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HERMITIAN STRUCTURES AND EQUI-HARMONIC TORI ON NON-SYMMETRIC COMPLEX FLAG MANIFOLDS *

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Dedicated to Prof. M. do Carmo on the occasion of his 70th birthday

Abstract

In this note we give a analyse Hermitian structures on flag manifolds and its relationship with tournament theory. We also discuss equi-harmonic maps into flag manifolds (i.e maps which are harmonic for any left invariant Borel type metric), as well as some stability properties of a very important class of maps, namely: the Eells-Wood's maps.

Resumo

Nesta nota fazemos uma exposição sobre as estruturas Hermitianas em variedades bandeira e a relação destas com a Teoria de torneios. Discutimos também aplicações equi-harmônicas em variedades bandeira (isto é, aplicações as quais são harmônicas relativamente a cada métrica invariante do tipo de Borel), assim como algumas propriedades de estabilidade relativas a uma classe muito importante de aplicações, a saber: as aplicações de Eells-Wood.

1. Introduction

In this paper we give a survey on some results obtained during the process of understanding harmonic maps from compact Riemann surfaces into flag manifolds. Since minimal surfaces are harmonic in some conformal structure, special examples are provided by them.

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Calabi [8], Chern [9] and Eells-Sampson [13] initiated the modern study of harmonic maps into Riemannian manifolds. It was given a special attention to the case of 2-spheres in S^n (or $\mathbb{R}^n$). Important results were obtained in [10], [12] and [1].

The physicists Din-Zakarewski and Glaser-Stora showed that it could be very useful to complexify the problem, i.e. to consider the natural and totally geodesic embedding of $\mathbb{RP}^n$ into $\mathbb{CP}^n$.

Eells-Wood [14] classified all harmonic maps $\phi : S^2 \to G_1(\mathbb{C}^n) \approx (\mathbb{CP}^{n-1}$, metric of Fubini-Study) in terms of full holomorphic maps $h : S^2 \approx \mathbb{CP}^1 \to \mathbb{CP}^{n-1}$. Chern-Wolfson [11] and independently Burstall-Wood [7] classified every harmonic map $\phi : S^2 \to G_k(\mathbb{C}^n)$, $k = 2, 3, 4$ and 5 in terms of the holomorphic maps between these manifolds. More generaly, using Cartan's embedding, Uhlenbeck in [25] classified all harmonic maps $\phi : S^2 \to G_k(\mathbb{C}^n)$ for arbitrary k also in terms of holomorphic data.

In this note we describe some properties of Hermitian structures on flag manifolds. We will give some results relative to f-structures. Such structures appear naturally in the theory of twistors.

In [20] is derived the harmonic map equations for the energy functional defined in the space of maps from surfaces into flag manifolds in order to construct an infinite family of examples which show that a converse of a well known theorem due to Black [2] is not true. We recall that Black's theorem states that a holomorphic map with respect to a horizontal f-structure from a surface into a flag manifold must be equi-harmonic.

Finally, we discuss some stability properties of a very important class of equi-harmonic maps, namely, the Eells-Wood's maps.

2. Complex Geometry of $F(n)$

We can see geometrically the maximal flag manifold $F(n)$ as the space formed by n-tuples $(L_1, ..., L_n)$ such that L_i is a rank one subspace of $\mathbb{C}^n$, $L_i \vdash L_j, i \neq j$ and $\bigoplus_{i=1}^{n} L_i = \mathbb{C}^n$.

Algebraically we can describe $F(n)$ as $\dfrac{U(n)}{T}$ where $U(n) = \{A \in M(n \times n, \mathbb{C});$
$A.\overline{A}^t = A.A^* = I\}$ and T is an maximal torus of $U(n)$ i.e., $T = \underbrace{U(1) \times \ldots \times U(1)}_{n-\text{times}}$.
Let $p = T(F(n))_{(T)}$ the tangent space of $F(n)$ at (T) and $u(n) = \{X \in M(n \times n, \mathbb{C}); X + X^* = 0\} = p \oplus \underbrace{u(1) \oplus \ldots \oplus u(1)}_{u-\text{times}}$.

