Matemática Contemporânea, Vol. 18, 97–109
http://doi.org/10.21711/231766362000/rmc186

INITIAL BOUNDARY VALUE PROBLEM FOR THE KURAMOTO-SIVASHINSKY EQUATION

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Abstract

We consider the initial-boundary value problem for the one-dimensional Kuramoto-Sivashinsky equation,

$$u_t + uu_x + \eta u_{xxx} + \beta u_{xx} + \delta u_{xxxx} = f,$$

where η, β, δ are positive constants, in the non-cylindrical domain $Q = \{(x,t); \alpha_1(t) < x < \alpha_2(t), t \in (0,T)\}$. We prove the existence and uniqueness of global weak and strong solutions, and the exponential decay of solutions as $t \to \infty$.

Resumo

Neste artigo abordamos o problema de valor inicial e de fronteira para a equação de Kuramoto-Sivashinsky unidimensional

$$u_t + uu_x + \eta u_{xxx} + \beta u_{xx} + \delta u_{xxxx} = f,$$

onde η, β, δ são constantes positivas, no domínio não cilíndrico $Q = \{(x,t); \alpha_1(t) < x < \alpha_2(t), t \in (0,T)\}$. Nós provamos a existência e unicidade de soluções globais fracas e fortes, e também o decaimento exponencial das soluções quando $t \to \infty$.

1. Introduction

The Kuramoto-Sivashinsky (K-S) equation was derived independently by Sivashinsky [6] , who studied flame propagation processes in turbulent flow of a gaseous combustible mixture, and by Kuramoto [5], who studied wave fronts in reaction-diffusion systems.

*The authors were supported by a research grant from CNPq-Brazil
AMS Subject Classification: 35Q35, 35Q53.
Key words and phrases: Kuramoto-Sivashinsky Equation, Domains with Moving Boundaries, Galerkin Method.

Untill recently, most publications were dedicated to physical aspects of K-S equation.

A systematic study of mathematical problems was started in the paper of H. Biagioni, J. Bona, R. Iorio and M. Scialom [2] where the Cauchy problem for the generalized K-S equation,

$$u_t + uu_x + \eta u_{xxx} + \beta u_{xx} + \delta u_{xxxx} = 0 \tag{1}$$

was treated. They proved the existence of local and global in t smooth solutions exploiting the Fourier-transform in x. Moreover, the asymptotic behavior of the solutions was studied when $\eta \to 0$ or $\beta = \delta \to 0$. The Cauchy problem for the multi-dimensional analogue of the K-S equation was studied by H. Biagioni and T. Gramchev (1998) [3].

In the paper of E. Tadmor [7] the well-posedness of the Cauchy problem was proved for the one-dimensional K-S equation. It was shown that the Cauchy problem admits a unique smooth solution depending continuously on initial data.

Here we study the K-S equation in domains with moving boundaries and prove the existence and uniqueness of global weak and strong solutions, and the stability of solutions as $t \to \infty$.

2. Statement of the problem

Let

$$\alpha_1(t) < x < \alpha_2(t),\ t \in [0,T],\ \gamma(t) = \alpha_2(t) - \alpha_1(t) \geq \delta_0 > 0;$$

and

$$\alpha_1, \alpha_2 \in \mathbf{C}^1[0,\infty), \quad \text{with } |\alpha_1'(t)| + |\alpha_2'(t)| \leq M < \infty.$$

We denote by Q:

$$Q = \{(x,t);\ \alpha_1(t) < x < \alpha_2(t),\ t \in (0,T)\}.$$

In Q we consider the generalized Kuramoto-Sivashinsky equation,

$$Lu = u_t + uu_x + \eta u_{xxx} + \beta u_{xx} + \delta u_{xxxx} = f, \tag{1.1}$$

where $\eta, \beta, \delta > 0$, with the initial data,

$$u(x,0) = u_0(x),\ \alpha_1(0) < x < \alpha_2(0). \tag{1.2}$$

The following conditions are given on moving boundaries:

$$u(\alpha_1(t),t) = u(\alpha_2(t),t) = u_{xx}(\alpha_1(t),t) = u_{xx}(\alpha_2(t),t) = 0,\ t \in [0,T]. \tag{1.3}$$

