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SUB-RIEMANNIAN SYMMETRIC SPACES OF ENGEL TYPE

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Abstract

A sub-Riemannian manifold is a smooth manifold which carries a metric defined only on a smooth distribution. In this paper we will restrict our attention to sub-Riemannian manifolds where the associated distribution is an Engel distribution. We obtain a parallelism on a sub-Riemannian structure of Engel type, and then we classify all simply-connected four-dimensional sub-Riemannian manifolds which are sub-symmetric by reducing them to some algebraic structure.

Resumo

Uma variedade sub-Riemanniana é uma variedade diferenciável que carrega uma métrica definida somente sobre uma distribuição diferenciável. Neste artigo restringiremos nossa atenção às variedades sub-Riemannianas cuja distribuição associada é uma distribuição de Engel. Obteremos um paralelismo para uma estrutura sub-Riemanniana de tipo Engel, e então classificaremos todas as variedades sub-Riemannianas 4-dimensionais simplesmente conexas, as quais são sub-simétricas, por redução a uma estrutura algébrica.

0. Introduction

A *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, g)$ where M is an oriented smooth manifold, $\mathcal{D}$ is an oriented smooth distribution on M (i.e. $\mathcal{D}$ is a subbundle of the tangent bundle TM) and g is a smoothly varying positive definite symmetric bilinear form defined on $\mathcal{D}$. *Sub-Riemannian symmetric spaces* constitute a class of sub-Riemannian manifolds $(M, \mathcal{D}, g)$ with special symmetric properties. The concept of sub-Riemannian symmetric space was introduced by Strichartz in [4], which is the analogue of a Riemannian symmetric space in the context of sub-Riemannian geometry.

The principal result of this work is the classification of simply-connected sub-Riemannian symmetric spaces with Engel distribution (see Table 1 for a complete list of spaces and their invariants). We use the existence of a parallelism on a sub-Riemannian structure of Engel type to define geometric invariants. Then we classify all 4-dimensional sub-Riemannian symmetric spaces by using a canonical linearization of the structure via a special class of involutive Lie algebras.

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1. Engel distribution

Let T_xM denote the tangent space at a point $x \in M$. The distribution $\mathcal{D}$ is said to be *bracket-generating* if for each $x \in M$ the local sections of $\mathcal{D}$ together with all its iterated brackets evaluated at a point x span the vector space T_xM.

Let $(\mathcal{D}^2)_x$ denote the subspace of T_xM spanned by all brackets of sections of $\mathcal{D}$ evaluated at a point x. Let $(\mathcal{D}^3)_x$ denote the subspace of T_xM spanned by all vector fields $[[X,Y],Z]_x$, where X, Y, Z varies over all sections of $\mathcal{D}$.

In this paper we shall consider only the case in which $\mathcal{D}$ is an Engel distribution. That means that $\mathcal{D}$ is a codimension two distribution in a 4-dimensional manifold M,*regular* and *bracket-generating*, or equivalently $\mathcal{D}$ is a distribution of codimension two in a 4-manifold M and for every $x \in M$ we have $\dim(\mathcal{D}^2)_x = 3$ and $(\mathcal{D}^3)_x = T_xM$.

Let $\mathcal{D}^2$ denote the distribution defined on M, which associates for every $x \in M$ the subspace $(\mathcal{D}^2)_x$ of T_xM. Consider the Levi form

$$\mathcal{L} : \mathcal{D}^2 \times \mathcal{D}^2 \to TM/\mathcal{D}^2, \quad \mathcal{L}(X,Y) = [\tilde{X}, \tilde{Y}] \mod \mathcal{D}^2 \tag{1}$$

where $\tilde{X}$, $\tilde{Y}$ are extensions of X, Y to sections of $\mathcal{D}^2$. Once we assume that D is an Engel distribution we have that $\mathcal{L}$ is a skew-symmetric bilinear form on $\mathcal{D}^2$, the kernel of $\mathcal{L}$ is 1-dimensional and is contained in $\mathcal{D}$.

