Matemática Contemporânea, Vol. 17, 251–266
http://doi.org/10.21711/231766361999/rmc1714
©1999, Sociedade Brasileira de Matemática

CONFORMAL METRICS AND RICCI TENSORS IN THE HYPERBOLIC SPACE

Romildo Pina

Abstract

We consider the hyperbolic space $H^n(-1) = (R^n_+, g)$, T a symmetric tensor given by $T = \text{Ric } g + \sum_{i,j} \frac{c_{ij}}{x_n^2} dx_i \otimes dx_j$ with $c_{ij} \in R$ and we study the problem of finding metrics $\bar{g}$ conformal to the hyperbolic metric g such that $\text{Ric}\bar{g} = T$. We show that such tensors are null or diagonal and we obtain explicitly such metrics $\bar{g}$. As a consequence of these results we show that for certain unbounded functions $\bar{K}$ defined on R^n_+ there exist metrics conformal to the hyperbolic metric, with scalar curvature $\bar{K}$.

Resumo

Consideramos o espaço hiperbólico $H^n(-1) = (R^n_+, g)$, T um tensor simétrico dado por $T = \text{Ric } g + \sum_{i,j} \frac{c_{ij}}{x_n^2} dx_i \otimes dx_j$ com $c_{ij} \in R$ e estudamos o problema de encontrar métricas $\bar{g}$, conformes à métrica hiperbólica g, tal que $\text{Ric}\bar{g} = T$. Mostramos que tais tensores são nulos ou diagonais e obtemos explicitamente tais métricas $\bar{g}$. Como conseqüência destes resultados, mostramos que, para certas funções ilimitadas $\bar{K}$, definidas em R^n_+, existem métricas, conformes à métrica hiperbólica, com curvatura escalar $\bar{K}$.

1. Introduction

Over the last few years several authors have considered the following problem : Given a symmetric tensor of order two T defined on a manifold M^n, is there a Riemannian metric g such that $\text{Ric}g = T$? (P)

Finding solutions to this problem is equivalent to solving a nonlinear system of second-order partial differential equations. Deturck showed in [D1], that if $n \geq 3$, problem (P) has a local solution, when the given tensor T is nonsingular.

Results on the existence and uniqueness of solutions for the problem (P), when M^n is a bi-dimensional manifold, can be found in [D2] and [CD1]. For compact manifolds, some results can be found in [DK] and [H].

Cao and Deturck [CD2] studied the existence and uniqueness of global solutions in R^n and S^n, for rotationally symmetric and nonsingular tensors. In this case, they showed that problem (P) has a unique solution (up to homothety) and that for certain tensors in R^n, there is a complete metric $\bar{g}$, globally defined on R^n, such that $\mathrm{Ric} g = T$. On the sphere S^n, they proved some nonexistence results and found necessary conditions on a given tensor T , for the existence of a metric g on S^n satisfying $\mathrm{Ric}\ g = T$.

Uniqueness and even local existence may fail (see [DK] and [D1]) if nonsingular tensores T are considered as in [CD2].

Our main purpose in this work is to study problem (P) in the hyperbolic space $H^n(-1) = \{(R^n_+, g)$ with $n \geq 3,\ \ g_{ij} = \dfrac{\delta_{ij}}{x_n^2}$ and $R^n_+ = \{(x_1, \ldots, x_n) \in R^n;\ x_n > 0\}\}$,for symmetric tensors of the form

$$T = \mathrm{Ric}\ g + \sum_{i,j} \frac{c_{ij}}{x_n^2} dx_i \otimes dx_j \quad \text{with} \quad c_{ij} \in R \tag{1}$$

We want to find conformal metric to g such that

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g \\ \mathrm{Ric}\ \bar{g} = T, \end{cases} \tag{2}$$

where $\mathrm{Ric}\ \bar{g} = \bar{R}_{ij} dx_i \otimes dx_j$.

Considering T given by (1), we show in the Theorem 1.1 the existence of metrics $\bar{g}$, conformal to g, satisfying $\mathrm{Ric}\ \bar{g} = T$, if and only if, $T = 0$ or

$$T = \mathrm{Ric}\ g + \left(\frac{2n-3}{n-1} c - \frac{(n-2)}{(n-1)^2} c^2 \right) \sum_{i<n} \frac{dx_i^2}{x_n^2} + \frac{c}{x_n^2} dx_n^2$$

with $c \in R$.

As a consequence of Theorem 1.1, we find explicit solutions of C^∞ class, defined on R^n_+ for the equation

$$-\varphi \Delta_g \varphi + \frac{n}{2} ||\nabla_g \varphi||^2 + \lambda \varphi^2 = 0$$

where Δ_g and ∇_g are the laplacian and gradient in the metric g respectively and $\lambda \in \left(-\infty, \dfrac{(n-1)^2}{2(n-2)}\right)$.

