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GLOBAL HYPOELLIPTICITY FOR SUBLAPLACIANS

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Abstract

We discuss the open problem of global hypoellipticity for sublaplacians, which may not satisfy the bracket condition, and we prove global regularity for a new family of such operators.

Resumo

Neste trabalho tratamos o problema aberto da hipoeliticidade global para sublaplacianos, os quais podem não satisfazer a condição do colchete e provamos a regularidade global para uma nova familia de tais operadores.

1 Introduction and Main Result

Let $\mathcal{M}$ be a C^∞ (C^ω) manifold of dimension n and let $X = \{X_1, \dots, X_m\}$ be a collection of real C^∞ (C^ω) vector fields with coefficients defined on $\mathcal{M}$. Their *sum of squares operator* or *sublaplacian*, Δ_X, is the following second order operator

$$\Delta_X \doteq -(X_1^2 + \cdots + X_m^2).$$

Necessary and sufficient conditions for the C^∞ (C^ω) local or global hypoellipticity of Δ_X are open problems. We shall need the following definitions to be more precise and to be able to state our result clearly. We recall that the operator

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Δ_X is said to be hypoelliptic (analytic hypoelliptic) in $\mathcal{M}$ if for any open set $U \subset \mathcal{M}$ the conditions $u \in D'(U)$ and $\Delta_X u \in C^\infty(U)$ ($\Delta_X u \in C^\omega(U)$) imply that $u \in C^\infty(U)$ ($u \in C^\omega(U)$). The operator Δ_X is said to be globally hypoelliptic in $\mathcal{M}$ (globally analytic hypoelliptic) if the conditions $u \in D'(\mathcal{M})$ and $\Delta_X u \in C^\infty(\mathcal{M})$ ($\Delta_X u \in C^\omega(\mathcal{M})$) imply that $u \in C^\infty(\mathcal{M})$ ($u \in C^\omega(\mathcal{M})$). Observe that hypoellipticity (analytic hypoellipticity) implies global hypoellipticity (global analytic hypoellipticity). Also, we recall that a point $x_0 \in \mathcal{M}$ is said to be of finite type if the Lie algebra generated by the vector fields $X_1, \ldots, X_m$ span the tangent space of $\mathcal{M}$ at x_0. Otherwise it is said to be of infinite type. By the celebrated theorem of Hörmander [19] (see also Kohn [20], Oleinik and Radkevic [22], and Rothschild and Stein [24]) the finite type condition is sufficient for the hypoellipticity of Δ_X. Therefore the finite type condition is also sufficient for the global hypoellipticity of Δ_X. In the analytic category Derridj [7] proved that the finite type condition is also necessary for hypoellipticity. Baouendi and Goulaouic [2] proved that the finite type condition is not sufficient for the analytic hypoellipticity of Δ_X. Many authors, including Helffer [14], Pham The Lai and Robert [23], Metivier [21], Hanges and Himonas [12], [13] and Christ [4], [5], found different classes of operators satisfying the finite type condition and failing to be analytic hypoelliptic (see Treves [26] for a survey, and conjectures). While, Amano [1], Fujiwara and Omori [9], Gramchev, Popivanov and Yoshino [10], Greenfield and Wallach [11], Bell and Mohammed [3], Fedii [8], Himonas [15], Himonas and Petronilho [16], [17], [18], and many other authors, found different classes of operators that are locally or globally hypoelliptic but the finite type condition does not hold. A well studied model in $\mathbb{T}^3$ is the operator P defined by

$$P = -\partial_t^2 - (\partial_{x_1} + a(t)\partial_{x_2})^2.$$

Cordaro and Himonas [6] proved that P is globally analytic hypoelliptic in $\mathbb{T}^3$ if the function a is in $C^w(\mathbb{T}^1)$, real valued, and not constant on $\mathbb{T}^1$ (see also Tartakoff [25]). In [5] Christ proved that P is not analytic hypoelliptic near 0 for any analytic function a with $a(0) = a'(0) = 0$. In [15] it has been shown that if $a \in C^\infty(\mathbb{T}^1)$ is real valued then P is globally hypoelliptic in $\mathbb{T}^3$ if and only if the range of the function a contains a non-Liouville number. Then in [18] the analogue of this operator was studied in higher dimensions. More precisely, it was shown that the operator

$$P = -\Delta_t - \left(\sum_{j=1}^{n} a_j(t) \partial_{x_j} \right)^2,$$

where $(t_1, \ldots, t_m, x_1, \ldots, x_n) = (t, x) \in \mathbb{T}^{m+n}$ and $a_j, j = 1, \ldots, n$, are real-valued functions in $C^\infty(\mathbb{T}^m)$, is globally hypoelliptic in $\mathbb{T}^{m+n}$ if and only if a Diophantine condition on the coefficients is satisfied.

