L^2(\Omega)), \\ v_{tt} &\in L^2(0, T; L^2(\Omega) \cap L^2(\Gamma)), \\ v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x) \end{aligned}$$

is a generalized solution to (2.1)-(2.3) if for any functions $h \in H^1(\Omega) \cap L^{\rho+2}(\Gamma)$ and $\varphi \in C^1(0, T)$ with $\varphi(T) = 0$ the following identity holds

$$\begin{aligned} \int_0^T \left\{ (Pv_{tt}, h)(t) + (\nabla v, \nabla h)(t) + \alpha(v_t, h)(t) + \int_\Gamma [g(v_t) - K'(v)v_t^2] h \, d\Gamma \right\} \varphi(t) \, dt \\ + \int_\Gamma K(v_0)v_1\varphi(0)h \, d\Gamma - \int_0^T \varphi'(t) \int_\Gamma K(v)v_th \, d\Gamma \, dt = \int_0^T (f, h)\varphi(t) \, dt. \end{aligned} \quad (2.12)$$

The main result of this paper is the following.

Theorem. Let the conditions (2.4)-(2.11) hold and suppose that $f \in H^1(0, T; L^2(\Omega))$. Then for any $v_0 \in H^2(\Omega)$, $v_1 \in H^2(\Omega) \cap L^{\rho+2}(\Gamma)$ and for all $T > 0$ there exists at least one generalized solution of the problem (2.1)-(2.3). If $n = 1$, this solution is unique.

Proof. We prove the Theorem by reducing the original problem to a homogeneous one [8]. The existence of solutions of the transformed problem is proved by the Faedo-Galerkin method. First, using the regularization $P_\varepsilon = P + \varepsilon$ with $\varepsilon > 0$, we construct approximations of the generalized solution. Then we obtain a priori estimates necessary to guarantee convergence of approximations. Finally, we prove the uniqueness in the one-dimensional case.

3. Approximate solutions

First of all we transform the problem (2.1)-(2.3) into an equivalent one with zero initial conditions. In fact, the change of variables

$$u(x, t) = v(x, t) - \phi(x, t), \quad (3.1)$$

where

$$\phi(x, t) = v_0(x) + v_1(x) \cdot t, \quad (x, t) \in Q \quad (3.2)$$

gives rise to the equivalent problem for the unknown $u(x, t)$:

$$P(x, t)u_{tt} - \Delta u + \alpha u_t = F(x, t) \quad \text{in } Q; \quad (3.3)$$

$$\frac{\partial u}{\partial \nu} + K(u + \phi)u_{tt} + g(u_t + \phi_t) = G(x, t) \quad \text{on } \Sigma; \quad (3.4)$$

$$u(x, 0) = u_t(x, 0) = 0 \quad \text{in } \Omega. \quad (3.5)$$

Here $F(x, t) = f + \Delta\phi - \alpha\phi_t$ and $G(x, t) = -\partial\phi/\partial\nu$ are given functions. If $u(x, t)$ is a solution of (3.3)-(3.5) at any interval $[0, T]$, then $v = u + \phi$ is a solution of the original problem (2.1)-(2.3) in the same interval. Thus, in order to prove the Theorem, it is sufficient to consider the problem (3.3)-(3.5). This is done by the following algorithm.

Let $\{w_j(x)\}$ be a basis in $H^1(\Omega) \cap L^{\rho+2}(\Gamma)$ and ε be an arbitrary positive number. We define approximations as follows,

$$P_\varepsilon = P + \varepsilon, \quad u_\varepsilon^N(x, t) = \sum_{i=1}^N g_{\varepsilon i}^N(t) w_i(x), \quad (3.6)$$

where $g_{\varepsilon i}^N(t)$ are solutions of the Cauchy problem,

$$\begin{aligned} & (P_\varepsilon u_{\varepsilon tt}^N, w_j)(t) + (\nabla u_\varepsilon^N, \nabla w_j)(t) + \alpha(u_{\varepsilon t}^N, w_j)(t) \\ & + \int_\Gamma \left\{ K(u_\varepsilon^N + \phi)u_{\varepsilon tt}^N + g(u_{\varepsilon t}^N + \phi_t) \right\} w_j d\Gamma \\ & = (F, w_j)(t) + (G, w_j)_\Gamma(t); \end{aligned} \quad (3.7)$$

$$g_{\varepsilon j}^N(0) = (g_{\varepsilon j}^N)'(0) = 0; \quad j = 1, \dots, N. \quad (3.8)$$