Finally, we show that for certain functions $\bar{K}$ defined on R^n_+, there are metrics $\bar{g}$, conformal to g, with scalar curvature $\bar{K}$. These provide examples of unbounded functions which have positive answers to the following problem: Given a smooth function $\bar{K} : M^n \to R$ on a manifold (M, g) is there a metric $\bar{g}$ conformal to g whose scalar curvature is $\bar{K}$?

This problem has been studied by various authors . Particularly , when $\bar{K}$ is a constant it is know as the Yamabe Problem . In the hyperbolic space , various results can be found in [BK] ,[CKY], [LTY], [MR], [RV] and in their references.

We can now state our results.

Theorem 1.1. *Let $H^n(-1)$ be the hyperbolic space and T given by (1). Then there is a metric $\bar{g} = \dfrac{1}{\varphi^2} g$ such that $Ric\ \bar{g} = T$, if and only if, ($c_{ij} = 0$ for all $i \neq j$) and T satisfies the following conditions:*

I) $c_{ii} = n-1$ *for all* $1 \leq i \leq n$ $(T = 0)$. *In this case, all the solutions are given by*

$$\varphi(x_1, \dots x_n) = \frac{1}{x_n}\left(\sum_i \left(\frac{a}{2}x_i^2 + a_i x_i\right) + r\right) \tag{3}$$

where the constants satisfy $2ar = \sum_{i=1}^{n} a_i^2$.

II) $T = Ric\ g + \left(\dfrac{(2n-3)}{n-1}c - \dfrac{(n-2)}{(n-1)^2}c^2\right)\sum_{i<n}\dfrac{dx_i^2}{x_n^2} + \dfrac{c^2}{x_n^2}dx_n^2$ *with* $c \in R$. *In this case, all the solutions are given by* $\varphi(x_n) = kx_n^{\frac{-c}{n-1}}$ *where k is a nonzero constant.*

Corollary 1.2. *Let $H^n(-1)$ be the hyperbolic space and T given by (1). Then there are no complete metrics $\bar{g}$ conformal and nonhomothetic to g, such that $Ric\ \bar{g} = T$.*

Corollary 1.3. *Let $H^n(-1)$ be the hyperbolic space and $\lambda \in R$. Consider the* *differential equation*

$$-\varphi(\Delta_g\varphi) + \frac{n}{2}\left(||\nabla_g\varphi||^2\right) + \lambda\varphi^2 = 0. \tag{4}$$

I) *If* $\lambda = \frac{n}{2}$ *and exist* $i \neq n$ *such that* $\varphi_{x_i} \neq 0$, *then the functions given by (3) are solutions of (4).*

II) *If* $\lambda \in \left(-\infty, \frac{(n-1)^2}{2(n-2)}\right]$, *then the equation (4) has at least two solutions given by* $\varphi(x_n) = kx_n^{\frac{-c}{n-1}}$ *with* $c = \frac{n-1}{n-2}\left((n-1) \pm \sqrt{(n-1)^2 - 2\lambda(n-2)}\right)$.

Corollary 1.4. *Let* $H^n(-1)$ *be the hyperbolic space. For each* $c \in R$, *consider the functions* $\bar{K} : R^n_+ \to R$ *given by*

$$\bar{K}(x_1, \ldots, x_n) = \rho x_n^{\frac{-2c}{n-1}}$$

with

$$\rho = 2(n-1)c - \frac{(n-2)}{(n-1)}c^2 - 1.$$

Then the metric $\bar{g} = \frac{1}{\varphi^2}g$, *where* φ *is given in (I) or (II) of the Theorem 1.1, has scalar curvature* $\bar{K}$.

Before proving the main results, we observe that a similar theory in the pseudo-euclidean was treated in [PT].

2. Proof of the main results

We will start with some lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.1. *Solving the problem (2) is equivalent to studying the following system of equations*

$$\begin{cases} \varphi_{x_ix_i} = (-2\delta_{in} + 1)\dfrac{\varphi_{x_n}}{x_n} + \dfrac{1}{x_n^2}\left(\lambda_i\varphi + \dfrac{||\nabla_g\varphi||^2}{2\varphi}\right), \\[2ex] \varphi_{x_ix_j} = \dfrac{c_{ij}\varphi}{(n-2)x_n^2} - \delta_{jn}\dfrac{\varphi_{x_i}}{x_n} - \delta_{in}\dfrac{\varphi_{x_j}}{x_n}, \end{cases} \tag{5}$$

