Matemática Contemporânea, Vol. 18, 31–42
http://doi.org/10.21711/231766362000/rmc182

SEMIGLOBAL SOLVABILITY OF A CLASS OF PLANAR VECTOR FIELDS OF INFINITE TYPE

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Abstract

We consider a class of planar vector fields having the unit circle as characteristic set. We assume that each vector field L is of infinite type on the unit circle. We study the C^ω and C^∞ solvability of the equation $Lu = f$ near the characteristic set.

Resumo

Consideramos uma classe de campos vetoriais no plano tendo a circunferência unitária como conjunto característico. Supomos que cada campo vetorial L é de tipo infinito nos pontos do conjunto característico. Estudamos a resolubilidade C^ω and C^∞ da equação $Lu = f$ perto do conjunto caracteristico.

1. Introduction

In this note we study the C^ω and C^∞ solvability of the equation

$$T_\lambda u = f \tag{1.1}$$

near the characteristic circle $\Sigma = \{0\} \times S^1 \subset \mathbb{R} \times S^1$, where

$$T_\lambda = \lambda \frac{\partial}{\partial \theta} - ir\frac{\partial}{\partial r}, \tag{1.2}$$

$\lambda = a + ib \in \mathbb{R}^* + i\mathbb{R}$, and (r, θ) are the coordinates of $\mathbb{R} \times S^1$. Note that since the change of variables $r' = r$, $\theta' = -\theta$ transforms T_λ into $T_{-\lambda}$, then it suffices to study (1.1) only for $\mathrm{Re}\lambda > 0$. The vector field T_λ is of infinite type along Σ and it satisfies the Nirenberg-Treves condition (P). It follows from classical results (see [NT], [T1] or [T2]) that for a given C^ω (resp. C^∞) function f and for every $p \in \Sigma$, there exist an open set $U \ni p$, $U \subset \mathbb{R} \times S^1$, and a function $u \in C^\omega(U)$

(resp. $u \in C^\infty(U)$) such that (1.1) holds in U. The problem is therefore relevant when one seeks solutions in a full neighborhood of Σ. Equation (1.1) has in fact been studied ([BM2]) in the C^0 category. By using Fourier series, it is easy to see that the period of f on Σ must be zero in order for a C^0 solution to exist. Thus from now on, we will assume that f satisfies

$$\int_0^{2\pi} f(0,\theta)d\theta = 0. \tag{1.3}$$

Equation (1.1) has also been studied in [M3]. It is proved, in particular, that if $\lambda \notin \mathbb{Q}$, f is C^∞ and satisfies (1.3), then for every $k \in \mathbb{Z}^+$, equation (1.1) has a C^k solution defined in a neighborhood of Σ.

In section 2, we study the problem of finding analytic solutions when f is analytic. It turns out that analytic solutions exist for every analytic function f satisfying (1.3) if and only if λ is not well approximable by rational numbers (exponential Liouville number). In section 3, we prove that for every λ, there exist C^∞ functions f satisfying (1.3) such that equation (1.1) has no C^∞ solutions. The C^∞ solvability in the region $r \geq 0$ is also addressed. The approach and the motivation for this work are related to those contained in the papers [B1,2], [BCH], [BCM], [BHS], [BM 1,2], [CH], [M1,2,3], [T1,2] and in many others.

This note grew out of discussions between the authors and by a question of Paulo Cordaro during the VI Workshop on PDE at IMPA. The authors are grateful for the support provided by the Brazilian agencies CNPq, FAPESP and FINEP.

2. Analytic solvability and Liouville numbers

We seek analytic solvability of (1.1) when f is real analytic and satisfies (1.3). The solvability will depend on whether the invariant λ satisfies a diophantine condition. It must be noted that, when $\lambda \notin \mathbb{R}$, the analytic solvability is contained in [BM2] and that, when $\lambda = a \in \mathbb{Q}^+$, there are analytic functions f satisfying (1.3) for which (1.1) does not have analytic solutions (in fact not even formal solutions). The only remaining case is therefore when $\lambda = a \in \mathbb{R}^+\backslash\mathbb{Q}$.