It is easy to see that (3.7) is not a normal system of ODE. However, using the method of [6] and the positivity of P_ε , we conclude that (3.7) can be reduced to normal form and, by the Caratheodory theorem, problem (3.7),(3.8) has solutions $g_{\varepsilon j}^N(t) \in H^2(0, t_N(\varepsilon))$, so that all the approximations (3.6) are defined in $(0, t_N(\varepsilon))$ for each $\varepsilon > 0$.

4. A priori estimates

Next, we need a priori estimates to show that $t_N(\varepsilon) = T$ and to pass to the limit as $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$. To simplify the exposition, we omit the indices N and ε whenever it is unambiguous.

Multiplying (3.7) by g'_j and summing from $j = 1$ to $j = N$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\sqrt{P}u_t\|^2 + \|\nabla u\|^2 \right) (t) &+ \frac{1}{2} \int_{\Omega} (2\alpha - P_t) u_t^2 dx + \int_{\Gamma} g(u_t + \phi_t) u_t d\Gamma \\ &+ \frac{1}{2} \int_{\Gamma} \left\{ \frac{d}{dt} \left(K(u + \phi) u_t^2 \right) - K'(u + \phi) (u_t + \phi_t) u_t^2 \right\} d\Gamma \\ &= (F, u_t)(t) + (G, u_t)_\Gamma(t). \end{aligned} \quad (4.1)$$

Estimating integrals over Γ from below, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Gamma} K(u + \phi) u_t^2 d\Gamma &+ \int_{\Gamma} \{g(u_t + \phi_t)(u_t + \phi_t) - g(u_t + \phi_t) \phi_t \\ &- \frac{1}{2} K'(u + \phi) (u_t + \phi_t) u_t^2 - G u_t\} d\Gamma \geq \frac{1}{2} \frac{d}{dt} \int_{\Gamma} K(u + \phi) u_t^2 d\Gamma \\ &+ \int_{\Gamma} \{ \alpha_1 |u_t + \phi_t|^{\rho+2} + \alpha_2 |u_t + \phi_t|^2 - g(u_t + \phi_t) \phi_t \\ &- \frac{1}{2} K'(u + \phi) (u_t + \phi_t) u_t^2 - \epsilon_1 u_t^2 - C_{\epsilon_1} G^2 \} d\Gamma. \end{aligned} \quad (4.2)$$

Here and later in this paper, ϵ_i are arbitrary small positive numbers and $C_{\epsilon_i} = C(\epsilon_i)$ are positive constants independent on N .

The Young inequality, together with (2.7) and (2.10) imply that

$$|g(u_t + \phi_t) \phi_t| \leq C \left(1 + \epsilon_2 |u_t + \phi_t|^{\rho+2} + C_{\epsilon_2} |\phi_t|^{\rho+2} \right); \quad (4.3)$$

$$\alpha_2 |u_t + \phi_t|^2 \geq \alpha_2 u_t^2 - \epsilon_3 u_t^2 - C_{\epsilon_3} \left(1 + |\phi_t|^{\rho+2} \right); \quad (4.4)$$

$$\begin{aligned}
-\frac{1}{2}|K'(u+\phi)(u_t+\phi_t)u_t^2| &\geq -u_t^2 \left(\epsilon_4 |u_t+\phi_t|^\rho + C_{\epsilon_4} |K'(u+\phi)|^{\frac{\rho}{\rho-1}} \right) \\
&\geq -u_t^2 \epsilon_4 |u_t+\phi_t|^\rho - \eta C_4 u_t^2 - C_{\epsilon_4} C(\eta) K(u+\phi) u_t^2.
\end{aligned} \tag{4.5}$$

We observe that the first term of the right hand in (4.5) satisfies

$$-\epsilon_4 u_t^2 |u_t+\phi_t|^\rho \geq -\epsilon_5 |u_t+\phi_t|^{\rho+2} - C_{\epsilon_5} |\phi_t|^{\rho+2}. \tag{4.6}$$