It follows from the above affirmation the existence of a Riemannian structure on $(M, \mathcal{D}, g)$: choose a unit vector field $Y_1 \in \mathcal{D}$, such that Y_1 generates the kernel of $\mathcal{L}$; take Y_2 as the unique unit vector field on $\mathcal{D}$ such that $\{Y_1, Y_2\}$ is a positive orthonormal frame on $\mathcal{D}$. Define $\xi_3 = [Y_1, Y_2]$, $\xi_4 = [\xi_3, Y_2]$ and extend g to a Riemannian metric $< \cdot, \cdot >$ on M, by setting ξ_3, ξ_4 orthonormal to $\mathcal{D}$. Notice that the choice of Y_1 is uniquely determined if we have that $\{Y_1, Y_2, \xi_3, \xi_4\}$ is a positive orthonormal frame on M under the above conditions.

The existence of this canonical frame defined on the manifold implies immediately the existence of a flat connection and makes possible to define geometric invariants.

We set the invariants that follows

$$T_0 =< [\xi_3, Y_1], Y_2 > \quad , \quad T_1 =< [\xi_4, Y_2], \xi_3 > \quad \text{and} \quad T_2 = | < [\xi_4, Y_1], \xi_3 > |$$

Note that T_0, T_1 and T_2 are independent of the chosen orientation on $\mathcal{D}$ and M.

2. Sub-Riemannian symmetric spaces and involutive Lie algebras

A *local isometry* between two sub-Riemannian manifolds $(M, \mathcal{D}, g)$ and $(M', \mathcal{D}', g')$ is a diffeomorphism between open sets $\psi : U \subset M \to U' \subset M'$ such that $\psi_*(\mathcal{D}) = \mathcal{D}'$ and $\psi^* g' = g$. In the Engel distribution case ψ will be a local Riemannian isometry relative to the extended Riemannian metrics on M and M' (since $\psi_* Y_1 = \pm Y_1'$, $\psi_* Y_2 = \pm Y_2'$ and $\psi_*[\cdot, \cdot] = [\psi_* \cdot, \psi_* \cdot]$). If ψ is globally defined on M to M', we say simply that ψ is an *isometry*.

A *Sub-Riemannian homogeneous space* (or *sub-homogeneous space*, for short) is a sub-Riemannian manifold $(M, \mathcal{D}, g)$ which admits a transitive Lie group of isometries acting smoothly on M. A *sub-Riemannian symmetric space* (or *sub-symmetric space*) is a sub-homogeneous space $(M, \mathcal{D}, g)$ such that for every point $x_0 \in M$ there is an isometry ψ, called the *sub-symmetry* at x_0, with $\psi(x_0) = x_0$ and $\psi_*|_{\mathcal{D}_{x_0}} = -1$ (recall that $\mathcal{D}$ is an Engel distribution; for a more general definition see [4]).

An *involutive Lie algebra* is a real Lie algebra $\mathfrak{g}$ equipped with an automorphism $s : \mathfrak{g} \to \mathfrak{g}$ satisfying $s^2 = 1$ (s is called the *involution of the Lie algebra* $\mathfrak{g}$). It is clear that if s is an involution of a Lie algebra $\mathfrak{g}$ then $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ where $\mathfrak{g}^+ = \{X \in \mathfrak{g} | s(X) = X\}$, $\mathfrak{g}^- = \{X \in \mathfrak{g} | s(X) = -X\}$. The next result says that a sub-Riemannian homogeneous space can be identified with a Lie group G which has a left-invariant sub-Riemannian structure $(\mathcal{D}, g)$ and whose Lie algebra $\mathfrak{g}$ has a special decomposition; in particular, if the space is sub-symmetric then $\mathfrak{g}$ is involutive.

Theorem 2.1. *Let $(M, \mathcal{D}, g)$ be a 4-dimensional simply-connected sub-homogeneous space where $\mathcal{D}$ is an Engel distribution. Then:*

a. there is a connected, simply-connected Lie group G of sub-Riemannian isometries of M which acts simply transitively on M;

b. the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]$, where $\mathfrak{p}$ corresponds to $\mathcal{D}_{x_0}$ under the identification of $\mathfrak{g}$ with $T_{x_0}M$ for a chosen base-point x_0, and $\mathfrak{p}$ does not depend on the chosen x_0;

c. the inner product B induced on $\mathfrak{p}$ by the identification of $\mathfrak{p}$ with $\mathcal{D}_{x_0}$ does not depend on the chosen $x_0 \in M$.