It turns out that (1.1) is solvable in the analytic category if and only if λ is not too well approximable by rationals (see [G] and [B1], [B2]). We now describe a diophantine condition for $a \in \mathbb{R}^+$

$$\begin{aligned}
&(\mathrm{DC})_1 \qquad \exists C>0 \ \ |\exp(i\tfrac{2\pi j}{a})-1| \geq C^{j+1} \quad \forall j \in \mathbb{Z}^+. \\
&(\mathrm{DC})_2 \qquad \exists C>0 \ \ |j+ak| \geq C^{j+1} \quad \forall j \in \mathbb{Z}^+ \forall k \in \mathbb{Z}.
\end{aligned}$$

Lemma 2.1. *For $a \in \mathbb{R}^+ \backslash \mathbb{Q}$, the diophantine conditions $(DC)_1$ and $(DC)_2$ are equivalent.*

Proof: Suppose that (DC_2) does not hold. Then,

$$\forall l \in \mathbb{Z}^+,\ \exists j_l \in \mathbb{Z}^+,\ k_l \in \mathbb{Z}\ :\ \ |j_l + ak_l| < l^{-(j_l+1)}.$$

For each l we have then

$$\begin{aligned}
\left|\exp(i\tfrac{2\pi j_l}{a})-1\right|^2 &= \left|\exp(i\tfrac{2\pi(j_l+ak_l)}{a})-1\right|^2 \\
&= 2\left(1-\cos\tfrac{2\pi(j_l+ak_l)}{a}\right) \\
&= 2\tfrac{2\pi(j_l+ak_l)}{a}\sin\theta_l
\end{aligned}$$

for some θ_l. It follows immediately that (DC_1) does not hold.

Conversely, suppose that (DC_1) does not hold. Then

$$\forall l \in \mathbb{Z}^+,\ \exists j_l \in \mathbb{Z}^+\ :\ \ \left|\exp(i\frac{2\pi j_l}{a})-1\right| < l^{-(j_l+1)}.$$

For each l, let $k_l = [\frac{j_l}{a}]$ be the integral part of $\frac{j_l}{a}$. It follows from the above assumption that

$$\left|\exp(i\frac{2\pi(j_l-ak_l)}{a})-1\right| < l^{-(j_l+1)}.$$

Since $0 \leq \frac{j_l}{a} - k_l < 1$, we have $\lim(j_l - ak_l) = 0$. Consequently,

$$\left|\exp(i\frac{2\pi(j_l-ak_l)}{a})-1\right|^2 = 2\left(1-\cos(\frac{2\pi(j_l-ak_l)}{a})\right) \geq (\frac{2\pi(j_l-ak_l)}{a})^2,$$

for l large, and thus (DC_2) does not hold.

$\square$

As in [B2], we make the following definition for an irrational number. An irrational number α is said to be an exponential Liouville number if there exists $\epsilon > 0$ such that the inequality

$$|\alpha - \frac{p}{q}| \leq \exp(-\epsilon q) \tag{2.1}$$

has infinitely many rational solutions p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$. The following lemmas are easy to prove.

Lemma 2.2. *The number $a \in \mathbb{R}^+ \backslash \mathbb{Q}$ is an exponential Liouville number if and only if the diophantine condition (DC) does not hold.*

Lemma 2.3. *The number $a \in \mathbb{R}^+ \backslash \mathbb{Q}$ satisfies (DC) if and only if $\frac{1}{a}$ satisfies (DC).*

For a construction of exponential Liouville numbers by means of continued fractions see [B2] and [GPY]. In [G] one finds an example of a Liouville number which is not exponential Liouville; recall that an irrational number α is said to be a Liouville number if for every positive integer N there exists $K \geq 0$ such that the inequality $|\alpha - p/q| \leq K q^{-N}$ has infinitely many rational solutions p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$

We are now ready to state the main result of this section.

Theorem 2.1. *Let $\lambda = a \in \mathbb{R}^+ \backslash \mathbb{Q}$. Equation (1.1) has a real analytic solution u defined near Σ for every real analytic function f satisfying (1.3) if and only if the invariant a is not an exponential Liouville number.*

Proof: Let $f(r, \theta)$ be a real analytic function defined near Σ. We use the Taylor expansion about $r = 0$ and write

$$f(r, \theta) = \sum_{j \geq 0} f_j(\theta) \frac{r^j}{j!}, \tag{2.2}$$

where each $f_j \in C^\omega(S^1)$. If $u(r, \theta)$ is a real analytic solution of (1.1), then

$$u(r, \theta) = \sum_{j \geq 0} u_j(\theta) \frac{r^j}{j!}, \tag{2.3}$$

where each $u_j \in C^\omega(S^1)$ satisfies the ode

$$a\frac{du_j}{d\theta} - iju_j = f_j. \tag{2.4$_j$}$$

Note that since f satisfies (1.3), i.e.,

$$\int_0^{2\pi} f_0(\theta)d\theta = 0, \tag{2.5}$$

each equation $(2.4)_j$ can be solved with

$$u_0(\theta) = \frac{1}{a}\int_0^\theta f_0(s)ds + K_0, \tag{2.6$_0$}$$

where $K_0 \in \mathbb{C}$ is arbitrary, and for $j > 0$, the solution u_j is uniquely determined by the formula