In the rest of this article we study the global hypoellipticity of a class of operators with coefficients depending on more variables than those in the operator P given above. For simplicity we shall consider the case of $\mathbb{T}^3$ only.

Theorem. *Let P be the operator*

$$P = -\partial_{t_1}^2 - \partial_{t_2}^2 - (\partial_{t_2} + a(t_1, t_2)\partial_x)^2,$$

where $(t_1, t_2, x) \in \mathbb{T}^3$ and $a \in C^\infty(\mathbb{T}^2)$ is real-valued. Then the operator P is globally hypoelliptic in $\mathbb{T}^3$ if and only if the function a is not identically zero.

2 Proof of Theorem

Since it is easy to check the necessity, we shall only present the proof of the sufficiency. If the function $a(t_1, t_2)$ is a non-zero constant then the operator P is elliptic and therefore it is locally and globally hypoelliptic in $\mathbb{T}^3$. Therefore, we shall assume that $a(t_1, t_2)$ is non-constant. Then, there exists a point t^0, which we may take to be $t^0 = 0 = (0, 0)$, such that either $\frac{\partial a}{\partial_{t_1}}(0) \neq 0$ or $\frac{\partial a}{\partial_{t_2}}(0) \neq 0$. Thus there exists $\delta > 0$ such that either $\frac{\partial a}{\partial_{t_1}}(t) \neq 0,\ \ t \in [-\delta, \delta]^2$ or $\frac{\partial a}{\partial_{t_2}}(t) \neq 0,\ \ t \in [-\delta, \delta]^2$. Next, let $u \in D'(\mathbb{T}^3)$ be such that

$$Pu = f,\ \ f \in C^\infty(\mathbb{T}^3). \tag{2.1}$$

If in (2.1) we take partial Fourier transform with respect to $x \in \mathbb{T}$ we obtain

$$\left[-\partial_{t_1}^2 - \partial_{t_2}^2 - (\partial_{t_2} + i\xi a(t_1, t_2))^2\right] \hat{u}(t, \xi) = \hat{f}(t, \xi), \quad \text{for all } \xi \in \mathbb{Z}. \tag{2.2}$$

For any $\xi \in \mathbb{Z}$ fixed $\hat{u}(t, \xi)$ is in $C^\infty(\mathbb{T}^2)$ since (2.2) is elliptic in t. Therefore, if we multiply (2.2) with $\bar{\hat{u}}$ and integrate by parts with respect to $t \in \mathbb{T}^2$, then we obtain

$$\begin{aligned} \|Y_1 \hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} + \|Y_2 \hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} &+ \|Y_3 \hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} \\ &= \int_{\mathbb{T}^2} \hat{f}(t, \xi) \bar{\hat{u}}(t, \xi) dt, \end{aligned} \tag{2.3}$$

where $Y_1 = \partial_{t_1}$, $Y_2 = \partial_{t_2}$ and $Y_3 = \partial_{t_2} + i\xi a(t_1, t_2)$. Also, note that

$$[Y_1, Y_3] = i\xi \frac{\partial a}{\partial_{t_1}} \ \text{ and } \ [Y_2, Y_3] = i\xi \frac{\partial a}{\partial_{t_2}}. \tag{2.4}$$

Let χ be a function such that $\chi \in C^\infty(\mathbb{T}^2)$, $\chi \geq 0$, $\chi \equiv 1$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]^2$ and supp $\chi \subset [-\delta, \delta]^2$. We will prove our result considering only the case $\frac{\partial a}{\partial_{t_1}}(t) \neq 0$ on $[-\delta, \delta]^2$ (the other case being similar). For $\xi \in \mathbb{Z} - 0$ and $\phi \in C^\infty(\mathbb{T}^2)$ we write