Proof. Let G be the connected component containing the identity of the Lie group of all sub-Riemannian isometries of M. Choose $x_0 \in M$, let K be the isotropy subgroup at x_0. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the respective Lie algebras of G and K. Then M is represented as the coset space G/K. Since G is a group of Riemannian isometries relative to the canonical extended Riemannian metric on M, then K is a compact subgroup of G, so there is an Ad_K-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. Consider the projection $\pi : G \to M$, $\pi(\phi) = \phi(x_0)$. The induced map π_* identifies $\mathfrak{m}$ with the tangent space $T_{x_0}M$ and is easily seen to be an equivalence between the Ad_K-action on $\mathfrak{m}$ and the K-action on $T_{x_0}M$ (i.e. $\pi_*(Ad_k X) = (dk)_{x_0}(\pi_* X) \forall k \in K, X \in \mathfrak{m}$). Define $\mathfrak{p}$ to be the inverse image of $\mathcal{D}_{x_0}$ in $\mathfrak{m}$ under π_*. Therefore $Ad_K(\mathfrak{p}) \subset \mathfrak{p}$ and this implies that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} + [\mathfrak{p},\mathfrak{p}] + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]$ is a Ad_K-invariant decomposition, since $\mathcal{D}$ is an Engel distribution. Let B be the inner product g_{x_0} lifted to $\mathfrak{p}$ by π. Then B is an Ad_K-invariant inner product on $\mathfrak{p}$. Now the existence of parallelism on M, obtained in the section 1, implies that isotropy is trivial. Thus, we have $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p},\mathfrak{p}] + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]$ and dim $\mathfrak{g} = \dim G = 4$, so G acts simply transitively on M, since we assumed that M is simply connected. Finally, the subspace $\mathfrak{p}$ of $\mathfrak{g}$ and the inner product B on $\mathfrak{p}$ induced by g_{x_0} do not depend on x_0 since G acts on M preserving $\mathcal{D}$ and g.

□

To a four-dimensional sub-homogeneous space $(M,\mathcal{D},g)$ we have now associated a triple $(\mathfrak{g},\mathfrak{p},B)$ where $\mathfrak{g}$ is a four-dimensional Lie algebra, $\mathfrak{p}$ is a two-dimensional subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p},\mathfrak{p}] + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]$ and B is an inner product on $\mathfrak{p}$; if the space is sub-symmetric we have the additional properties $[[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}],\mathfrak{p}] \subset [\mathfrak{p},\mathfrak{p}]$ and $\dim(\mathfrak{p} + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]) = 3$.

Conversely, given a triple $(\mathfrak{g},\mathfrak{p},B)$ with properties as in the above paragraph, we can construct a simply-connected sub-homogeneous space as follows. Let $M = G$ be the simply-connected group with Lie algebra $\mathfrak{g}$, and $\mathcal{D}$ a G-invariant distribution on G such that $\mathcal{D}_1 = \mathfrak{p}$ and $\mathcal{D}_m = d(L_m)(\mathcal{D}_1)$, $\forall m \in G$, and g a G-invariant metric on $\mathcal{D}$ determined by B such that $g_1 = B$, that is $g_m(X_m, Y_m) = g_1(d(L_m)^{-1}{}_m(X_m), d(L_m)^{-1}{}_m(Y_m))$; $\forall m \in G$, where L_m denotes the automorphism given by the left-translation by m on G. The distribution $\mathcal{D}$, so constructed, is an Engel distribution because of the hypotheses on $\mathfrak{g}$ an $\mathfrak{p}$. If the triple has the additional properties $[[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}],\mathfrak{p}] \subset [\mathfrak{p},\mathfrak{p}]$ and $\dim(\mathfrak{p} + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]) = 3$ then the involutive automorphism s of $\mathfrak{g}$, which is $+1$ on $[\mathfrak{p},\mathfrak{p}]$ and is -1 on $\mathfrak{p} + [[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]$, induces an automorphism ψ of G, which is an isometry of G and $\psi_* = s$. Thus ψ is the sub-symmetry at 1 and $L_m \circ \psi \circ L_{m^{-1}} : G \to G$ is the sub-symmetry at m, so G is sub-symmetric.