Indeed,

$$|u_t+\phi_t|^{\rho+2} = |u_t+\phi_t|^\rho \left(u_t^2 + 2u_t\phi_t + \phi_t^2 \right),$$

therefore

$$\begin{aligned}
u_t^2 |u_t+\phi_t|^\rho &= |u_t+\phi_t|^{\rho+2} - (2u_t\phi_t + \phi_t^2) |u_t+\phi_t|^\rho \\
&\leq |u_t+\phi_t|^{\rho+2} + 2 \left(\epsilon_6 |u_t|^2 + C_{\epsilon_6} |\phi_t|^2 \right) |u_t+\phi_t|^\rho \\
&\leq |u_t+\phi_t|^{\rho+2} + \frac{1}{2} u_t^2 |u_t+\phi_t|^\rho + \epsilon_7 |u_t+\phi_t|^{\rho+2} + C_{\epsilon_7} |\phi_t|^{\rho+2}
\end{aligned}$$

which gives (4.6).

Taking into account (2.11), (4.2)-(4.6), setting all the $\epsilon_i > 0$ sufficiently small, we conclude from (4.1) that

$$\begin{aligned}
&\frac{d}{dt} \left(\|\sqrt{P}u_t\|^2 + \|\nabla u\|^2 \right) (t) + \frac{d}{dt} \int_{\Gamma} K(u+\phi) u_t^2 d\Gamma + \delta(u_t, u_t)(t) \\
&\quad + \frac{\alpha_1}{2} \|u_t+\phi_t\|_{L^{\rho+2}(\Gamma)}^{\rho+2}(t) + \frac{\alpha_2}{2} \|u_t\|_{L^2(\Gamma)}^2(t) \\
&\leq C \left\{ 1 + \int_{\Gamma} K(u+\phi) u_t^2 d\Gamma + \|F\|^2(t) + \|G\|_{L^2(\Gamma)}^2(t) + \|\phi_t\|_{L^{\rho+2}(\Gamma)}^{\rho+2}(t) \right\}.
\end{aligned}$$

Integrating the last inequality over $t \in [0, T]$ and using Gronwall's lemma, we obtain the first a priori estimate:

$$\begin{aligned}
&\left(\|\sqrt{P}u_t\|^2 + \|\nabla u\|^2 \right) (t) + \int_{\Gamma} K(u+\phi) u_t^2 d\Gamma \\
&\quad + \int_0^t \left\{ \|u_t\|^2(\tau) + \|u_t+\phi_t\|_{L^{\rho+2}(\Gamma)}^{\rho+2}(\tau) + \|u_t\|_{L^2(\Gamma)}^2(\tau) \right\} d\tau \leq C,
\end{aligned} \tag{4.7}$$

where the constant C does not depend on N .

In order to obtain the second a priori estimate, we observe that

$$(Pu_{tt}, u_{tt})(0) + \int_{\Gamma} K(v_0) u_{tt}^2(x, 0) d\Gamma = 0. \tag{4.8}$$

Indeed, multiplying (3.7) by $g_j''(0)$, summing over j and setting $t = 0$, we obtain

$$(Pu_{tt}, u_{tt})(0) + \int_{\Gamma} \left(K(\phi)u_{tt}^2 + g(\phi_t)u_{tt} \right) (x, 0) d\Gamma = (F, u_{tt})(0) + (G, u_{tt})_{\Gamma}(0).$$

Using (2.4) and (2.5), we conclude that

$$(F, u_{tt})(0) = (G, u_{tt})_{\Gamma}(0) - (g(\phi_t), u_{tt})_{\Gamma}(0) = 0$$

which implies (4.8).

Differentiating (3.7) with respect to t , multiplying by g_j'' and summing over j , we are led to the identity:

$$\begin{aligned} & \int_{\Omega} \left(P_t u_{tt}^2 + Pu_{tt}u_{ttt} + \nabla u_t \nabla u_{tt} + \alpha u_{tt}^2 \right) dx \\ & + \int_{\Gamma} \left\{ K'(u + \phi)(u_t + \phi_t)u_{tt}^2 + K(u + \phi)u_{tt}u_{ttt} + g'(u_t + \phi_t)u_{tt}^2 \right\} d\Gamma \\ & = (F_t, u_{tt})(t) + (G_t, u_{tt})_{\Gamma}(t). \end{aligned} \quad (4.9)$$

