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DUPIN HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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1 Introduction

Let M be an oriented hypersurface of S^{n+1} with second fundamental form h. The eigenvalues, $k_1, ..., k_n$ of h are the principal curvatures of M. The hypersurface M is said to be Dupin if each of its principal curvature has constant multiplicity and is constant along the leaves of its principal foliation. In [T], Thorbergsson proved that if a compact Dupin hypersurface M has p distinct principal curvatures then $p \in \{1, 2, 3, 4, 6\}$. This is the same restriction found by Münzner for the isoparametric hypersurfaces having p distinct principal curvatures. Grove and Halperin [GH] also found topological relations between the isoparametric and Dupin hypersurfaces.

Associated to the second fundamental form there are n functions $H_1, ..., H_2$ given by

$$H_r = \sum_{i_1 < ... < i_r} k_{i_1} ... k_{i_r}.$$

Note that $H = H_1$ is the mean curvature and H_2 is, up to a constant, the scalar curvature of M. In this work we are interested in the following:

Question 1. *Let $\mathcal{F}_{r,s}$ be the family of closed oriented Dupin hypersurfaces $M \subset S^{n+1}$ having $dH_r = dH_s = 0$. Determine $\mathcal{F}_{r,s}$, for all $r \neq s$.*

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The above question seems to be extremely difficult. For this reason we restrict ourselves to the family $\mathcal{F}$ of closed Dupin hypersurfaces $M \subset S^{n+1}$ having constant mean curvature H and constant scalar curvature R_M. Especifically we are interested in the following:

Question 2. *Let $R : \mathcal{F} \to \mathbf{R}$ be given by $R(M) = R_M$. Then $R(\mathcal{F})$ is a discrete set.*

We will refer to question 2 as Chern-Do Carmo-Kobayashi conjecture for Dupin hypersurfaces. In this direction we obtain the following results.

Theorem 1.1 *Let $M \subset S^{n+1}$, $n \leq 4$ be a closed Dupin hypersurface with constant mean curvature and constant scalar curvature $R \geq 0$. Then M is isoparametric.*

In particular we have the following theorem:

Theorem 1.2 *Let $M \subset S^5$ be a closed Dupin hypersurface with constant mean curvature and constant scalar curvature $R \geq 0$. Then M is isoparametric.*

The case $n = 3$, holds even without the assumption that M is a Dupin hypersurface [?].

Theorem 1.3 *([?]) Let $M \subset S^4$ be a closed hypersurface with constant mean curvature and constant scalar curvature $R \geq 0$. Then M is isoparametric.*

By using results of T. Otsuki ([?]), S. S. Chang ([?]) proved the following result:

Theorem 1.4 *Let $M \subset S^{n+1}$ be a closed hypersurface with constant mean curvature and constant scalar curvature. Suppose in addition that M has three distinct principal curvatures. Then M is isoparametric.*

This is one more evidence that Chern-Do Carmo-Kobayashi conjecture may be true.

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2 Preliminaries

In this section we give definitions and the basic results that we will be used through out the paper.

2.1 The structure equations of hypersurfaces of S^{n+1}

Let M be a hypersurface of the unit $(n+1)$-dimensional sphere S^{n+1}. We choose a local orthonormal frame $e_1, ..., e_{n+1}$ in S^{n+1}, such that when restricted to M, $e_1, ..., e_n$ are tangent to M. We will denote by $\omega_1, ..., \omega_{n+1}$ the dual coframe. The structural equations of S^{n+1} are given by

$$\begin{cases} d\omega_A &= \sum_B \omega_{AB} \wedge \omega_B, \\ d\omega_{AB} &= \sum_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \end{cases} \tag{2.1}$$

where $\omega_{AB} + \omega_{BA} = 0$ and

$$\Omega_{AB} = -\frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D. \tag{2.2}$$