Changing variables,

$$(x,t) \leftrightarrow (y,t),\ u(x(y,t),t) = v(y,t),$$

where

$$y = \frac{x - \alpha_1(t)}{\gamma(t)},$$

we transform Q into the rectangle $\tilde{Q} = (0,1) \times (0,T)$, and the problem (1.1) – (1.3) into the following problem;

$$Lv = v_t + \frac{1}{\gamma(t)} v v_y - \frac{y\gamma'(t) + \alpha_1'(t)}{\gamma(t)} v_y + \frac{\beta}{\gamma^2(t)} v_{yy} +$$

$$\frac{\eta}{\gamma^3(t)} v_{yyy} + \frac{\delta}{\gamma^4(t)} v_{yyyy} = \tilde{f}(y,t); \tag{1.4}$$

$$v(0,t) = v(1,t) = v_{yy}(0,t) = v_{yy}(1,t) = 0, \tag{1.5}$$

$$v(y,0) = v_0(y) = u_0(\alpha_1(0) + y\gamma(0)), \tag{1.6}$$

where $\tilde{f}(y,t) \equiv f(x(y,t),t)$.

Because the transformation $(x,t) \leftrightarrow (y,t)$ is a diffeomorphism, by solving (1.4) – (1.6), we solve the problem (1.1) – (1.3).

To solve (1.4) – (1.6) we use the method of Faedo-Galerkin.

3. Strong solutions

Let $y \in (0,1),\ t \in (0,T)$ and $\tilde{Q} = (0,1) \times (0,T)$. We define $W_k(0,1)$ as the subspace of functions g from $H^k(0,1)$ such that

$$\left.\frac{\partial^{2j} g}{\partial y^{2j}}\right|_{y=0,1} = 0,\ j = 0, \cdots \left[\frac{k}{2}\right] - 1.$$

Theorem 2.1. Let $v_0 \in W_2(0,1)$. Then there exists a function $v(y,t)$,

$$v \in L^\infty(0,T;W_2(0,1)) \cap L^2(0,T;W_4(0,1)), \quad v_t \in L^2(D)$$

which is a unique strong solution to $(1.4) - (1.6)$.

Proof: Let $w_j(y)$ be eigenfunctions of the following problem

$$\begin{cases} w_{jyy} + \lambda_j w_j &= 0, \quad \text{in } (0,1), \\ w_j|_{y=0,1} &= 0. \end{cases} \tag{2.0}$$

It is known that $w_j(y)$ create a basis in W_k which is orthonormal in $L^2(0,1)$. We seek approximate solutions to $(1.4) - (1.6)$ in the form,

$$v^N(y,t) = \sum_{j=1}^{N} g_j^N(t) w_j(y),$$

where $g_j^N(t)$ are solutions of the following Cauchy problem for the normal system of N ordinary differential equations,

$$\begin{cases} (Lv^N, w_j)(t) = (\tilde{f}, w_j)(t), \quad (u,v)(t) = \int_0^1 u(y,t)v(y,t)dy, \\ \\ g_j^N(0) = (v_0, w_j), \quad j = 1, \cdots, N. \end{cases} \tag{2.1}$$

Obviously, solutions of (2.1) exist for some interval $(0, T_N)$. To prolong them to any interval $(0,T)$ and to pass to the limit as $N \to \infty$, we need a priori estimates.

Estimates

From now on, C represents any positive constant and C_ε is any positive constant depending on $\varepsilon > 0$.

Substituting in (2.1) v^N for w_j, we obtain the inequality,

$$\frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yy}^N(t)|^2 \le \frac{M}{\delta_0}|v_y^N(t)||v^N(t)|$$

$$+\frac{\beta}{\delta_0^2}|v_{yy}^N(t)||v^N(t)| + \frac{\eta}{\delta_0^3}|v_{yy}^N(t)||v_y^N(t)| + |\tilde{f(t)}||v^N(t)|. \tag{2.2}$$

Due to the Ehrling inequalities (see Adams [1]), for any $\varepsilon > 0$,

$$|v_y^N(t)| \le \varepsilon |v_{yy}^N(t)| + C_\varepsilon |v^N(t)|.$$

Then

$$|v_y^N(t)||v^N(t)| \le \varepsilon |v_{yy}^N(t)|^2 + C_\varepsilon |v^N(t)|^2$$

and

$$|v_{yy}^N(t)||v_y^N(t)| \le \varepsilon |v_{yy}^N(t)|^2 + C_\varepsilon |v^N(t)|^2.$$