$$u_j(\theta) = \frac{1}{a}\int_0^\theta \exp[i\frac{j(\theta - s)}{a}]f_j(s)ds \,+\, K_j \exp(i\frac{j\theta}{a}), \tag{2.6$_j$}$$

where

$$K_j = \left[1 - \exp(i\frac{2\pi j}{a})\right]^{-1} \frac{1}{a}\int_0^{2\pi} \exp[i\frac{j(2\pi - s)}{a}]f_j(s)ds. \tag{2.7$_j$}$$

Note that K_j is well defined since $a \notin \mathbb{Q}$.

For a real analytic function $f(r,\theta)$ as in (2.3) and satisfying (2.5), the series (2.3), where u_j is given by (2.6_j), defines a formal solution of (1.1). We will show that when the invariant a satisfies (DC), the series (2.3) defines a real analytic function near Σ. Before we proceed with the estimation of $|u_j(\theta)|$, recall that since f is real analytic, then there exists $C_1 > 0$ such that

$$|f_j(\theta)| \le C_1^{j+1}, \quad \forall j \in \mathbb{Z}^+,\ \forall \theta \in S^1. \tag{2.8$_j$}$$

In view of $(DC)_1$, we have

$$\left|\left[1 - \exp(i\frac{2\pi j}{a})\right]^{-1}\right| \le C^{-(j+1)} \quad \forall j \in \mathbb{Z}^+. \tag{2.9$_j$}$$

We also have

$$\left|\frac{1}{a}\int_0^{2\pi} \exp[i\frac{j(2\pi - s)}{a}]f_j(s)ds \exp(i\frac{j\theta}{a})\right| \le \frac{2\pi}{a}C_1^{j+1} \le C_2^{j+1}, \quad \forall j \in \mathbb{Z}^+, \tag{2.10$_j$}$$

for some $C_2 > 0$. Similarly, we can show that

$$\left| \frac{1}{a} \int_0^\theta \exp[i\frac{j(\theta - s)}{a}] f_j(s) ds \right| \leq C_3^{j+1}, \quad \forall j \in \mathbb{Z}^+, \qquad (2.11_j)$$

for some $C_3 > 0$. Hence, it follows at once from (2.6), (2.9), (2.10), and (2.11), that

$$|u_j(\theta)| \leq C^{-(j+1)} C_2^{j+1} + C_3^{j+1} \leq C_4^{j+1}, \qquad (2.12_j)$$

for some $C_4 > 0$. If we let $\hat{\theta} = \theta + i\theta'$ and $\hat{r} = r + ir'$ with $\theta', r' \in \mathbb{R}$, $|\theta'| < \epsilon$, $|r'| < \epsilon$, then it follows, after complexifying f and the formal series (2.3), that estimates (2.12_j) remain valid for the complexified functions $u_j(\hat{\theta})$ (with possibly different constants and small ϵ). It follows then that the series $\sum u_j(\hat{\theta})\hat{r}^j/j!$ converges uniformly to a holomorphic function in the variables $(\hat{\theta}, \hat{r})$. The restriction of this holomorphic function to $\theta' = r' = 0$ defines a real analytic function near Σ.

We prove the necessity of (DC) by contradiction. When (DC) fails, we construct a real analytic function f satisfying (1.3) such that equation (1.1) has no real analytic solution. Suppose then that $(DC)_2$ does not hold. We can find a sequence

$$\{(j_l, k_l)\}_{l \in \mathbb{Z}^+} \subset \mathbb{Z}^+ \times \mathbb{Z}^+$$

such that

$$|j_l - ak_l| < l^{-(j_l+1)} \quad \forall l \in \mathbb{Z}^+. \qquad (2.13)$$

We have then $\lim(j_l/k_l) = a$ and this implies that we may take j_l to be an increasing sequence with $\lim j_l = \infty$. We can also assert the existence of constants $c_1, c_2 > 0$ such that

$$c_1 j_l \leq k_l \leq c_2 j_l, \quad \forall l \in \mathbb{Z}^+. \qquad (2.14)$$

Now set

$$f(r, \theta) = \sum_{l \geq 0} \mathrm{e}^{ik_l\theta} r^{j_l}. \qquad (2.15)$$