In (??) $K_{ABCD} + K_{ABDC} = 0$. The Ricci tensor and the scalar curvature are given respectively by

$$K_{AB} = K_{BA} = \sum_C K_{ACBC} \tag{2.3}$$

$$K = \sum_A K_{AA} = \sum_{AC} K_{ACAC}. \tag{2.4}$$

In S^{n+1},

$$K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} \tag{2.5}$$

$$K_{AB} = n\delta_{AB} \tag{2.6}$$

$$K = n(n+1). \tag{2.7}$$

If we restrict those formulas to M,

$$0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1,i} \wedge \omega_i,$$

and from Cartan's lemma we have

$$\omega_{n+1,i} = \sum_{j=1}^{n} h_{ij}\omega_j, \tag{2.8}$$

where $h_{ij} = h_{ji}$. From now on we assume that $1 \leq i \leq n$ and write

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \ \ \omega_{ij} = -\omega_{ji}, \tag{2.9}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l \tag{2.10}$$

In (**??**),

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}. \tag{2.11}$$

The tensor h defined by

$$h = \sum_{ij} h_{ij} \ \omega_i \otimes \omega_j \tag{2.12}$$

and the function

$$H = \sum_i h_{ii} \tag{2.13}$$

are called the second fundamental form and the mean curvature of M, respectively. The covariant derivative ∇h of h, with components h_{ijk}, is given by

$$\nabla h = \sum_{i,j,k} h_{ijk} \ \omega_i \otimes \omega_j \otimes \omega_k,$$

where

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_m h_{im}\omega_{mj} + \sum_m h_{mj}\omega_{mi}. \tag{2.14}$$

By exterior differentiating (??), we get from one side

$$d\omega_{n+1,i} = \sum_j dh_{ij} \wedge \omega_j + \sum_{jm} h_{im}\omega_{mj} \wedge \omega_j$$

and from the other side,

$$d\omega_{n+1,i} = \sum_m \omega_{n+1,m} \wedge \omega_{mi} - \frac{1}{2}\sum_{ml} R_{(n+1)iml}\omega_m \wedge \omega_l.$$

We also have from (??) and (??)

$$d\omega_{n+1,i} = -\sum_{jm} h_{mj}\omega_{mi} \wedge \omega_j.$$

Therefore,

$$\sum_j dh_{ij} \wedge \omega_j = -\sum_{jm} h_{mj}\omega_{mi} \wedge \omega_j - \sum_{jm} h_{im}\omega_{mj} \wedge \omega_j.$$

From this last identity and from (??) we get

$$\sum_{kj} h_{ijk}\omega_k \wedge \omega_j = 0 \tag{2.15}$$

and therefore, h_{ijk} is symmetric in all indices.

Exterior differentiating the equation (??) and defining h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk} \tag{2.16}$$

we obtain

$$\sum_{kl}(h_{ijkl} - \frac{1}{2}\sum_m h_{im}R_{mjkl} - \frac{1}{2}\sum_m h_{mj}R_{mikl})\omega_k \wedge \omega_l = 0 \tag{2.17}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.18}$$

We denote by S the square of the norm of the second fundamental form. Therefore

$$S = \sum_{ij} h_{ij}^2. \tag{2.19}$$

It is easy to see the for a hypersurface in S^{n+1},

$$S = n(n-1) + H^2 - R \tag{2.20}$$

where R is the scalar curvature and H is the mean curvature of M.

We now compute the laplacian $(\triangle h)$ of h. By definition

$$(\triangle h)_{ij} = \triangle h_{ij} = \sum_{ij} h_{ijkk}. \tag{2.21}$$

From (**??**) and (**??**) we obtain

$$\sum_k h_{ijkk} = \sum_k h_{kijk}.$$

Therefore,

$$\triangle h_{ij} = \sum_k h_{kijk}.$$

Using (**??**) we obtain

$$\begin{array}{rcl} \triangle h_{ij} & = & \sum_k h_{kikj} + \sum_k (\sum_m h_{mi} R_{mkjk} + \sum_m h_{km} R_{mijk}) \\ & = & (n-S)h_{ij} + H\sum_m h_{mi}h_{mj} - H\delta_{ij} \end{array} \tag{2.22}$$

From (**??**), we see that

$$\frac{1}{2}\triangle S = \sum_{ij} h_{ij}\triangle h_{ij} + \sum_{ijk} h_{ijk}^2 = (n-S)S + Hf - H^2 + \sum_{ijk} h_{ijk}^2$$

where $f =$ trace (h^3). Note that when S is constant,

$$\sum_{ijk} h_{ijk}^2 = (S-n)S + H^2 - Hf. \tag{2.23}$$

When M is minimal,

$$|\nabla h|^2 = (S-n)S. \tag{2.24}$$

Note that when $S = n$, h is covariantly constant over M^n. In this direction we should mention a result of H. B. Lawson Jr.([?])