where $1 \leq i \neq j \leq n$ *and*

$$\lambda_i = \frac{2(n-1)c_{ii} - \sum c_{kk}}{2(n-1)(n-2)} \tag{6}$$

Proof. We know (see by example [KR]), that if (M, g) is a semi-Riemannian manifold and $\bar{g} = \frac{1}{\varphi^2} g$, then the Ricci tensors satisfy the relation

$$\mathrm{Ric}\bar{g} - \mathrm{Ric}g = \frac{1}{\varphi^2}\left\{(n-2)\varphi \mathrm{Hess}_g(\varphi) + (\varphi\Delta_g\varphi - (n-1)||\nabla_g\varphi||^2)g\right\} \tag{7}$$

Using (7) we obtain that (2) is equivalent to studying the following system of equations

$$\frac{1}{\varphi^2}\{(n-2)\varphi \ \mathrm{Hess}_g(\varphi)_{ij} + (\varphi\Delta_g\varphi - (n-1)||\nabla_g\varphi||^2)g_{ij}\} = \frac{c_{ij}}{x_n^2} \tag{8}$$

where, $1 \leq i, j \leq n$ and Hess_g, Δ_g and ∇_g are the Hessian, Laplacian and gradient in the metric g respectively.

The system of equations (8) is given by

$$\left\{\begin{array}{l} \frac{1}{\varphi^2}\left\{(n-2)\varphi\left[\varphi_{x_ix_i} + (-1+\delta_{in})\frac{\varphi_{x_n}}{x_n}\right] + \left(\varphi\Delta_g\varphi - (n-1)||\nabla_g\varphi||^2\right)\frac{1}{x_n^2}\right\} = \frac{c_{ii}}{x_n^2} \\ \frac{1}{\varphi^2}\left\{(n-2)\varphi\left[\varphi_{x_ix_j} + \delta_{jn}\frac{\varphi_{x_i}}{x_n} + \delta_{in}\frac{\varphi_{x_j}}{x_n}\right]\right\} = \frac{c_{ij}}{x_n^2} \end{array}\right. \tag{9}$$

Substituing $\Delta_g\varphi = x_n^2\left(\sum_j \varphi_{x_jx_j}\right) - (n-2)x_n\varphi_{x_n}$, in the first n equations of (9) we have

$$\begin{array}{l} \sum_{j\neq i}\varphi_{x_jx_j} + (n-1)\varphi_{x_ix_i} = 2(n-2)\frac{\varphi_{x_n}}{x_n} - 2(n-2)\delta_{in}\frac{\varphi_{x_n}}{x_n} \\ + \frac{c_{ii}\varphi^2 + (n-1)||\nabla_g\varphi||^2}{\varphi x_n^2} \quad \forall 1 \leq i \leq n. \end{array} \tag{10}$$

For a fixed i ,multiplying the equation (10) by $(2n-3)$ and adding with the $(n-1)$ remaining equations we obtain

$$\varphi_{x_kx_k} = (-1+2\delta_{kn})\frac{\varphi_{x_n}}{x_n} + \frac{1}{x_n^2}\left(\lambda_k\varphi + \frac{||\nabla_g\varphi||^2}{2\varphi}\right),$$

where λ_k is given by (6). The proof of the Lemma follows from (9) and (10).

$\square$

Lemma 2.2. *If $\varphi : R^n_+ \to R$ is a solution of the system of equations (5) then the following equations are satisfied.*

$$c_{ij}\left(\varphi_{x_k} - \delta_{kn}\frac{\varphi}{x_n}\right) = c_{ik}\left(\varphi_{x_j} - \delta_{jn}\frac{\varphi}{x_n}\right) \quad 1 \le i \ne j \ne k \le n \tag{11}$$

$$(-1+\lambda_i+\lambda_j)\varphi_{x_j} + \sum_{\substack{\ell\ne i\\ \ell\ne j,\ell\ne n}} \frac{c_{j\ell}}{n-2}\varphi_{x_\ell} + \frac{c_{nj}}{n-2}\left(\varphi_{x_n} + \frac{\varphi}{x_n}\right) = 0 \tag{12}$$

$1 \le i \ne j < n$

$$(-1+\lambda_j+\lambda_n)\varphi_{x_j} + \sum_{\substack{\ell\ne j\\ \ell\ne n}} \frac{c_{\ell j}}{n-2}\varphi_{x_\ell} + \frac{2c_{nj}}{(n-2)}\frac{\varphi}{x_n} = 0 \quad 1 \le j < n \tag{13}$$

$$(-1+\lambda_i+\lambda_n)\varphi_{x_n} + \sum_{\substack{\ell\ne i\\ \ell\ne n}} \frac{c_{n\ell}}{n-2}\varphi_{x_\ell} + (\lambda_n-\lambda_i)\frac{\varphi}{x_n} = 0 \quad 1 \le i < n \tag{14}$$

Proof. The proof follows from the comutativity of the derivatives of third order of φ.