Remark 2.1. *We can prove that all simply connected sub-symmetric space of Engel type is also an odd-contact sub-symmetric space as follows: Given*

$(M, \mathcal{D}, g)$ Engel sub-symmetric manifold, simply-connected, take their its associated $(\mathfrak{g}, \mathfrak{p}, B)$. Since $\mathfrak{g}$ has an involutive structure, i.e. $\mathfrak{g}^+ = \mathfrak{p} + [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]$, $\mathfrak{g}^- = [\mathfrak{p}, \mathfrak{p}]$ and $\dim \mathfrak{g}^+ = 3$, consider the abstract sub-OIL algebra $(\mathfrak{g}, \mathfrak{p}', B')$, where $\mathfrak{p}' = \mathfrak{g}^+$ and B' is the restriction of the canonical metric defined in $\mathfrak{g}$ to $\mathfrak{p}'$ via the induced map π_*. The triple $(\mathfrak{g}, \mathfrak{p}', B')$ corresponds to a simply connected, sub-symmetric space of odd contact $(G \equiv M, \mathcal{D}', g')$, where G is the simply-connected group with Lie algebra $\mathfrak{g}$, $\mathcal{D}'$ is a G-invariant distribution on G such that $\mathcal{D}'_1 = \mathfrak{p}'$ and g' is a G-invariant metric on $\mathcal{D}'$ such that $g'_1 = B'$.

3. The classification of 4-dimensional sub-symmetric spaces

In this section we shall establish the classification of simply connected sub-symmetric spaces with Engel distribution. More precisely, we shall see that there are exactly ten classes of examples of these spaces, they are the universal coverings (to be denoted with a tilde) of: the direct product of the Euclidean proper motion group $Euc_2^+ = SO(2) \bowtie \mathbb{R}^2$ with $\mathbb{R}$; the direct product of the Poincaré orthochronological proper group $Poinc_2^+ = SO_e(1,1) \bowtie \mathbb{R}^2$ with $\mathbb{R}$; the unitary indefinite group $U(1,1)$ (this space admits four distinct distributions which are not isometric as sub-symmetric spaces - this follow by looking invariants in table 1 - we distinguish these spaces one from the other by "primes"); the Engel group E^4 whose Lie algebra e_4 has one basis $\beta = \{x_1, x_2, y = [x_1, x_2], z = [x_2, y]\}$ satisfying $[y, x_1] = [z, x_1] = [y, z] = [z, x_2] = 0$; the unitary group $U(2)$; the semidirect product of the special orthogonal group $SO(2)$ with the Heisenberg group H^3; the semidirect product of the special orthogonal indefinite group $SO_e(1,1)$ with the Heisenberg group H^3.

Remark 3.1. *The detailed list of these spaces with the respective distributions, metrics and their invariants can be found in Table 1.*

Remark 3.2. *Although the Theorem (2.1) and its converse allow us to deal with the classification problem in a more general context for the sub-homogeneous*