Notice that

$$P_t u_{tt}^2 + Pu_{tt}u_{ttt} \geq \frac{1}{2} \frac{d}{dt} \left(Pu_{tt}^2 \right) - \frac{1}{2} |P_t| u_{tt}^2; \quad (4.10)$$

$$g'(u_t + \phi_t)u_{tt}^2 \geq (\alpha_3 |u_t + \phi_t|^\rho + \alpha_4) u_{tt}^2; \quad (4.11)$$

$$K(u + \phi)u_{tt}u_{ttt} = \frac{1}{2} \frac{d}{dt} \left(K(u + \phi)u_{tt}^2 \right) - \frac{1}{2} K'(u + \phi)(u_t + \phi_t)u_{tt}^2; \quad (4.12)$$

$$(F_t, u_{tt})(t) \leq \int_{\Omega} (\epsilon_8 u_{tt}^2 + C_{\epsilon_8} F_t^2) dx \quad (4.13)$$

$$(G_t, u_{tt})_{\Gamma}(t) \leq \int_{\Gamma} (\epsilon_9 u_{tt}^2 + C_{\epsilon_9} G_t^2) d\Gamma; \quad (4.14)$$

$$\begin{aligned} & \left| K'(u + \phi)(u_t + \phi_t)u_{tt}^2 \right| \leq u_{tt}^2 \left(\epsilon_{10} |u_t + \phi_t|^\rho + C_{\epsilon_{10}} |K'(u + \phi)|^{\frac{\rho}{\rho-1}} \right) \\ & \leq \epsilon_{10} u_{tt}^2 |u_t + \phi_t|^\rho + \eta C_{\epsilon_{10}} 3u_{tt}^2 + C(\eta, \epsilon_{10}) K(u + \phi)u_{tt}^2. \end{aligned} \quad (4.15)$$

Using (4.10)-(4.15) we obtain from (4.9) that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(Pu_{tt}^2 + |\nabla u_t|^2 \right) dx + \frac{d}{dt} \int_{\Gamma} K(u + \phi)u_{tt}^2 d\Gamma + (\delta - 2\epsilon_8) \int_{\Omega} u_{tt}^2 dx \\ & + (2\alpha_3 - \epsilon_{10}) \int_{\Gamma} |u_t + \phi_t|^\rho u_{tt}^2 d\Gamma + (\alpha_4 - 2\epsilon_9 - \eta C_{\epsilon_{10}}) \int_{\Gamma} u_{tt}^2 d\Gamma \end{aligned}$$

$$\leq C(\eta, \epsilon_{10}) \int_{\Gamma} K(u + \phi) u_{tt}^2 d\Gamma + C_{\epsilon_9} \int_{\Gamma} |G_t|^2 d\Gamma + C_{\epsilon_8} \int_{\Omega} |F_t|^2 dx. \quad (4.16)$$

Choosing $\epsilon_8, \dots, \epsilon_{10}$ and then $\eta > 0$ sufficiently small, integrating (4.16) over t , using (4.9) and Gronwall's lemma, we obtain the second a priori estimate

$$\begin{aligned} & \left(\|\sqrt{P} u_{tt}\| + \|\nabla u_t\| \right) (t) + \int_{\Gamma} K(u + \phi) u_{tt}^2 \\ & + \int_0^t \left\{ \|u_{tt}\|^2(\tau) + \|u_{tt}\|_{L^2(\Gamma)}^2(\tau) + \int_{\Gamma} |u_t + \phi_t|^\rho u_{tt}^2 d\Gamma \right\} d\tau \leq C. \end{aligned} \quad (4.17)$$

Thus, we obtain the following a priori estimates

$$\begin{aligned} u_\varepsilon^N & \in L^\infty(0, T; H^1(\Omega)); \\ u_{\varepsilon t}^N & \in L^\infty(0, T; H^1(\Omega)) \cap L^{\rho+2}(\Sigma); \\ P_\varepsilon u_{\varepsilon tt}^N & \in L^\infty(0, T; L^2(\Omega)); \\ u_{\varepsilon tt}^N & \in L^2(Q) \cap L^2(\Sigma); \\ \frac{\partial}{\partial t} |u_{\varepsilon t}^N|^{1+\rho/2} & \in L^2(\Sigma); \\ K^{1/2}(u_\varepsilon^N) u_{\varepsilon tt}^N & \in L^\infty(0, T; L^2(\Gamma)). \end{aligned} \quad (4.18)$$