Using the Young inequality, we rewrite (2.2) for any $\varepsilon > 0$ as follows,

$$\frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yy}^N(t)|^2$$

$$\le \frac{M}{\delta_0}\left[\frac{\varepsilon^2}{2}|v_{yy}^N(t)|^2 + (C_\varepsilon + \frac{1}{2})|v^N(t)|^2\right] + \frac{\beta}{\delta_0^2}\left[\varepsilon|v_{yy}^N(t)|^2 + C_\varepsilon|v^N(t)|^2\right]$$

$$+\frac{\eta}{\delta_0^3}\left[2\varepsilon|v_{yy}^N(t)|^2 + C_\varepsilon|v^N(t)|^2\right] + \frac{1}{2}|\tilde{f}(t)|^2 + \frac{1}{2}|v^N(t)|^2.$$

Rearranging terms, we can write this inequality as

$$\frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yy}^N(t)|^2 \le \left[\frac{M}{2\delta_0}\varepsilon^2 + \frac{\beta}{\delta_0^2}\varepsilon + \frac{2\eta}{\delta_0^3}\varepsilon\right]|v_{yy}^N(t)|^2$$

$$+\left[(C_\varepsilon + \frac{1}{2}) + \frac{\beta}{\delta_0^2}C_\varepsilon + \frac{\eta}{\delta_0^3}C_\varepsilon + \frac{1}{2}\right]|v^N(t)|^2 + \frac{1}{2}|\tilde{f}(t)|^2. \qquad (2.3)$$

Choosing $\varepsilon > 0$ such that

$$\frac{\delta}{\gamma^4(t)} - \left[\frac{M}{2\delta_0}\varepsilon^2 + \frac{\beta}{\delta_0^2}\varepsilon + \frac{2\eta}{\delta_0^3}\varepsilon\right] \ge \frac{\delta}{2\gamma^4(t)},$$

we obtain from (2.3),

$$\frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yy}(t)|^2 \le C(|v^N(t)|^2 + |\tilde{f}(t)|^2),$$

where C is a constant independent of N, v^N and t.

Integrating (2.3) over $[0,t],\ t < T$, we have by the Gronwall lemma,

$$|v^N(t)|^2 + \int_0^t |v_{yy}^N(\tau)|^2 d\tau \le C(|v_0|^2 + \|\tilde{f}\|^2_{L^2(Q)}), \quad \forall t \in (0,T). \qquad (2.4)$$

This estimate allows us to extend the local solution to the whole interval $[0,T]$. On the other hand, by Rolle's theorem,

$$v_y^N(y,t) = \int_\xi^y v_{ss}^N(s,t)ds$$

for some $\xi \in (0,1)$. Then

$$|v_y^N(t)|^2 \le |v_{yy}^N(t)|^2. \tag{2.5}$$

This and (2.4) imply

$$\int_0^t |v_y^N(\tau)|^2 d\tau \le C(|v_0|^2 + \|\tilde{f}\|^2_{L^2(\tilde{Q})}). \tag{2.6}$$

Estimate 2:

Multiplying Lv^N by $\lambda_j^2 g_j^N(t)$ and summing over $j=1,\cdots N$, we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}|v_{yy}^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yyyy}^N(t)|^2 \le \frac{2M}{\delta_0}|v_y^N(t)||v_{yyyy}^N(t)| + \frac{1}{\delta_0}|(v^N v_y^N, v_{yyyy}^N)(t)|$$

$$+\frac{\eta}{\delta_0^3(t)}|v_{yyy}^N(t)||v_{yyyy}^N(t)| + \frac{\beta}{\delta_0^2(t)}|v_{yy}^N(t)||v_{yyyy}^N(t)| + |\tilde{f}(t)||v_{yyyy}^N(t)|, \tag{2.7}$$

By the Ehrling inequalities,

$$|v_y^N(t)| \le \varepsilon|v_{yyyy}^N(t)| + C_\varepsilon|v^N(t)|,$$

$$|v_{yyy}^N(t)| \le \varepsilon|v_{yyyy}^N(t)| + C_\varepsilon|v^N(t)|, \quad \varepsilon > 0.$$