$$\begin{aligned} \int_{[-\frac{\delta}{2}, \frac{\delta}{2}]^2} |\phi(t)|^2 dt &= \int_{[-\frac{\delta}{2}, \frac{\delta}{2}]^2} \left(\frac{1}{i\xi \frac{\partial a}{\partial_{t_1}}(t)} [Y_1, Y_3] \right) |\phi(t)|^2 dt \\ &= \int_{[-\frac{\delta}{2}, \frac{\delta}{2}]^2} \chi(t) \left(\frac{1}{i\xi \frac{\partial a}{\partial_{t_1}}(t)} [Y_1, Y_3] \right) |\phi(t)|^2 dt \\ &\leq \int_{[-\delta, \delta]^2} \chi(t) \left(\frac{1}{i\xi \frac{\partial a}{\partial_{t_1}}(t)} [Y_1, Y_3] \right) |\phi(t)|^2 dt \\ &= \int_{\mathbb{T}^2} \left(\chi(t) \frac{1}{i\xi \frac{\partial a}{\partial_{t_1}}(t)} \right) [Y_1, Y_3] |\phi(t)|^2 dt \\ &= \left| \int_{\mathbb{T}^2} \left(\chi(t) \frac{1}{i\xi \frac{\partial a}{\partial_{t_1}}(t)} \right) [Y_1, Y_3] |\phi(t)|^2 dt \right| \\ &= \frac{1}{|\xi|} \left| (b(t)[Y_1, Y_3]\phi, \phi) \right| \leq \left| (b(t)[Y_1, Y_3]\phi, \phi) \right|, \end{aligned} \tag{2.5}$$

where

$$b(t) = \chi(t) \frac{1}{\frac{\partial a}{\partial_{t_1}}} \in C^\infty(\mathbb{T}^2).$$

Since for any $b \in C^\infty(\mathbb{T}^2)$ there exists $C > 0$ such that

$$|(b(t)[Y_1, Y_3]\phi, \phi)| \leq C \left(\|Y_1\phi\|^2 + \|Y_3\phi\|^2 + \|\phi\|\|Y_1\phi\| + \|\phi\|\|Y_3\phi\|\right),$$

it follows from this and (2.5) that

$$\int_{[-\frac{\delta}{2}, \frac{\delta}{2}]^2} |\phi(t)|^2 dt \leq C \left(\|Y_1\phi\|^2 + \|Y_3\phi\|^2 + \|\phi\|\|Y_1\phi\| + \|\phi\|\|Y_3\phi\|\right).$$

Since

$$\|\phi\|^2_{L^2(\mathbb{T}^2)} \leq C \left(\int_{[-\frac{\delta}{2}, \frac{\delta}{2}]^2} |\phi(s)|^2 ds + \|\phi_{t_1}\|^2 + \|\phi_{t_2}\|^2\right),$$

the last inequality applied with $\phi = \hat{u}(t, \xi)$ gives

$$\begin{aligned}
\|\hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} &\leq C_1 \left(\|Y_1\hat{u}\|^2 + \|Y_3\hat{u}\|^2 + \|\hat{u}\|\|Y_1\hat{u}\| + \|\hat{u}\|\|Y_3\hat{u}\|\right) \\
&+ C_1 \left(\|\hat{u}_{t_1}\|^2 + \|\hat{u}_{t_2}\|^2\right) \\
&= C_1 \left(\|Y_1\hat{u}\|^2 + \|Y_3\hat{u}\|^2 + \|\hat{u}\|\|Y_1\hat{u}\| + \|\hat{u}\|\|Y_3\hat{u}\|\right) \\
&+ C_1 \left(\|Y_1\hat{u}\|^2 + \|Y_2\hat{u}\|^2\right) \\
&\leq C_1 \left(\|Y_1\hat{u}\|^2 + \|Y_3\hat{u}\|^2 + \frac{\epsilon^2}{2}\|\hat{u}\|^2 + \frac{1}{2\epsilon^2}\|Y_1\hat{u}\|^2\right) \\
&+ C_1 \left(\frac{\epsilon^2}{2}\|\hat{u}\|^2 + \frac{1}{2\epsilon^2}\|Y_3\hat{u}\|^2 + \|Y_1\hat{u}\|^2 + \|Y_2\hat{u}\|^2\right).
\end{aligned}$$

Choosing ϵ such that $C_1\epsilon^2 < 1$ the last inequality gives

$$\begin{aligned}
\|\hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} &\leq C_2 \left(\|Y_1\hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} + \|Y_2\hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)} + \|Y_3\hat{u}(\cdot, \xi)\|^2_{L^2(\mathbb{T}^2)}\right) \\
&\leq C_2 \int_{\mathbb{T}^2} \hat{f}(t, \xi)\bar{\hat{u}}(t, \xi).
\end{aligned}$$

Thus we have

$$\|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^2)} \leq C_2\|\hat{f}(\cdot, \xi)\|_{L^2(\mathbb{T}^2)}.$$

Finally, using a standard microlocalization argument (see [15]) we obtain that $u \in C^\infty(\mathbb{T}^3)$. This shows that P is globally hypoelliptic in $\mathbb{T}^3$, which completes the proof of Theorem.

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