Proposition 2.1 *([?]) Let M be a minimal hypersurface of the unit sphere S^{n+1}. Suppose in addition that h is covariantly constant over M^n. Then M is an open submanifold of one of the minimal products of spheres*

$$S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right); \quad k = 0, ..., \left[\frac{n}{2}\right].$$

2.2 Dupin Hypersurfaces of S^{n+1}

Let $x : M \to S^{n+1}$ be a compact Dupin hypersurface in S^{n+1} with global field of unit normals e_{n+1}. Suppose h is the second fundamental form of the immersion x. Associated to h there are n functions $H_1, ..., H_n$, defined by

$$H_r = \sum_{i_1 < ... < i_r} k_{i_1} ... k_{i_r}.$$

Here $k_1, ..., k_n$ are the principal curvatures of M. They are the eigenvalues of the second fundamental form h. We note that H_2 is, up to a constant, the scalar curvature of M. In general H_r is the so called r-mean curvature function of the immersion x.

To fix notation from now on we will assume that M has p distinct principal curvatures $k_{i_1}, ..., k_{i_p}$ of constant multiplicities $m_1, ..., m_p$ respectively. With this notation we have the following result.

Theorem 2.2 *([?]) The number p of distinct principal curvatures of a compact Dupin hypersurface M is $1, 2, 3, 4$ or 6.*

Remark 2.1 *The restriction given by the above result is the same found by Münzner for the isoparametric hypersurfaces.*

The isoparametric hypersurfaces are interesting examples of Dupin hypersurfaces. Its principal curvatures $k_1, ..., k_n$ are constant everywhere on M. As usual we will write $k_i = \cot \alpha_i$, where $0 < \alpha_1 \leq \alpha_2 ... \leq \alpha_n < \pi$. For a parameter t to be specified later, we let $x_t : M \to S^{n+1}$ be given by $x_t(x) = \cos t \ x + \sin t \ e_{n+1}$. The following result is well known

Theorem 2.3 *Let $x : M \to S^{n+1}$ be an isoparametric hypersurface and $I = [0, \pi]$. For each $t \in I - \{\alpha_1, ..., \alpha_n\}$, x_t is an immersion. The principal curvatures of x_t are also constant and given by $k_i = \cot(\alpha_i - t)$*

Remark 2.2 *The family x_t is said to be an isoparametric family of hypersurfaces.*

Since $H'(t) = \sum(1 + k_i^2) > 0$ and $\lim_{t \to \pm\alpha_i} H(t) = \mp\infty$ we have the following result.

Theorem 2.4 *There exists $t^* \in I - \{\alpha_1, ..., \alpha_n\}$ such that $x_{t^*} : M \to S^{n+1}$ is a minimal immersion.*

Definition 2.5 *A hypersurface of S^{n+1} is called isoparametric of type p if it has p distinct constant principal curvatures.*

Back in the thirties Cartan obtained very interesting results about isoparametric hypersurfaces. Cartan was able to show, that for $p \leq 3$ all hypersurfaces are homogeneous. Around the same time Cartan constructed a whole family of isoparametric hypersurfaces having three distinct principal curvatures. This family was obtained as level hypersurfaces of a harmonic homogeneous polynomial $F : S^{n+1} \to R$. Such hypersurfaces exists only in S^4, S^7, S^{13} and S^{25}. Those hypersurfaces are unique in each such dimensions and are called Cartan's hypersurfaces. The classification of homogeneous hypersurfaces in the spheres given by Hsiang and Lawson ([?]) solved the classification of homogeneous isoparametric hypersurfaces. Later in the seventies, H. F. Münzner, in his paper Isoparametrische Hyperfläche in Sphären ([?]) proved the following result:

Theorem 2.6 *Let $M \subset S^{n+1}$ be a type p isoparametric hypersurface and $\lambda_1, ..., \lambda_p$ distinct principal curvatures with multiplicities $m_1, ..., m_p$ respectively. Then* *(a)* $p \in \{1, 2, 3, 4, 6\}$

(b) If $p = 3$, *then* $m_1 = m_2 = m_3$

(c) If $p = 4$ *or* 6, *then* $m_1 = m_3 = m_5$ *and* $m_2 = m_4 = m_6$

In Cartan's theory, the isoparametric hypersurfaces are closely related to families of level hypersurfaces of a certain class of functions. Cartan observed that if a given function F defined on an open set of S^{n+1} satisfies:

a) $\| \bigtriangledown F\|^2$ is a function of F

b) ΔF is a function of F.