$\square$

The proof of Theorem 1.1 will be completed after various steps.

Proposition 2.3. *Let $H^n(-1)$ be the hyperbolic space and T given by (1). If there is $\bar g = \dfrac{1}{\varphi^2}g$ with φ non-constant, such that $Ric\,\bar g = T$, then $c_{ij} = 0$ for all $1 \le i \ne j \le n$.*

Proof. We shall consider two cases:

I) Case: Suppose that $\varphi_{x_n} - \dfrac{\varphi}{x_n} \ne 0$ in a open subset $U \subset R^n_+$. By equation (11) we have that in U

$$c_{in}\varphi_{x_k} = c_{ik}\left(\varphi_{x_n} - \frac{\varphi}{x_n}\right) \quad \forall 1 \le i \ne k < n. \tag{15}$$

Taking derivative of (15) wiht respect to the variable x_i and using the system (5) we obtain

$$c_{ik}\varphi_{x_i} = 0 \quad \forall 1 \le i \ne k < n. \tag{16}$$

If there is at least one pair (i_0, k_0) with $1 \leq i_0 \neq k_0 < n$ such that $c_{i_0k_0} \neq 0$, it follows from (16) that $\varphi_{x_{i_0}} = 0$ and since φ satisfies the system (5) we obtain that $\varphi = 0$. (a contradition). Therefore, $c_{ik} = 0 \; \forall \; 1 \leq i \neq k < n$. In this case, the equation (15) is given by

$$c_{in}\varphi_{x_k} = 0 \quad \forall 1 \leq i \neq k < n. \tag{17}$$

If there is $i_0 < n$ such that $c_{i_0n} \neq 0$, using (17) and the system (5) we prove that $c_{kn} = 0 \; \forall \; 1 \leq i_0 \neq k < n$. Considering $i = i_0$ and $i \neq i_0$ fixed in (14) we obtain the following equations

$$\begin{aligned} &(-1 + \lambda_{i_0} + \lambda_n)\varphi_{x_n} + (\lambda_n - \lambda_{i_0})\frac{\varphi}{x_n} = 0 \\ &(-1 + \lambda_i + \lambda_n)\varphi_{x_n} + (\lambda_n - \lambda_i)\frac{\varphi}{x_n} + \frac{c_{i_0n}}{n-2}\varphi_{x_{i_0}} = 0 \end{aligned}$$

Considering the difference of the equations above we obtain

$$\frac{c_{i_0n}}{n-2}\varphi_{x_{i_0}} + (\lambda_i - \lambda_{i_0})\left(\varphi_{x_n} - \frac{\varphi}{x_n}\right) = 0. \tag{18}$$

Now considering $i \neq i_0$ and $j = i_0$ in (12) we obtain

$$(-1 + \lambda_i + \lambda_{i_o})\varphi_{x_{i_0}} + \frac{c_{i_0n}}{n-2}\left(\varphi_{x_n} + \frac{\varphi}{x_n}\right) = 0.$$

If $j = i_0$ in (13), we have

$$(-1 + \lambda_n + \lambda_{i_0})\varphi_{x_{i_0}} + \frac{2c_{i_0n}}{n-2}\left(\frac{\varphi}{x_n}\right) = 0.$$

Considering the difference of the equatios above we obtain

$$(\lambda_i - \lambda_n)\varphi_{x_{i_0}} + \frac{c_{i_0n}}{n-2}\left(\varphi_{x_n} - \frac{\varphi}{x_n}\right) = 0. \tag{19}$$

$\varphi_{x_{i_0}} = 0$, otherwise if $\varphi_{x_{i_0}} \neq 0$ in a open subset $\bar{U} \subset U$, it follows from (18) and (19) that

$$\left(\frac{c_{i_0n}}{n-2}\right)^2 = (\lambda_i - \lambda_n)(\lambda_i - \lambda_{i_0}). \tag{20}$$

Taking derivative of (18) with respect to the variable x_{i_0} we have

$$\frac{c_{i_0n}}{n-2}\varphi_{x_{i_0}x_{i_0}} + (\lambda_i - \lambda_{i_0})\varphi_{x_{i_0}x_n} + (\lambda_{i_0} - \lambda_i)\frac{\varphi_{x_{i_0}}}{x_n} = 0 \tag{21}$$