5. Passage to the limit

Let us multiply (3.7) by $\varphi \in C^1(0, T)$ with $\varphi(T) = 0$ and integrate with respect to t from 0 to T . After integration by parts, we obtain

$$\begin{aligned} & \int_0^T \left\{ (P_\varepsilon u_{\varepsilon tt}^N, w_j) + (\nabla u_\varepsilon^N, \nabla w_j) + \alpha(u_{\varepsilon t}^N, w_j) + \int_{\Gamma} g(u_{\varepsilon t}^N + \phi_t) w_j d\Gamma \right\} \varphi(t) dt \\ & - \int_0^T \varphi'(t) \int_{\Gamma} K(u_\varepsilon^N + \phi) u_{\varepsilon t}^N w_j(x) d\Gamma dt + \varphi(t) K(u_\varepsilon^N + \phi) u_{\varepsilon t}^N \Big|_0^T \\ & - \int_0^T \varphi(t) \int_{\Gamma} K'(u_\varepsilon^N + \phi) (u_{\varepsilon t}^N + \phi_t) u_{\varepsilon t}^N w_j d\Gamma dt \\ & = \int_0^T [(F, w_j) + (G, w_j)_\Gamma] \varphi(t) dt. \end{aligned} \quad (5.1)$$

The estimates (4.18) imply that a subsequence u_ε^μ can be extracted from u_ε^N

such that:

$$\begin{aligned}
u_\varepsilon^\mu &\rightarrow u \text{ weakly star in } L^\infty(0, T; H^1(\Omega)); \\
u_{\varepsilon t}^\mu &\rightarrow u_t \text{ weakly star in } L^\infty(0, T; H^1(\Omega)) \cap L^{\rho+2}(\Sigma); \\
P_\varepsilon u_{\varepsilon tt}^\mu &\rightarrow Pu_{tt} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)); \\
u_{\varepsilon tt}^\mu &\rightarrow u_{tt} \text{ weakly in } L^2(Q) \cap L^2(\Sigma); \\
u_\varepsilon^\mu, u_{\varepsilon t}^\mu &\rightarrow u, u_t \quad \text{a.e. on } \Sigma;
\end{aligned}$$

Therefore,

$$\begin{aligned}
g(u_{\varepsilon t}^\mu + \phi_t) &\in L^q(\Sigma), \quad q = (\rho + 2)/(\rho + 1) \text{ and converges a.e. on } \Sigma; \\
K(u_\varepsilon^\mu + \phi)u_{\varepsilon t}^\mu &\in L^q(\Sigma), \quad \text{and converges a.e. on } \Sigma; \\
K'(u_\varepsilon^\mu + \phi)(u_{\varepsilon t}^\mu + \phi_t)u_{\varepsilon t}^\mu &\in L^q(\Sigma), \quad \text{and converges a.e. on } \Sigma.
\end{aligned}$$

Thus, we are able to pass to the limit in (5.1) to obtain

$$\begin{aligned}
&\int_0^T \left\{ (Pu_{tt}, w_j) + (\nabla u, \nabla w_j) + \alpha(u_t, w_j) + \int_\Gamma g(u_t + \phi_t)w_j \, d\Gamma \right\} \varphi(t) \, dt \\
&- \int_0^T \left\{ \varphi'(t) \int_\Gamma K(u + \phi)u_t w_j \, d\Gamma + \varphi(t) \int_\Gamma K'(u + \phi)(u_t + \phi_t)u_t w_j \, d\Gamma \right\} dt \\
&= \int_0^T [(F, w_j) + (G, w_j)_\Gamma] \varphi(t) \, dt. \tag{5.2}
\end{aligned}$$

It can be seen that all the integrals in (5.2) are defined for any function $\varphi(t) \in C^1(0, T)$ with $\varphi(T) = 0$. Taking into account that $\{w_j(x)\}$ are dense in $H^1(\Omega) \cap L^{\rho+2}(\Gamma)$, we conclude that for any $h \in H^1(\Omega) \cap L^{\rho+2}$ the equality (2.12) holds.