type	G	A'	$\{x_1, x_2\}$ is o.n. basis of $\mathfrak{p}$	T_0	T_1	T_2
$(1a)$	E^4	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & x_1' \\ x_2 & = & a x_1' + x_2' \end{array}$	0	0	0
$(2ab)$	$\widetilde{Euc_2^+} \times \mathbb{R}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' \\ x_2 & = & a\sqrt{b} x_1' - \frac{1}{\sqrt{b}} y' - \frac{1}{b} z' \end{array}$	0	$-b(1+a^2)$	0
$(3ab)$	$Poinc_2^+ \times \mathbb{R}$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' \\ x_2 & = & a\sqrt{b} x_1' + \frac{1}{\sqrt{b}} y' + \frac{1}{b} z' \end{array}$	0	$b(1+a^2)$	0
$(4bc)$	$\widetilde{U(1,1)}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \frac{1}{\sqrt{2}}(-1 - \frac{c}{2}) x_2' + z' \\ & & + \frac{1}{\sqrt{2}}(1 - \frac{c}{2}) y' \\ x_2 & = & \frac{b}{\sqrt{2}}(x_2' + y') \end{array}$	0	c	b
$(5ab)$	$\widetilde{SO(2)} \bowtie H^3$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' + a x_2' \\ x_2 & = & x_2' \end{array}$	$\frac{b}{a^2+1}$	$-\frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$
$(6ab)$	$SO_e(1,1) \bowtie H^3$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' + a x_2' \\ x_2 & = & x_2' \end{array}$	$-\frac{b}{a^2+1}$	$\frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$
$(7abd)$	$\widetilde{U(2)}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' + a\sqrt{d} x_2' + \frac{1}{d} z' \\ x_2 & = & \sqrt{d} x_2' \end{array}$	$\frac{b}{a^2+1}$	$-d(a^2+1) - \frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$
$(8abd)$	$\widetilde{U(1,1)}'$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' + a\sqrt{d} x_2' - \frac{1}{d} z' \\ x_2 & = & \sqrt{d} x_2' \end{array}$	$-\frac{b}{a^2+1}$	$d(a^2+1) + \frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$
$(9abd)$	$\widetilde{U(1,1)}''$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & a\sqrt{d} x_2' + \sqrt{b} y' - \frac{1}{d} z' \\ x_2 & = & \sqrt{d} x_2' \end{array}$	$\frac{b}{a^2+1}$	$d(a^2+1) - \frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$
$(10abd)$	$\widetilde{U(1,1)}'''$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{array}{rcl} x_1 & = & \sqrt{b} x_1' + a\sqrt{d} y' + \frac{1}{d} z' \\ x_2 & = & \sqrt{d} y' \end{array}$	$-\frac{b}{a^2+1}$	$-d(a^2+1) + \frac{a^2 b}{a^2+1}$	$\frac{ab}{a^2+1}$

Table 1: Four-dimensional sub-symmetric spaces: $\{x_1', x_2', y' = [x_1', x_2'], z'\}$ is a basis of the Lie algebra of G, such that z' is central; A' is the transposed matrix of $\mathrm{ad}_{y'}$ restricted to the basis $\{x_1', x_2', z'\}$; T_0, T_1 and T_2 are invariants; a, b, c, d are parameters such that b, $d > 0$, $a \geq 0$ and $c \in I\!R$.

spaces, a more detailed analysis of the decomposition of the occurring Lie algebras shows that the most illustrative examples (in the sense that they are not essentially algebraic and help us to understand their geometric properties) are restricted to the class of the sub-symmetric spaces. Therefore, we prefer to restrict our attention to the classification given in this paper before we can increase our comprehension of the geometric characteristics of the larger category.

Now we take up the classification problem. Let $(G, \mathcal{D}, g)$ be a simply-connected sub-symmetric space of dimension 4 with an Engel distribution.

Consider its associated $(\mathfrak{g}, \mathfrak{p}, B)$. Then, $\dim \mathfrak{g} = 4$, $\dim \mathfrak{p} = 2$, $\dim[\mathfrak{p}, \mathfrak{p}] = 1$, $\dim(\mathfrak{p} + [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]) = 3$, $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]$, $[[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}], \mathfrak{p}] \subset [\mathfrak{p}, \mathfrak{p}]$ and B is an inner product on $\mathfrak{p}$. We have to consider two cases

<u>First case:</u> $\dim\, [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] = 1$

In this case there is $\beta = \{x_1, x_2, y, z\}$ basis of $\mathfrak{g}$, where $\{x_1, x_2\}$ is orthonormal basis of p, satisfying:

$$\begin{array}{rclcrclc} [x_1, x_2] &=& y & \qquad & [z, x_1] &=& by & \\ [y, x_1] &=& z & & [z, x_2] &=& aby & \qquad (a \geq 0, b \in I\!R) \\ [y, x_2] &=& az & & [z, y] &=& 0 & \end{array}$$

If $b = 0$, the transformation

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= -ax_1 + x_2 \\ y' &= y \\ z' &= -z \end{aligned}$$

is such that

$$\begin{array}{rclcrcl} [x_1', x_2'] &=& y' & \qquad & [z', x_1'] &=& 0 \\ [y', x_1'] &=& -z' & & [z', x_2'] &=& 0 \\ [y', x_2'] &=& 0 & & [z', y'] &=& 0 \end{array}$$

thus, it is an isomorphism onto the Lie algebra of Engel group E^4.