Using these and the Gagliardo-Nirenberg inequalities, we estimate the terms of (2.7) as follows,

$$\frac{1}{\delta_0}|(v^N v_y^N, v_{yyyy}^N)(t)| \le C|v^N(t)||v_y^N(t)|^{\frac{1}{2}}|v_{yy}^N(t)|^{\frac{1}{2}}|v_{yyyy}^N(t)|$$

$$\le C_\varepsilon(|v_y^N(t)|^2 + |v_{yy}^N(t)|^2) + \epsilon|v_{yyyy}^N(t)|^2\varepsilon^2 + \varepsilon|v_{yyyy}^N(t)|^2; \tag{2.8}$$

$$\frac{2M}{\delta_0}|v_y^N(t)||v_{yyyy}^N(t)| \le \varepsilon|v_{yyyy}^N(t)|^2 + C_\varepsilon|v_y^N(t)|^2; \tag{2.9}$$

$$\frac{\eta}{\delta_0^3(t)}|v_{yyy}^N(t)||v_{yyyy}^N(t)| \le C_\varepsilon|v_{yyy}^N(t)|^2 + \varepsilon|v_{yyyy}^N(t)|^2$$

$$\le C_\varepsilon|v^N(t)|^2 + 2\varepsilon|v_{yyyy}^N(t)|^2; \tag{2.10}$$

$$\frac{\beta}{\delta_0^2(t)}|v_{yy}^N(t)||v_{yyyy}^N(t)| \le C_\varepsilon |v_{yy}^N(t)|^2 + \varepsilon |v_{yyyy}^N(t)|^2; \tag{2.11}$$

$$|\tilde{f}(t)||v_{yyyy}^N(t)| \le C_\varepsilon |\tilde{f}(t)|^2 + \varepsilon |v_{yyyy}^N(t)|^2, \ \forall \varepsilon > 0. \tag{2.12}$$

Taking into account (2.4) and choosing ε sufficiently small, we obtain the inequality,

$$\frac{d}{dt}|v_{yy}^N(t)|^2 + |v_{yyyy}^N(t)|^2 \le C(|\tilde{f}(t)|^2 + |v_{yy}^N(t)|^2). \tag{2.13}$$

By the Gronwall lemma,

$$|v_{yy}^N(t)|^2 + \int_0^T |v_{yyyy}(\tau)|^2 d\tau \le C\left(|v_0|^2_{H^2(0,1)} + \|\tilde{f}\|^2_{L^2(\tilde{Q})}\right). \tag{2.14}$$

From estimates (2.4) and (2.14), we conclude that

$$v^N \ \text{ is bounded in } \ L^\infty(0,T;W_2(0,1) \cap L^2(0,T;W_4(0,1)). \tag{2.15}$$

On the other hand, from (2.1), we obtain

$$\int_0^t |v_\tau^N(\tau)|^2 d\tau \le \int_0^t \left[\frac{1}{\delta_0}|(v^N v_y^N, v_\tau^N)(\tau)| + \frac{2M}{\delta_0}|v_y^N(\tau)||v_\tau^N(\tau)| + \frac{\beta}{\delta_0}|(v_{yy}^N, v_\tau^N)(\tau)|\right.$$

$$\left. + \frac{\eta}{\delta_0^3}|v_{yyy}^N(\tau)||v_\tau^N(\tau)| + \frac{\delta}{\delta_0^4}|v_{yyyy}^N(\tau)||v_\tau^N(\tau)| + |\tilde{f}(\tau)||v_\tau^N(\tau)|\right] d\tau. \tag{2.16}$$

We estimate the first term in the right-hand side of (2.16) as follows,

$$\int_0^t \frac{1}{\delta_0}|(v^N v_y^N, v_\tau^N)(\tau)| d\tau \le C \int_0^t |v_y^N(\tau)|^{\frac{1}{2}} |v_{yy}^N(\tau)|^{\frac{1}{2}} |v^N(\tau)||v_\tau^N(\tau)| d\tau. \tag{2.17}$$

Taking into account (2.15), and (2.17) we get from (2.16),

$$\int_0^t |v_\tau^N(\tau)|^2 d\tau \le \varepsilon \int_0^t |v_\tau^N(\tau)|^2 d\tau + C_\varepsilon, \ \varepsilon > 0.$$

Then for $\varepsilon > o$ sufficiently small

$$v_t^N \ \text{ is bounded in } \ L^2(0,T;L^2(0,1)). \tag{2.18}$$

Estimates (2.15) and (2.18) allow us to pass to the limit in (2.1) as $N \to \infty$, and therefore to prove the existence result of Theorem 2.1

□

Uniqueness of strong solutions follows from uniqueness of weak solutions proved in Theorem 4.1.