$\forall\ 1 \leq i_0 \neq i < n$. As $\varphi_{x_i} = 0$ for all $i \neq i_0,\ i < n$, it follows from system (5) that

$$\frac{||\nabla_g \varphi||^2}{2\varphi_{x_n}} + \varphi_{x_n} = -\frac{\lambda_i \varphi}{x_n}. \tag{22}$$

Using (22) and the system (5), we have that the equation (21) is given by

$$(\lambda_{i_0} - \lambda_i)\frac{\varphi_{x_{i_0}}}{x_n} = 0.$$

Therefore, $\lambda_{i_0} - \lambda_i = 0$ in $\bar{U}$. It follows by (20) that $c_{i_0 n} = 0$; that is a contradiction. Therefore $\varphi_{x_{i_0}} = 0$. Since, $c_{i_0 n} \neq 0$ we obtain from system (5) that $\varphi = 0$. In this case, we conclude that $c_{in} = 0\ \forall\ 1 \leq i < n$.

II) Case: If $\varphi_{x_n} - \dfrac{\varphi}{x_n} = 0$ in a open subset $V \subset R^n_+$, we obtain from (15) that

$$c_{in}\varphi_{x_k} = 0 \quad \forall 1 \leq i \neq k < n. \tag{23}$$

If there is $i_0 < n$ such that $c_{i_0 n} \neq 0$, it follows from (23) that $\varphi_{x_k} = 0$ for all $k < n,\ k \neq i_0$. Then from system (5) $c_{kn} = 0\ \forall\ 1 \neq k \neq i_0 < n$. Considering $i = i_0$ in (14) we obtain that $(-1 + 2\lambda_n)\dfrac{\varphi}{x_n} = 0$. In this case, $-1 + 2\lambda_n = 0$.

Now considering $i = i_0$ in (14) we obtain that $\varphi_{x_{i_0}} = 0$ and consequentely $\varphi_{x_{i_0} x_n} = 0$. It follows from system (5) that $\varphi = 0$ in $V \subset R^n_+$, which is a contradiction. Therefore $c_{in} = 0$ for all $i < n$. Since $\varphi_{x_n} - \dfrac{\varphi}{x_n} = 0$ and $c_{in} = 0$ for all $i < n$, it follows from system (5), that $c_{ij} = 0$ for all $1 \leq i \neq \leq n$. c_{ij} are constants hence we obtain from (I) and (II) that $c_{ij} = 0$ for all $1 \leq i \neq j \leq n$ in R^n_+.

□

We showed that if problem (2) has a solution then the given tensor T is necessarily diagonal.

Lemma 2.4 *Let $H^n(-1)$ be the hyperbolic space and $T = Ric\ g + \sum \dfrac{c_{ii}}{x_n^2} dx_i^2$. Suppose that there is $\bar{g} = \dfrac{1}{\varphi^2} g$ satisfying $Ric\ \bar{g} = T$. If there is at least one $j_0 < n$, such that $\varphi_{x_{j_0}} \neq 0$, then $c_{ii} = n - 1$, $\forall\ 1 \leq i \leq n$.*

Proof. It follows directly from equations (12), (13) and (14).

□

Now we shall study the problem (2) when $c_{ii} = n-1$, $\forall\ 1 \le i \le n$. In this case, $T = 0$.

Proposition 2.5. *Let $H^n(-1)$ be the hyperbolic space and $T = 0$. Then there is $\bar{g} = \frac{1}{\varphi^2} g$ such that $Ric\bar{g} = 0$, if and only if,*

$$\varphi(x_1, \dots, x_n) = \frac{1}{x_n}\left(\sum_{i=1}^{n}\left(\frac{a}{2}x_i^2 + a_i x_i\right) + r\right) \quad \text{where} \quad 2ar = \sum_{i=1}^{n} a_i^2,$$

$a, r, a_i \in R$.

Proof. Since $T = 0$, the system (5) is given by

$$\begin{cases} \varphi_{x_i x_i} = (-2\delta_{in} + 1)\dfrac{\varphi_{x_n}}{x_n} + \dfrac{1}{x_n^2}\left(\dfrac{\varphi}{2} + \dfrac{||\nabla_g \varphi||^2}{2\varphi}\right) \\ \varphi_{x_i x_j} = 0\ ,\ 1 \le i \ne j < n \\ \varphi_{x_i x_n} = -\dfrac{\varphi_{x_i}}{x_n} \end{cases} \tag{24}$$

Since $\varphi_{x_i x_j} = 0$, $\forall\ 1 \le i \ne j < n$, we have that

$$\varphi(x_1, \dots, x_n) = \sum_{i<n} \psi_i(x_i, x_n).$$

It follows from (24) that $\varphi_{x_i x_i} = \varphi_{x_j x_j}\ \forall\ 1 \le i \ne j < n$. Therefore

$$(\psi_i)_{x_i x_i} = h(x_n). \tag{25}$$

Using (24) and (25) we obtain that

$$\psi_i(x_i, x_n) = \frac{1}{x_n}\left(\frac{a}{2}x_i^2 + a_i x_i\right) + h_i(x_n) \tag{26}$$

where a and a_i are constants.