Condition (1.3) is trivially satisfied and f is continuous near Σ since $|\exp(ik_l\theta)| = 1$ for each l. Condition (2.14) guarantees that f is also real analytic. To see why, we can complexify r and θ into $\hat{r} = r + ir'$, $\hat{\theta} = \theta + i\theta'$ with $|r'| < \epsilon$, $|\theta'| < \epsilon$ and $\epsilon > 0$ small enough. The series

$$\sum_{l\geq 0} \mathrm{e}^{ik_l\hat{\theta}}\hat{r}^{j_l}$$

defines a continuous function near $\Sigma \subset \mathbb{C}^2$ and it is holomorphic with respect to $(\hat{r}, \hat{\theta})$. The holomorphy of the series follows from the estimates

$$\begin{aligned}\left|\sum_{l\geq 0} \mathrm{e}^{ik_l\hat{\theta}}\hat{r}^{j_l}\right| &\leq \sum_{l\geq 0} \mathrm{e}^{-\theta' k_l}|\hat{r}^{j_l}| \leq \sum_{l\geq 0} \mathrm{e}^{\epsilon k_l}|2\epsilon|^{j_l} \\ &\leq \sum_{l\geq 0} \mathrm{e}^{\epsilon(k_l - a^{-1}j_l)}|2\epsilon \mathrm{e}^{a^{-1}\epsilon}|^{j_l} \leq C\sum_{l\geq 0} |2\epsilon \mathrm{e}^{a^{-1}\epsilon}|^{j_l},\end{aligned}$$

for some positive constant C (since $\lim(ak_l - j_l) = 0$).

We claim that for such a function f, equation (1.1) has no real analytic solution. Indeed, if $u(r,\theta)$ were such a C^ω solution, then a straightforward computation would give

$$u(r,\theta) = K + i\sum_{l\geq 0} \frac{1}{j_l - ak_l}\mathrm{e}^{ik_l\theta}r^{j_l}, \tag{2.16}$$

with $K \in \mathbb{C}$. But the above series has radius of convergence equal to 0 since it follows from (2.13) that

$$\left|\sum_{l\geq 0} \frac{1}{j_l - ak_l}\mathrm{e}^{ik_l\theta}r^{j_l}\right| \geq \sum_{l\geq 0} l^{j_l+1}|r|^{j_l}. \tag{2.17}$$

3. Nonexistence of C^∞ solutions

We prove here that in general equation (1.1) does not have C^∞ solutions. More precisely, we have the following theorem.

Theorem 3.1. *Let $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$. Then there exists C^∞ functions f satisfying (1.3) so that equation (1.1) does not have C^∞ solutions in any neighborhood of Σ.*

Proof: For $\epsilon > 0$, let

$$\begin{aligned} A_\epsilon &= \{(r,\theta) \in \mathbb{R} \times S^1 :\ |r| < \epsilon\}; \\ A_\epsilon^+ &= \{(r,\theta) \in A_\epsilon :\ r > 0\}; \\ A_\epsilon^- &= \{(r,\theta) \in A_\epsilon :\ r < 0\}; \end{aligned} \tag{3.1}$$

The function $z = r^\lambda e^{i\theta}$ is a first integral of T_λ in A_ϵ^+ and in A_ϵ^-. It is continuous on A_ϵ and maps $\overline{A_\epsilon^+}$ and $\overline{A_\epsilon^-}$ onto the disc $\overline{D(0,\epsilon^a)}$, sending Σ into 0. Since

$$T_\lambda \overline{z} = -2ia\overline{z}, \tag{3.2}$$

then the pushforward of the equations

$$T_\lambda u = f \quad \text{in} \quad A_\epsilon^\pm \tag{3.3}$$

give rise to the inhomogeneous CR equations

$$-2ia\overline{z}\frac{\partial \hat{u}^\pm}{\partial \overline{z}} = \hat{f}^\pm(z) \quad \text{in} \quad D(0,\epsilon^a)\backslash 0, \tag{3.4}$$

where $\hat{f}^\pm$ and $\hat{u}^\pm$ are the pushforwards of f and u via the map z. Let

$$\sum_{j\geq 0} \alpha_j z^j \tag{3.5}$$

be a series in one variable, with coefficients in $\mathbb{C}$, and with radius of convergence equal to zero. Let $g(z)$ be a C^∞ function in the disc $D(0,\epsilon^a)$ whose Taylor series at 0 coincides with the series (3.5). The function $\frac{\partial g(z)}{\partial \overline{z}}$ is then a C^∞ function in $D(0,\epsilon^a)$ and is flat at 0 (its partial derivatives of all orders vanish at 0). Define a function $f(r,\theta)$ in A_ϵ by

$$\begin{aligned} f(r,\theta) &= -2iar^{\overline{\lambda}}e^{-i\theta}\tfrac{\partial g(z)}{\partial \overline{z}}(r^\lambda e^{i\theta}) \quad \text{for } r \geq 0 \quad \text{and} \\ f(r,\theta) &= 0 \quad \text{for } r < 0. \end{aligned} \tag{3.6}$$

The function f is C^∞ in A_ϵ and is flat along the circle Σ. Condition (1.3) is trivially satisfied for this function.