4. Weak solutions

In this section we prove that if $v_0 \in L^2(0,1)$, that is $u_0 \in L^2(\alpha_1(0), \alpha_2(0))$, then $(1.4) - (1.6)$ has a unique weak solution.

Theorem 4.1. *Let $v_0 \in L^2(0,1)$ and $\tilde{f} \in L^2(0,T; H^{-2}(0,1)$. Then there exists a unique weak solution $v(y,t)$ for the problem ,*

$$Lv = \tilde{f}, \quad in\ L^2(0,T;H^{-2}(0,1),$$

$$v(0,t) = v(1,t) = v_{yy}(0,t) = v_{yy}(1,t) = 0, \quad t \in (0,T),$$

$$v(y,0) = v_0(y), \; y \in (0,1)$$

such that

$$v \in L^\infty(0,T,L^2(0,1)) \cap L^2(0,T;H^2(0,1)),$$

$$v_t \in L^2(0,T;H^{-2}(0,1)).$$

Proof: Taking into account density theorems, we can find sequences $\{v_0^\nu\}$ in $W_2 = H_0^1(0,1) \cap H^2(0,1)$, $f^\nu \in L^2(Q)$ which converge to v_0 in $L^2(0,1)$ and to $\tilde{f}$ in $L^2(0,T;H^{-2}(0,1))$ respectively.

By Theorem 3.1, for each ν we have a solution v^ν to the problem,

$$Lv^\nu = f^\nu \ \text{ in } \tilde{Q}, \tag{3.1}$$

$$v^\nu(0,t) = v^\nu(1,t) = v^\nu_{yy}(0,t) = v^\nu_{yy}(1,t) = 0, \quad t \in [0,T], \tag{3.2}$$

$$v^\nu(y,o) = v_0^\nu(y), \quad y \in (0,1). \tag{3.3}$$

Multiplying equation (3.1) by $v^\nu(t)$, and proceeding as in section 2, we obtain the estimate,

$$|v^\nu(t)|^2 + \int_0^T |v^\nu_{yy}(\tau)|^2 d\tau \le C\left(|v_0^\nu|^2 + \|f^\nu\|^2_{L^2(\tilde{Q})}\right). \tag{3.4}$$

Therefore,

$$v^\nu \text{ is bounded in } L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;W_2) \tag{3.5}$$

uniformly in ν. Now we can estimate v_t^ν directly from (3.1) and obtain that

$$v_t^\nu \text{ is bounded in } L^2(0,T,H^{-2}(0,1)). \tag{3.6}$$

Taking into account compactness arguments and embedding theorems, we can see that v^ν converges strongly in $L^2(Q)$. Therefore, there exists a subsequence of $\{v^\nu\}$ which converges a.e. in Q. Then $v^\nu v_x^\nu$ converges to vv_x in the sense of distributions in Q. From (3.5) and (3.6), we conclude that

$$Lv = v_t + \frac{1}{\gamma(t)}vv_y - \frac{(y\gamma'(t)+\alpha_1'(t))}{\gamma(t)}v_y + \frac{\beta}{\gamma^2(t)}v_{yy} +$$

$$\frac{\eta}{\gamma^3(t)}v_{yyy} + \frac{\delta}{\gamma^4(t)}v_{yyyy} = \tilde{f}, \quad \text{in } L^2(0,T;H^{-2}(0,1)) \tag{3.7}$$

$$v(y,0) = v_0(y), \quad y \in (0,1). \tag{3.8}$$

Proof of uniqueness. Let v_1, v_2 be two solutions of (3.7)$-$(3.8) corresponding to the same initial data v_0, and $z = v_1 - v_2$.