If $b \neq 0$, the transformation

$$\begin{aligned} x_1' &= \frac{1}{\sqrt{|b|}} x_1 \\ x_2' &= y \\ y' &= -\frac{1}{\sqrt{|b|}} z \\ z' &= b(-ax_1 + x_2) + z \end{aligned}$$

is such that

$$\begin{array}{lcl} [x'_1, x'_2] & = & y' \\ [y', x'_1] & = & -sgn(b)x'_2 \\ [y', x'_2] & = & 0 \end{array} \qquad \begin{array}{lcl} [z', x'_1] & = & 0 \\ [z', x'_2] & = & 0 \\ [z', y'] & = & 0 \end{array}$$

thus, it is an isomorphism onto the Lie algebra of

$$\begin{cases} \widetilde{Euc_2^+} \times \mathbb{R} & \text{if } b < 0; \\ Poinc_2^+ \times \mathbb{R} & \text{if } b > 0. \end{cases}$$

Next, we compute the associated invariants in the first case, where the orientations on M and $\mathcal{D}$ are such that β and $\{x_1, x_2\}$ are positive bases. We get:

$$\begin{array}{rcl} Y_1 & = & \frac{1}{\sqrt{1+a^2}}(ax_1 - x_2) \\ Y_2 & = & \frac{1}{\sqrt{1+a^2}}(x_1 + ax_2) \\ \xi_3 & = & y \\ \xi_4 & = & \sqrt{1+a^2}z \\ [\xi_3, Y_1] & = & 0 \\ [\xi_4, Y_1] & = & 0 \\ [\xi_4, Y_2] & = & b(1+a^2)\xi_3 \\ [\xi_3, \xi_4] & = & 0 \end{array}$$

Second case: $\dim\, [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] = 2$

In this case there is $\{x_1, x_2\}$ orthonormal basis of $\mathfrak{p}$ such that $x_2 \in \mathfrak{p} \bigcap [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]$. Denote $y = [x_1, x_2]$ and we have to consider two possibilities:

Possibility A): $[y, x_2] \in \mathfrak{p} \bigcap [[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}]$

In this case using the Jacobi identity we have that $\beta = \{x_1, x_2, y, z = [y, x_1]\}$ is a basis of $\mathfrak{g}$ satisfying:

$$\begin{array}{lcl} [x_1, x_2] & = & y \\ [y, x_1] & = & z \\ [y, x_2] & = & bx_2 \end{array} \qquad \begin{array}{lcl} [z, x_1] & = & cy \\ [z, x_2] & = & -by \\ [z, y] & = & bcx_2 + bz \end{array} \qquad (b > 0, c \in I\!R)$$

The transformation

$$\begin{aligned} x'_1 &= \frac{1}{b}y \\ x'_2 &= \frac{1}{b\sqrt{2}}((1+\frac{c}{2})x_2+z) \\ y' &= \frac{1}{b\sqrt{2}}((1-\frac{c}{2})x_2-z) \\ z' &= x_1+\frac{c}{b}x_2+\frac{1}{b}z \end{aligned}$$

is such that

$$\begin{aligned} [x'_1,x'_2] &= y' & \qquad [z',x'_1] &= 0 \\ [y',x'_1] &= -x'_2 & \qquad [z',x'_2] &= 0 \\ [y',x'_2] &= x'_1 & \qquad [z',y'] &= 0 \end{aligned}$$

thus, it is an isomorphism onto the Lie algebra of $\widetilde{U(1,1)}$.

Next, we express the associated invariants for possibility A), where the orientations on M and $\mathcal{D}$ are such that β and $\{x_1,x_2\}$ are positive bases. We compute:

$$\begin{aligned} Y_1 &= -x_2 \\ Y_2 &= x_1 \\ \xi_3 &= y \\ \xi_4 &= z \\ [\xi_3,Y_1] &= bY_1 \\ [\xi_4,Y_1] &= b\xi_3 \\ [\xi_4,Y_2] &= c\xi_3 \\ [\xi_3,\xi_4] &= bcY_1-b\xi_4 \end{aligned}$$

<u>Possibility</u> B): $\qquad [y,x_2]\notin \mathfrak{p}\bigcap[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}]$