If $n = 1, 2$, one can get more regular solutions. In this case, $v \in L^\infty(0, T; L^q(\Gamma))$ for any $q \in [1, \infty)$. Hence, $K(v)v_{tt} \in L^\infty(0, T; L^p(\Gamma))$, with arbitrary $p \in [1, 2)$. This allows to rewrite (2.12) in the form

$$(Pv_{tt}, h)(t) + (\nabla v, \nabla h)(t) + \alpha(v_t, h) + \int_\Gamma \{K(v)v_{tt} + g(v_t)\}h \, d\Gamma = (f, h)(t),$$

where h is an arbitrary function from $H^1(\Omega)$.

6. Uniqueness

Let $n = 1$ and $Q = (0, 1) \times (0, T)$. Let u and v be two solutions of the problem (2.1)-(2.3) in Q and $z(x, t) = u(x, t) - v(x, t)$. Then for fixed t , for every function $\phi \in H^1(0, 1)$, we have

$$(Pz_{tt}, \phi)(t) + (z_x, \phi_x)(t) + \alpha(z_t, \phi)(t) + \left\{ [K(u)z_{tt} + v_{tt}(K(u) - K(v)) + g(u_t) - g(v_t)] \phi \right\} \Big|_0^1 = 0.$$

Since $(z, z_t)(x, t) \in L^\infty(0, T; H^1(0, 1))$, we may take $\phi = z_t$, and this equation can be reduced to the inequality,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [E(t) + (K(u)(z_t)^2)_0^1] + ((\alpha - P_t/2), z_t^2)(t) + \alpha_4 z_t^2 \Big|_0^1(t) \\ & + \left\{ v_{tt} z_t (K(u) - K(v)) - \frac{1}{2} K'(u) u_t (z_t)^2 \right\} \Big|_0^1(t) \leq 0. \end{aligned}$$

Here we set $E(t) = \|\sqrt{P}z_t\|^2(t) + \|z_x\|^2(t)$ and use (2.9), the differentiability of K and the regularity of $K(u)u_{tt}$ (see the end of previous section). Condition (2.7) then implies that

$$\begin{aligned} & \frac{d}{dt} [E(t) + (K(u)z_t^2)_0^1] + \delta \|z_t\|^2(t) + 2\alpha_4 z_t^2 \Big|_0^1(t) \leq \epsilon [1 + \max_{x=0,1} |u_t|^\rho] \cdot z_t^2 \Big|_0^1(t) \\ & + C_\epsilon \max_{x=0,1} |K(u) - K(v)| \cdot v_{tt}^2 \Big|_0^1(t) + C_\epsilon [(\eta + C_\eta K(u)) z_t^2] \Big|_0^1(t) \\ & \leq [\epsilon (1 + \max_{x=0,1} |u_t|^\rho) + \eta C_\epsilon] \cdot z_t^2 \Big|_0^1(t) + C_\epsilon C_\eta (K(u) z_t^2) \Big|_0^1(t) \\ & + C_\epsilon \max_{x=0,1} |K'(u)|^2 \cdot v_{tt}^2 \Big|_0^1(t) \cdot [\|z\|^2(t) + \|z_x\|^2(t)]. \end{aligned} \quad (6.1)$$

Taking into account that $\|z\|^2(t) \leq t \int_0^t \|z_t\|^2(\tau) d\tau$, choosing in (6.1) first $\epsilon > 0$ then $\eta > 0$ and $T_1 > 0$ sufficiently small, for all $t \in (0, T_1)$ we obtain the inequality

$$\begin{aligned} & \frac{d}{dt} [E(t) + (K(u)z_t^2)_0^1] + \frac{1}{2} \|z_t\|^2(t) + \alpha_4 z_t^2 \Big|_0^1 \\ & \leq C [v_{tt}^2 \cdot E(t) + K(u)z_t^2] \Big|_0^1. \end{aligned}$$

Since

$$\int_0^T [v_{tt}^2(1, t) - v_{tt}^2(0, t)] dt \leq C,$$

then by Gronwall's lemma,

$$E(t) + [K(u)z_t^2]_0^1 + \int_0^t \|z_t\|^2 d\tau = 0, \quad \forall t \in (0, T_1).$$

Therefore $z(x, t) \equiv 0$, in $(0, 1) \times (0, T_1)$. Splitting, if necessary, $(0, T)$ in a finite number of intervals $(0, T_1)$, we prove the uniqueness result and complete the proof of the Theorem.

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