The function $\varphi(x_1, \dots, x_n) = \sum_{i<n} \psi_i(x_i, x_n)$ with ψ_i given by (26) satisfies the last two equations of (24). We still have to see if the first equation of (24) is satisfied.

Since $\varphi_{x_i x_i} = \dfrac{a}{x_n}$ $\forall\, i < n$, this equation reduces to the following

$$\begin{cases} a = \varphi_{x_n} + \dfrac{1}{x_n}\left(\dfrac{\varphi}{2} + \dfrac{||\nabla_g \varphi||^2}{2\varphi}\right) \\ \varphi_{x_n x_n} = -\dfrac{\varphi_{x_n}}{x_n} + \dfrac{1}{x_n^2}\left(\dfrac{\varphi}{2} + \dfrac{||\nabla_g \varphi||^2}{2\varphi}\right) \end{cases} \tag{27}$$

Taking the difference of these equations we obtain

$$\varphi_{x_n x_n} = \frac{a}{x_n} - 2\frac{\varphi_{x_n}}{x_n}. \tag{28}$$

Substituing φ given by (26) in (28), we obtain that

$$\sum_{i<n} h_i(x_n) = \frac{1}{x_n}\left(\frac{a}{2}x_n^2 + a_n x_n + r\right).$$

Therefore the function φ is given by

$$\varphi(x_1, \ldots, x_n) = \frac{1}{x_n}\left(\sum_{i=1}^{n}\left(\frac{a}{2}x_i^2 + a_i x_i\right) + r\right).$$

Finally, it is easy to see that φ satisfies equations in (27), if and only if, $2ar = \sum_{i=1}^{n} a_i^2$.

$\square$

Remark 2.6. We see from Lemma 2.4 that if T is a nonzero tensor then $\varphi : R_+^n \to R$ satisfying (2), depend only on the variable x_n. In this case, we necessarily have that $c_{11} = \ldots = c_{n-1n-1}$. Therefore, we need to study the problem (2) only for tensors of the form

$$T = \text{Ric } g + \frac{c_{11}}{x_n^2}\sum_{i<n} dx_i^2 + \frac{c_{nn}}{x_n^2}dx_n^2. \tag{29}$$

with $c_{11}, c_{nn} \in R$.

Lemma 2.7. *Let $H^n(-1)$ be the hyperbolic space and let T be a tensor given by (29). If there is $\bar{g} = \dfrac{1}{\varphi^2}g$ such that Ric $\bar{g} = T$, then*

$$\varphi(x_n) = \begin{cases} k_1 x_n^{r_1} + k_2 x_n^{r_2} \;\; if \;\; \dfrac{c_{11} - c_{nn}}{n-2} < \dfrac{1}{4}, \\ (k_1 + k_2(\log x_n))x_n^{-\frac{1}{2}} \;\; if \;\; \dfrac{c_{11} - c_{nn}}{n-2} = \dfrac{1}{4}, \\ x_n^{-\frac{1}{2}}(k_1 cos(\mu \log x_n) + k_2 sen(\mu \log x_n)) \;\; if \;\; \dfrac{c_{11} - c_{nn}}{n-2} > \dfrac{1}{4}, \end{cases} \tag{30}$$

where

$$r_1 = \frac{-1+\sqrt{1-4(\lambda_1-\lambda_n)}}{2}, \quad r_2 = \frac{-1-\sqrt{1-4(\lambda_1-\lambda_n)}}{2},$$

$$\mu = \sqrt{4(\lambda_1-\lambda_n)-1}$$

and $k_1, k_2 \in R$.

Proof. In this case, it follows from Remark 2.6 that the system (5) is given by

$$\begin{cases} 0 = \varphi'(x_n) + \dfrac{1}{x_n}\left(\lambda_1\varphi + \dfrac{||\nabla_g\varphi||^2}{2\varphi}\right) \\ \varphi''(x_n) = -\dfrac{\varphi'(x_n)}{x_n} + \dfrac{1}{x_n^2}\left(\lambda_n\varphi + \dfrac{||\nabla_g\varphi||^2}{2\varphi}\right). \end{cases} \tag{31}$$

Considering the difference of these equations, we obtain

$$x_n^2\varphi''(x_n) + 2x_n\varphi''(x_n) + \frac{c_{11}-c_{nn}}{n-2}\varphi = 0. \tag{32}$$

Equation (32) is a particular case of the Euler equation. Its general solution is given by (30), (see [BD]).