Obviously,

$$z \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)),$$

$$z_t \in L^2(0,T;H^{-2}(0,1))$$

and

$$\int_0^t (z_\tau, w)(\tau)d\tau + \int_0^t \frac{1}{\gamma(\tau)}\left([v_1v_{1y} - v_2v_{2y}], w\right)(\tau)d\tau$$

$$-\int_0^t \left([\frac{(y\gamma'(\tau)+\alpha_1'(\tau)}{\gamma(\tau)}z_y - \frac{\beta}{\gamma^2(\tau)}z_{yy}], w\right)(\tau)d\tau - \int_0^t \frac{\eta}{\gamma^3(\tau)}(z_{yy}, w_y)(\tau)d\tau$$

$$+\int_0^t \frac{\delta}{\gamma^4(\tau)}(z_{yy}, w_{yy})(\tau)d\tau = 0,$$

where w is an arbitrary function from $L^2(0,T;W_2(0,1))$. Replacing w by z, we obtain the equality,

$$|z(t)|^2 + \int_0^t ([v_1^2 - v_2^2]_y, z)(\tau)d\tau + \int_0^t (\frac{\gamma'(\tau)}{\gamma(\tau)}|z(\tau)|^2 d\tau - 2\int_0^t \frac{\beta}{\gamma^2(\tau)}|z_y(\tau)|^2 d\tau$$

$$-2\int_0^t \frac{\eta}{\gamma^3(\tau)}(z_{yy}, z_y)(\tau)d\tau + 2\int_0^t \frac{\delta}{\gamma^4(\tau)}|z_{yy}(\tau)|^2 d\tau = 0. \tag{3.9}$$

Because

$$|([v_1^2 - v_2^2]_y, z)(t)| = |([v_1^2 - v_2^2], z_y)(t)| = |(z[v_1 + v_2], z_y)(t)|$$

$$\leq \max_{y \in [0,1]} |v_1(t) + v_2(t)||z(t)||z_y(t)| \leq C(|v_{1y}(t)| + |v_{2y}(t)|)|z(t)||z_y(t)|,$$

using the inequalities of Young and Ehrling, we obtain from (3.9),

$$|z(t)|^2 + 2\delta \int_0^t \frac{1}{\gamma^4(\tau)} |z_{yy}(\tau)|^2 d\tau \leq \varepsilon \int_0^t |z_{yy}(\tau)|^2 d\tau$$

$$+ C_\varepsilon \int_0^t (|v_{1y}(\tau)|^2 + |v_{2y}(\tau)|^2 + 1)|z(\tau)|^2 d\tau,$$

where ε is an arbitrary positive number. Choosing ε sufficiently small, we obtain the inequality,

$$|z(t)|^2 \leq C \int_0^t (1 + |v_{1y}(\tau)|^2 + |v_{2y}(\tau)|^2)|z(\tau)|^2 d\tau.$$

By Gronwall's lemma, $|z(t)| = 0$. This proves the uniqueness result of Theorem 4.1

$\square$

5. Stability

It is well-known that solutions of a parabolic equation

$$v_t + Av = 0$$

are stable as $t \to \infty$, provided A is a positive operator. In our case, A is nonlinear and depends on the parameters $\eta, \gamma(t), \beta, \delta$. But it is possible to find sufficient conditions which guarantee the asymptotic decay of $v(y, t)$.

Theorem 5.1. *Let $v(y, t)$ be a strong solution to problem $(1.4) - (1.6)$ and assume that for large t the following conditions hold*

5.1) $\sup_{t \in \mathbf{R}^+}(\gamma(t)) \leq \gamma_0 < \infty$,

5.2) $\delta - \gamma^2(t)\beta - \gamma(t)\eta \geq \sigma > o$,

5.3) $2\lambda_1\sigma + \gamma^3(t)\gamma'(t) \geq \nu > 0$,

5.4) $\int_0^t e^{\theta\tau} |\tilde{f}(\tau)|^2 d\tau \leq C e^{\theta_1 t}, \ \theta_1 \in [0, \frac{\nu}{\gamma_0^4})$, *where* λ_1 *is the first eigenvalue of the Dirichlet problem* (2.0). *Then there exist constants* $K, \lambda > 0$ *such that*

$$|v(t)|^2 \le Ke^{-\lambda t}, \quad \forall\, t > 0.$$

Proof: Multiplying equation (1.4) by v, we obtain the equality,

$$\frac{d}{dt}|v(t)|^2 + (\frac{\gamma'}{\gamma}, v^2)(t) - \frac{2\eta}{\gamma^3(t)}(v_{yy}, v_y)(t)$$

$$+\frac{2\delta}{\gamma^4(t)}|v_{yy}|^2 - \frac{2\beta}{\gamma^2(t)}|v_y(t)|^2 = 2(\tilde{f}, v)(t). \tag{5.1}$$