In this case using the Jacobi identity we have that $\beta=\{x_1,x_2,y,z=[y,x_2]\}$ is basis of $\mathfrak{g}$ satisfying:

$$\begin{array}{rclrcl} [x_1, x_2] & = & y & [z, x_1] & = & ady \\ [y, x_1] & = & bx_2 + az & [z, x_2] & = & dy \\ [y, x_2] & = & z & [z, y] & = & -bdx_2 \end{array} \qquad (b \neq 0, a \geq 0, d \in I\!R)$$

If $d = 0$, the transformation

$$\begin{aligned} x_1' &= \frac{1}{\sqrt{|b|}}(x_1 - ax_2) \\ x_2' &= x_2 \\ y' &= \frac{1}{\sqrt{|b|}}y \\ z' &= \frac{1}{\sqrt{|b|}}z \end{aligned}$$

is such that

$$\begin{array}{rclrcl} [x_1', x_2'] & = & y' & [z', x_1'] & = & 0 \\ [y', x_1'] & = & sgn(b)x_2' & [z', x_2'] & = & 0 \\ [y', x_2'] & = & z' & [z', y'] & = & 0 \end{array}$$

thus, it is an isomorphism onto the Lie algebra of

$$\begin{cases} \widetilde{SO(2)} \ltimes H^3 & \text{if } b > 0; \\ SO_e(1,1) \ltimes H^3 & \text{if } b < 0. \end{cases}$$

If $d \neq 0$, the transformation

$$\begin{aligned} x_1' &= \frac{1}{d\sqrt{|b|}}z \\ x_2' &= \frac{1}{\sqrt{|d|}}x_2 \\ y' &= \frac{1}{\sqrt{|bd|}}y \\ z' &= -d(x_1 - ax_2) + z \end{aligned}$$

is such that

$$\begin{array}{rclrcl} [x_1', x_2'] & = & y' & [z', x_1'] & = & 0 \\ [y', x_1'] & = & sgn(b)x_2' & [z', x_2'] & = & 0 \\ [y', x_2'] & = & sgn(d)x_1' & [z', y'] & = & 0 \end{array}$$

thus, it is an isomorphism onto the Lie algebra of

$$\begin{cases} \widetilde{U(1,1)'} & \text{if } b<0,\ d>0; \\ \widetilde{U(1,1)''} & \text{if } b>0,\ d>0; \\ \widetilde{U(1,1)'''} & \text{if } b<0,\ d<0; \\ \widetilde{U(2)} & \text{if } b>0,\ d<0. \end{cases}$$

The associated invariants for possibility B), where the orientations on M and $\mathcal{D}$ are such that β and $\{x_1, x_2\}$ are positive bases, are:

$$\begin{aligned} Y_1 &= \frac{1}{\sqrt{a^2+1}}(x_1 - a x_2) \\ Y_2 &= \frac{1}{\sqrt{a^2+1}}(a x_1 + x_2) \\ \xi_3 &= y \\ \xi_4 &= \frac{ab}{\sqrt{a^2+1}} x_2 + \sqrt{a^2+1} z \\ [\xi_3, Y_1] &= -\frac{ab}{a^2+1} Y_1 + \frac{b}{a^2+1} Y_2 \\ [\xi_4, Y_1] &= -\frac{ab}{a^2+1}\xi_3 \\ [\xi_4, Y_2] &= (d(a^2+1) - \frac{a^2 b}{a^2+1})\xi_3 \\ [\xi_3, \xi_4] &= (-abd + \frac{a^3 b^2}{(a^2+1)^2}) Y_1 + (bd - \frac{a^2 b^2}{(a^2+1)^2}) Y_2 + \frac{ab}{a^2+1}\xi_4 \end{aligned}$$

The complete classification is summarized in Table 1. In the first column, we list representatives G for the four-dimensional groups which are sub-symmetric. In the second column, we describe the Lie algebra structure of $\mathfrak{g}$: there is a basis $\{x_1', x_2', y', z'\}$ of the Lie algebra of G such that z' is central, $y' = [x_1', x_2']$ and A' is the transposed matrix of $\mathrm{ad}_{y'}$, restricted to the basis $\{x_1', x_2', z'\}$. In the third column, we give an orthonormal basis $\{x_1, x_2\}$ of $\mathfrak{p}$. The remaining columns list the invariants: T_0, T_1 and T_2.

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