Then if under those conditions

$$M_c^n = \Big\{x \in S^{n+1} : F(x) = c, \ \Delta F(x) \neq 0\Big\}$$

is a nonempty set the level hypersurface defined in this way is a hypersurface with constant principal curvatures.

Example 1 *On* $S^{n+1} \subset R^{n+2}$ *with rectangular coordinates* $(x_0, x_1, ..., x_{n+1})$, *let* F *be the restriction of* x_0 *to* S^{n+1}. *Then* $\|\bigtriangledown F\|^2 = 1 - F^2$ *and* $\Delta F = -(n+1)F$. *The level hypersurfaces*

$$M_s = \Big\{x \in S^{n+1} : F(x) = s\Big\}, \quad -1 < s < 1,$$

are spheres in S^{n+1}.

Example 2 *On* S^{n+1}, *we let* $F(x) = x_0^2 + x_1^2 + ... + x_k^2$, $x \in S^{n+1}$, *where* k *is a fixed integer,* $1 \leq k \leq n - 1$. *Note that* $\| \bigtriangledown F\|^2 = 4F(1 - F)$ *and* $\Delta F = 2(k + 1) - 2(n + 2)F$. *For each* s, $0 < s < 1$, *the hypersurfaces* M_s *are the well known product of spheres* $S^k(\sqrt{s}) \times S^{n-k}(\sqrt{1 - s})$ *embedded in* S^{n+1}.

Cartan showed that if M^n has p distinct principal curvatures with the same multiplicity, $m_1 = ... = m_p = m$, ($n = p.m$), then M is given as a level hypersurface

$$M = \{x \in S^{n+1} : F(x) = \cos pt\}, \tag{2.25}$$

where F is a degree p homogeneous harmonic polynomial over R^{n+2}, restricted to S^{n+1}. In the case $p = 3$, $m_1 = m_2 = m_3$. Therefore, there exists a homogeneous harmonic polynomial $F : R^{n+2} \to R$, such that M_t^n is a level hypersurface of F. In ([?]) Cartan explicitly exibited this polynomial. It is given by

$$\begin{aligned} F &= u^3 - 3uv^2 + \tfrac{3}{2}u\left(X\overline{X} + Y\overline{Y} - Z\overline{Z}\right) \\ &\quad + \tfrac{3\sqrt{3}}{2}v\left(X\overline{X} - Y\overline{Y}\right) + \tfrac{3\sqrt{3}}{2}\left(XYZ + \overline{ZYX}\right), \end{aligned} \tag{2.26}$$

where $u = x_{n+2}$, $v = x_{n+1}$ and X, Y and Z are real, complex quarternions or octonions of Graves-Cayley depending if $n = 3, 6, 12$ or 24 respectively. For the case $n = 24$, we adopt the convention that $XYZ = (XY)Z$ and $\overline{ZYX} = \overline{Z}(\overline{YX})$, because of the nonassociativity of the Cayley numbers. Those hypersurfaces are tubes of constant radius over the embedded Veronese FP^2, $F = R, C, Q, O$ in S^4, S^7, S^{13} and S^{25}, respectively. Here Q, O are the quaternions and the Cayley numbers. Those hypersurfaces are known as Cartan's isoparametric hypersurfaces. The three principal curvatures of Cartan's hypersurfaces are:

$$\frac{\cot t + \sqrt{3}}{\sqrt{3}\cot t - 1}, \frac{\cot t - \sqrt{3}}{-\sqrt{3}\cot t - 1}, -\cot t.$$

We will give explicit equations of Cartan minimal isoparametric hypersurface $M \subset S^4$. It is given by $M = P^{-1}(0) \cap S^4$, where $P : R^5 \to R$ is the polynomial

$$P(u, v, x, y, z) = \begin{vmatrix} u & x & y \\ x & v & z \\ y & z & -u - v \end{vmatrix}$$

2.3 The Differential Form Ψ

Let $x : M \to S^{n+1}$ be an orientable hypersurface in S^{n+1} and h its second fundamental form. We suppose in addition that M has distinct principal curvatures $\lambda_1 < \lambda_2 < ... < \lambda_n$. We say that (U, ω) is admissible if:

i) U is an open subset of M

ii) $\omega = (\omega_1, ..., \omega_n)$ is a smooth orthonormal coframe field on U

iii) $\omega_1 \wedge ... \wedge \omega_n$ is the volume form on M

iv) $h = \sum_{i \in I} \lambda_i \omega_i \otimes \omega_i$.