□

Using Lemma 2.7 we shall prove the following results.

Proposition 2.8. *Let $H^n(-1)$ be the hyperbolic space and let T be a tensor given by (29) with $\frac{c_{11}-c_{nn}}{n-2} \leq \frac{1}{4}$. Then there exists $\bar{g} = \frac{1}{\varphi^2}g$ such that Ric $\bar{g} = T$, if and only if,*

$$c_{11} = \frac{(2n-3)}{n-1}c_{nn} - \frac{(n-2)}{(n-1)^2}c_{nn}^2.$$

In this case, all the solutions $\varphi = \varphi(x_n)$ are give, explicitly by $\varphi(x_n) = kx_n^{-\frac{c_{nn}}{n-1}}$ where $c_{nn} \in R$ and k is nonzero constant.

Proof. In the proof of the previous Lemma we have seen that for a tensor given by (29), studying the problem (2) is equivalent to study the system (31). If $\frac{c_{11}-c_{nn}}{n-2} < \frac{1}{4}$, it follow from Lemma 2.7 that the solution of (31) is necessarily given by $\varphi(x_n) = k_1 x_n^{r_1} + k_2 x_n^{r_2}$. Substituing φ in (31) we see that a solution exist, if and only if,

$$c_{11} = \left(\frac{2n-3}{n-1}\right) c_{nn} - \frac{n-2}{(n-1)^2} c_{nn}^2, \; c_{nn} \neq \frac{n-1}{2}.$$

Moreover, all solutions are given by $\varphi(x_n) = k x_n^{-\frac{c_{nn}}{n-1}}$. If $\frac{c_{11} - c_{nn}}{n-2} = \frac{1}{4}$, then the solutions of (31) are necessarily given by $\varphi(x_n) = (k_1 + k_2 \log x_n) x_n^{-\frac{1}{2}}$.

Finally, we see that equations in (31) are satisfied, if and only if, $k_2 = 0, c_{11} = \frac{3n-4}{2}$, $c_{nn} = \frac{n-1}{2}$. Therefore, all the solutions are given by $\varphi(x_n) = k x_n^{-\frac{1}{2}}$.

□

Proposition 2.9. *Let $H^n(-1)$ be the hyperbolic space and let T be a tensor given by (29) with $\frac{c_{11} - c_{nn}}{n-2} > \frac{1}{4}$. Then there are no metrics $\bar{g} = \frac{1}{\varphi^2} g$ such that $Ric\, \bar{g} = T$.*

Proof. Since $\frac{c_{11} - c_{nn}}{n-2} > \frac{1}{4}$, we have from Lemma 2.7 that the solution of the system (31) is necessarily given by

$$\varphi(x_n) = x_n^{-\frac{1}{2}} (k_1 \cos(\mu \log x_n) + k_2 \operatorname{sen}(\mu \log x_n)).$$

Substituing this expression in (31), we see that the system does not admit non-null solution.

□

Proof of Theorem 1.1. It follows from Propositions 2.3, 2.5, 2.8, 2.9, Lemma 2.4, and Remark 2.6, denoting c_{nn} by c.

□

Proof of Corollary 1.2. One can prove that the metrics obtained in the Theorem 1.1 are not complete.

□

Proof of Corollary 1.3. If $\varphi : R_+^n \to R$ is a solution of the system (5), in particular it satisfies the following equations

$$\frac{1}{\varphi^2}\left\{(n-2)\varphi \mathrm{Hess}_g(\varphi)_{ii} + (\varphi\nabla_g\varphi - (n-1)||\nabla_g\varphi||^2)g_{ii}\right\} = \frac{c_{ii}}{x_n^2} \tag{33}$$

for all $1 \le i \le n$.

If $c_{ii} = n-1$, $1 \le i \le n$ it follows from Proposition 2.5 that

$$\varphi(x_1,\ldots,x_n) = \frac{1}{x_n}\left(\sum_i \left(\frac{a}{2}x_i^2 + x_i\right) + r\right)$$

where $2ar = \sum_{i=1}^{n} a_i^2$, are solutions of (33).

In this case, we obtain equation (4) with $\lambda = \frac{n}{2}$ by adding the equations above multiplied by x_n^2.