Using (2.5), we

get from (5.1),

$$\frac{d}{dt}|v(t)|^2 + \frac{\gamma'(t)}{\gamma(t)}|v(t)|^2 + \frac{2}{\gamma^4(t)}\left(\delta - \gamma^2(t)\beta - \gamma(t)\eta\right)|v_{yy}(t)|^2 \le 2|\tilde{f}(t)||v(t)|$$

which can be rewritten as follows,

$$\frac{d}{dt}|v(t)|^2 + \frac{\gamma'(t)}{\gamma(t)}|v(t)|^2 + \frac{2\sigma}{\gamma^4(t)}|v_{yy}(t)|^2 \le 2|\tilde{f}(t)||v(t)|. \tag{5.2}$$

If λ_1 is the first eingenvalue of (2.0), then

$$|v_{yy}(t)|^2 \ge \lambda_1 |v(t)|^2.$$

From (5.2), we obtain

$$\frac{d}{dt}|v(t)|^2 + \left(\frac{2\sigma\lambda_1}{\gamma^4(t)} + \frac{\gamma'(t)}{\gamma(t)}\right)|v(t)|^2 \le \varepsilon|v(t)|^2 + C_\varepsilon|\tilde{f}(t)|^2, \ \forall \varepsilon > 0.$$

Putting $\varepsilon = \frac{\nu}{2\gamma_0^4}$ and taking into account the condition 5.3 of Theorem 5.1, we obtain the inequality,

$$\frac{d}{dt}|v(t)|^2 + \theta|v(t)|^2 \le C(\theta)|\tilde{f}(t)|^2,$$

where $\theta = \frac{\nu}{2\gamma_0^4}$. Solving this inequality, we obtain

$$|v(t)|^2 \le C\left(\int_0^t e^{\theta\tau}|\tilde{f}(\tau)|^2 d\tau + |v_0|^2\right)e^{-\theta t}.$$

By the condition 5.4 of Theorem 5.1, there exist $K, \lambda > 0$ such that

$$|v(t)|^2 \leq Ke^{-\lambda t}.$$

We proved our results on existence, uniqueness and stability for the transformed problem (1.4) – (1.6). Because the transformation $(x, t) \leftrightarrow (y, t)$ is diffeomorphism, the same results are valid for the original problem (1.1) – (1.3).

The authors are grateful to the reviewer for the contructive remarks.

References

[**1**] Adams, R. A., *Sobolev Spaces,* Academic Press, (1975).

[**2**] Biagioni, H. A.; J.L. Bona; Iorio Jr., R. J.; Scialom, M., *On the Kortweg-de Vries-Kuramoto-Sivashinsky Equation,* Adv. Diff. Eqs. 1, No. 1, (1996) 1-20.

[**3**] Biagioni, H. A.; Gramchev, T., *Multidimensional Kuramoto-Sivashinsky type equations: Singular initial data and analytic regularity,* Matemática Contemporânea 15, (1998) 21-42.

[**4**] Cross, M. C., *Pattern formation outside of equilibrium,* Reviews of Modern Physics, 65, No. 3, (1993) 851-1086.

[**5**] Kuramoto, Y.; Tsuzuki, T., *On the formation of dissipative strutures in reaction-diffusion systems* , Prog. Theor. Phys., 54, (1975) 687-699.

[**6**] Sivashinsky, G. I., *Nonlinear analysis of hydrodinamic instability in laminar flames - I. Derivation of basic equations.,* Acta Astronautica, 4, (1977) 1177-1206.

[**7**] Tadmor, E., *The well-posedness of the Kuramoto-Sivashinsky equation,* SIAM J. Math. Anal., 17, No. 4, (1986) 884-893.

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