Now we claim that for such a function f, equation (1.1) does not have a C^∞ solution in any neighborhood of Σ. To see why, we transfer equations (3.3) via the first integral z and get the CR equations

$$\frac{\partial \hat{u}^+}{\partial \overline{z}} = \frac{\partial g(z)}{\partial \overline{z}} \quad \text{and} \quad \frac{\partial \hat{u}^-}{\partial \overline{z}} = 0 \quad \text{in } D(0,\epsilon^a). \tag{3.7}$$

It follows at once that

$$\hat{u}^+(z) = g(z) + h^+(z) \quad \text{and} \quad \hat{u}^-(z) = h^-(z),$$

where $h^{\pm}$ are holomorphic functions in the disc. Hence any solution of (1.1) in A_ϵ has the form

$$\begin{aligned} u(r,\theta) &= g(r^\lambda \mathrm{e}^{i\theta}) + h^+(r^\lambda \mathrm{e}^{i\theta}) \qquad && \text{for } r \geq 0 \\ u(r,\theta) &= h^-(r^\lambda \mathrm{e}^{i\theta}) && \text{for } r < 0 \end{aligned} \tag{3.8}$$

But such a function cannot be C^∞ on Σ since the Taylor series at 0 of g diverges and thus cannot be equal to that of the holomorphic function $(h^- - h^+)$

□

The above construction can be used to show the nonexistence of C^∞ solutions in $\overline{A}_\epsilon^+$ when $\lambda \notin \mathbb{Z}$.

Theorem 3.2. *If $\lambda = a + ib \in \mathbb{R}^+ + i\mathbb{R}$ and $\lambda \notin \mathbb{Z}$, then there exist C^∞ functions f in A_ϵ satisfying (1.3) such that there is no C^∞ function u defined in A_ϵ and satisfying*

$$T_\lambda u = f \quad \text{in } A_\epsilon^+. \tag{3.9}$$

Proof: First note that if $\lambda \notin \mathbb{Q}$, then

$$r^{\lambda j} \mathrm{e}^{ij\theta} \notin C^\infty(A_\epsilon) \quad \forall j \in \mathbb{Z}^+, \tag{3.10}$$

and if $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ with p, q relatively prime and $q > 1$, then

$$r^{\lambda j} \mathrm{e}^{ij\theta} \notin C^\infty(A_\epsilon) \quad \forall j \in \mathbb{Z}^+ \backslash q\mathbb{Z}^+. \tag{3.11}$$

Let f and g be as in the proof of Theorem 3.1. If $u \in C^\infty(A_\epsilon)$ satisfies (3.9), then

$$u(r,\theta) = g(r^\lambda \mathrm{e}^{i\theta}) + h^+(r^\lambda \mathrm{e}^{i\theta}) \quad \text{in } A_\epsilon^+ \tag{3.12}$$

with h^+ holomorphic.

Now suppose that $\lambda \notin \mathbb{Q}$. We show that any function u defined by (3.12) cannot be C^∞ on Σ. For this, it suffices to notice that since the series (3.5) has radius of convergence 0, then for every holomorphic function

$$h^+(z) = \sum_{j \geq 0} c_j z^j \tag{3.13}$$

there exists $n > 0$ such that

$$\alpha_n + c_n \neq 0. \tag{3.14}$$

Let $N \in \mathbb{Z}^+$ such that $N > n$. We have

$$u(r,\theta) = \sum_{j=0}^{N} (\alpha_j + c_j) r^{\lambda j} \mathrm{e}^{ij\theta} + O(|r|^{Na}). \tag{3.15}$$

It follows at once from (3.14) and (3.10) that

$$u \notin C^{[an]+1}(\overline{A}_\epsilon^+), \tag{3.16}$$

where $[x]$ denotes the integral part of the real number x. When $\lambda = p/q \in \mathbb{Q}^+ \backslash \mathbb{Z}$, the above argument works as well if one replaces the divergent series $\sum \alpha_j z^j$ by the series $\sum \alpha_j z^{qj+1}$ for example.

□

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