If $c_{ii} = \left(\frac{2n-3}{n-1}\right)c_{nn} - \frac{(n-2)}{(n-1)^2}c_{nn}^2$ for all $1 \le i < n$ and $c_{nn} \in R$, it follows from Propostion 2.8 that $\varphi(x_n) = kx_n^{-\frac{c_n n}{n-1}}$ are solutions of (33).

For to obtain the equation (4) with $\lambda = c - \frac{(n-2)}{2(n-1)^2}c^2$ it is enough to add the equations in (33) multiplied by x_n^2.

□

Proof of Corollary 1.4. It follows from the relation (7) that, if $H^n(-1) = (R_+^n, g)$ is the hyperbolic space and $\bar{K} : R_+^n \to R, \bar{K} = \sum_{i,j} \bar{g}^{ij}\bar{R}_{ij}$ is a smooth function, finding $\bar{g} = \frac{1}{\varphi^2}g$ with scalar curvature $\bar{K}$ is equivalent to solving the following differential equation

$$-\varphi\Delta_g\varphi + \frac{n}{2}||\nabla_g\varphi||^2 + \frac{1}{2(n-1)}(\varphi^2 + \bar{K}) = 0. \tag{34}$$

Considering $\bar{K} = \rho x_n^{-2\frac{c}{n-1}}$ we have that if $\rho = 0$, it follows from Corollary 1.3 that the solutions of equation (34) are following functions

$$\varphi(x_1,\ldots,x_n) = \frac{1}{x_n}\left(\sum_{i=1}^{n} \left(\frac{a}{2}x_i^2 + a_i x_i\right) + r\right)$$

where $2ar = \sum_{i=1}^{n} a_i^2$,

$$\varphi(x_n) = kx_n^{-1} \quad \text{and} \quad \varphi(x_n) = kx_n^{-\frac{n}{n-2}}.$$

If $\rho \neq 0$ we have that $\varphi(x_n) = kx_n^{-\frac{c}{n-1}}$ is solution of (34).

$\square$

Acknowledgement. I thank Professor Keti Tenenblat for her very helpful comments and constant encouragement.

References

[BD] Boyce, W. E.; Diprima, R. C., *Elementary Differential Equations and Boundary Value Problems* 3^a ed., John Wiley&Sons ,(1977).

[BK] Bland, J., Kalka, M., *Complete metrics conformal to the hyperbolic disc*, Proc. Amer. Math. Soc. 97 (1986), 128-132.

[CD1] Cao, J., Deturck, D., *Prescribing Ricci Curvature on open surfaces*, Hokkaido Math. J. 20 (1991), 265-278.

[CD2] ————-, *The Ricci curvature equation with rotational symmetry*, American Journal of Mathematics 116 (1994), 219-241.

[CKY] Chen, C. Y., Cheng, K.S., Yu, W. W., *Conformal deformations of metrics on $H^n(-1)$ with prescribed scalar curvature*, Chinese J. Math. 16, (1988), 157-187.

[D1] Deturck, D., *Existence of metrics with prescribed Ricci Curvature: Local Theory*, Invent. Math. 65 (1981), 179-207.

[D2] ————, *Metrics with prescribed Ricci curvature, Seminar on Differential Geometry*, Ann. of Math. Stud. Vol. 102, (S. T. Yau, ed.), Princeton University Press, (1982), 525-537.

[DK] Deturck, D., Koiso, W., *Uniqueness and Non-existence of Metrics with prescribed Ricci Curvature*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 351-359.

[H] Hamilton, R. S., *The Ricci curvature equation*, Seminar on nonlinear partial differential equations (Berkeley, California, 1983), 47-72.

[KR] Kühnel, W., Rademacher, H.B., *Conformal diffeomorphisms preserving the Ricci tensor*, Proc. of the American Mathematical Society, Vol 123, (1995), 2841-2848.

[LTY] Li, P., Tam, L., Yang, D., *On the elliptic equaton* $\Delta u + Ku - Ku^p = 0$ *on complete Riemannian Manifolds and their Geometric Applications*, Transactions of the American Mathematical Society, Vol. 350, (1998), 1045-1078.

[MR] McOwen, R., Aviles, P., *Conformal deformations of complete manifolds with negative curvature*, J. Diff. Geom. 21 (1985), 269-281.

[PT] Pina, R.,Tenenblat, K., *Conformal Metrics and Ricci Tensors in the pseudo-Euclidean space, to appear.*

[RV] Ratto, A., Rigoli, M., Veron, L. *Scalar curvature and conformal deformation of hiperbolic space*, Journal of Funcional Analisis 121 (1994), 15-77.

R. Pina
Instituto de Matemática e Estatística
Universidade Federal de Goiás
Goiânia, GO, Brazil
Email: romildo@mat.ufg.br