As in $[AB_2]$ there is one and only one n-form Ψ on M such that if (U, ω) is admissible then

$$\Psi = \sum_{i<j} \omega_{ij} \wedge *(\omega_i \wedge \omega_j)$$

A standard computation gives

$$- *(d\Psi) = \sum_{i<j} (1 + \lambda_i \lambda_j) - \sum_{k=1}^{n} \sum_{k \neq i < j \neq k} \frac{h_{iik} h_{jjk}}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)} \tag{2.27}$$

3 Proof of Theorem ??

First of all we will prove the following:

Theorem 3.1 *Let $M \subset S^{n+1}$ be a closed Dupin hypersurface with constant mean curvature H and constant scalar curvature R. Suppose in addition that M has $p < 4$ distinct principal curvatures. Then M is isoparametric .*

Proof: Let $\lambda_1 < ... < \lambda_p$ be the principal curvatures of M. The case $p = 1$ is simply the case of umbilic hypersurfaces, i.e. hyperspheres of $S^{n+1}(1)$. When $p = 2$,

$$m_1 \lambda_1 + m_2 \lambda_2 = H$$

$$m_1 \lambda_1^2 + m_2 \lambda_2^2 = S$$

where $m_1 = k$ and $m_2 = n - k$ are the multiplicities of λ_1 and λ_2 respectively. Using those equations we see that M is an isoparametric hypersurface obtained from a compact minimal isoparametric hypersurface $M_0 \subset S^{n+1}$. The principal curvatures μ_1 and μ_2 of M_0 are given by $\mu_2 = -1/\mu_1 = \sqrt{k/(n-k)}$. They also have multiplicities k and $n-k$ respectively. It follows from equation (**??**) and Proposition **??** that

$$M_0 = S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right).$$

As a consequence M is a product of spheres.

We will assume now that M has $p=3$ distinct principal curvatures $\lambda_1 < \lambda_2 < \lambda_3$ with multiplicities given by m_1, m_2, m_3. We will choose a local frame field $e_1, ..., e_n$ with dual coframe $\omega_1, ...\omega_n$ such that the second fundamental form $h = \sum_{ij} h_{ij}\omega_i \otimes \omega_j$ is given by

$$h = \begin{bmatrix} \lambda_1 I_{m_1} & 0 & 0 \\ 0 & \lambda_2 I_{m_2} & 0 \\ 0 & 0 & \lambda_3 I_{m_3} \end{bmatrix},$$

where I_s denotes the $s \times s$ identity matrix. Note that

$$m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = H$$

$$m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 = S$$

where S is the square of the norm of the second fundamental form of M. Since M^n is a Dupin hypersurface, $d\lambda_1(e_k) = 0$ for $1 \le k \le m_1$. Therefore

$$m_2 d\lambda_2(e_k) + m_3 d\lambda_3(e_k) = 0$$

$$m_2\lambda_2 d\lambda_2(e_k) + m_3\lambda_3 d\lambda_3(e_k) = 0.$$

Since $\lambda_2 - \lambda_3 \neq 0$ it follows that

$$d\lambda_1(e_k) = d\lambda_2(e_k) = d\lambda_3(e_k) = 0.$$

In an analogous way we can prove that

$$d\lambda_1(e_k) = d\lambda_2(e_k) = d\lambda_3(e_k) = 0,$$

for $k > m_1$. It follows that M is isoparametric. This concludes the proof of Theorem **??**

We will now consider the the case $p=4$. In this direction we have the following result:

Theorem 3.2 *Let $M \subset S^5$ be a closed Dupin hypersurface with constant mean curvature and constant scalar curvature $R \geq 0$. Then M is isoparametric.*

Proof: We need only consider the case $p = 4$. Let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be the principal curvatures of M. For each $i, j \in I = \{1,2,3,4\}$, we set $c_{ij} = \lambda_i - \lambda_j$. It follows from equation (**??**) that

$$2 * d\Psi = -R + 2\sum_{k=1}^{4} I_k,$$

where $R \geq 0$ is the scalar curvature of M and for a fixed $k \in I$

$$I_k = \sum_{k \neq i < j \neq k} \frac{h_{iik} h_{jjk}}{c_{ik} c_{jk}}. \tag{3.1}$$

Since H and S are constant functions and M is a Dupin hypersurface we have

$$\sum_{r \neq k} h_{rrk} = 0 = \sum_{r \neq k} \lambda_r h_{rrk},$$

which gives

$$c_{is} h_{iik} + c_{js} h_{jjk} = 0, \tag{3.2}$$

for distinct $i, j, k, s \in I$. Note that

$$I_4 = \frac{h_{114} h_{224}}{c_{14} c_{24}} + \frac{h_{114} h_{334}}{c_{14} c_{34}} + \frac{h_{224} h_{334}}{c_{24} c_{34}}.$$

On the other hand, from equation (**??**) we have

$$h_{334} = \frac{c_{21}}{c_{32}} h_{114} = -\frac{c_{21}}{c_{31}} h_{224},$$

which gives

$$I_4 = h_{114}^2 \left[\frac{-c_{13}}{c_{23} c_{14} c_{24}} + \frac{-c_{12}}{c_{32} c_{14} c_{34}} + \frac{c_{12} c_{13}}{c_{32} c_{23} c_{24} c_{34}} \right].$$

Since $c_{ji} > 0$ for $j > i$ we obtain

$$c_{41} c_{42} c_{43} c_{32}^2 I_4 = -\left[c_{21}(c_{41} c_{31} - c_{42} c_{32}) + c_{31} c_{32} c_{43} \right] h_{114}^2 \leq 0.$$

To evaluate I_1 we note that

$$I_1 = \frac{h_{221}h_{331}}{c_{21}c_{31}} + \frac{h_{221}h_{441}}{c_{21}c_{41}} + \frac{h_{331}h_{441}}{c_{31}c_{41}}.$$

Since

$$h_{441} = \frac{c_{32}}{c_{43}}h_{221} = -\frac{c_{32}}{c_{42}}h_{331},$$

we have

$$I_1 = h_{221}^2\left[\frac{-c_{42}}{c_{43}c_{21}c_{31}} + \frac{c_{32}}{c_{43}c_{21}c_{41}} + \frac{c_{32}c_{42}}{c_{43}c_{34}c_{41}c_{31}}\right]$$

and then

$$c_{21}c_{31}c_{41}c_{43}^2 I_1 = -\left[c_{43}(c_{41}c_{42} - c_{32}c_{31}) + c_{21}c_{32}c_{42}\right]h_{221}^2 \leq 0.$$

In the same way we prove that $I_k \leq 0$, for $k = 2, 3$. We also note that $I_k = 0$ if and only if $h_{iik} = 0$ for all $i \in I$. Finally, using Stokes's theorem we obtain

$$0 = \int_M -2d\Psi = \int_M \left(R - 2\sum_{k=1}^{4} I_k\right) dM,$$

where dM is the volume form of M. Since $R \geq 0$ it follows that $I_k = 0$ for all $k \in I$. Therefore $h_{11k} = h_{22k} = h_{33k} = h_{44k} = 0$, for $k = 1, 2, 3, 4$. It follows that all principal curvatures of M are constant and M^4 is isoparametric. This completes the proof.

Remark 3.1 *Theorem ?? of section 1 is an immediate consequence of Theorem ?? and Theorem ??*

4 Final Comments

Let M be a minimal, compact, Dupin hypersurface immersed in the unit $(n+1)$-dimensional sphere S^{n+1} with 6 distinct principal curvatures. In [MO], T. Ozawa and R. Miyaoka have shown that is possible to construct examples of Dupin hypersurfaces that are not equivalent (by a Lie transformation) to an isoparametric hypersurface. Their examples are immersed in S^7. The natural question now is if there exists a minimal, Dupin hypersurfaces of constant scalar curvature and 6 distinct principal curvatures that is not isoparametric.

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