Now we consider $U(n)$-invariant almost complex structures $J : p \to p$; $J^2 = -I$. Borel and Hirzebruch [4] showed that there are $2^{\binom{n}{2}}$ such invariant structures.

Example 2.1. We consider $n = 3$ and $J : p \to p$ defined in the following way:

$$J\left[\begin{pmatrix} 0 & a_{12} & a_{13} \\ -\overline{a}_{12} & 0 & a_{23} \\ -\overline{a}_{13} & -\overline{a}_{23} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \varepsilon_1\sqrt{-1}\,a_{12} & \varepsilon_2\sqrt{-1}\,a_{13} \\ \varepsilon_1\sqrt{-1}\,\overline{a}_{12} & 0 & \varepsilon_3\sqrt{-1}\,a_{23} \\ \varepsilon_2\sqrt{-1}\,\overline{a}_{13} & \varepsilon_3\sqrt{-1}\,\overline{a}_{23} & 0 \end{pmatrix}$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$ and 3. There are $2^{\binom{3}{2}} = 2^3 = 8$ distinct invariant almost complex structures.

Definition 2.1. *A Tournament τ consists of a finite set $T = \{1, 2, \ldots, n\}$ of players together with a dominance relation that assigns to every pair a winner. Thus, if $i, j \in T (i \neq j)$, then either $i \to j$ or $j \to i$. Let τ_1 and τ_2 be two tournaments. A map $\phi : T_1 \to T_2$ of their sets is called a homomorphism if $i \to j \Leftrightarrow \phi(i) \to \phi(j)$ or $\phi(i) = \phi(j)$. If ϕ is bijective, it is said to be an isomorphism from τ_1 to τ_2. We define the canonical tournament τ_n in the following way:*

$$i \xrightarrow{\tau_n} j \Longleftrightarrow i < j$$

We notice that there is a natural 1–1 correspondence between almost complex structures J on $F(n)$ and tournaments τ_j with n elements. More precisely, if $J(a_{ij}) = (a'_{ij})$, then τ_j is determined by

$$\begin{aligned} i \to j(i<j) &\Longleftrightarrow a'_{ij} = \sqrt{-1}\, a_{ij} \\ j \to i(i<j) &\Longleftrightarrow a'_{ij} = -\sqrt{-1}\, a_{ij} \end{aligned}$$

For example, if J is the invariant almost complex structure defined by

$$J \begin{pmatrix} 0 & a_{12} & a_{13} \\ -\overline{a}_{12} & 0 & a_{23} \\ -\overline{a}_{13} & -\overline{a}_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\, a_{12} & -\sqrt{-1}\, a_{13} \\ \sqrt{-1}\, \overline{a}_{12} & 0 & \sqrt{-1}\, a_{23} \\ -\sqrt{-1}\, \overline{a}_{13} & \sqrt{-1}\, \overline{a}_{23} & 0 \end{pmatrix}$$

then the associated tournament τ_j is

Theorem 2.1 (Burstall - Salamon [5]). *There is a 1–1 correspondence between invariant almost complex structures J on $F(n)$ and n-dimensional tournaments τ_J. Also,*

J is integrable $\Leftrightarrow$ τ_J is isomorphic to τ_n (canonical tournament) $\Leftrightarrow$ τ_J does not contain 3-cycles,

where 3-cycles are closed paths of the form

We define the following left invariant metrics on $F(n)$ (called Borel type metrics [3]):

$$\langle A, B \rangle := \sum_{i,j} \lambda_{ij} \left(\sum \lambda_{ij} A E_i B^* E_j\right),$$

Here $\Lambda = (\lambda_{ij})$ is a symmetric matrix with $\lambda_{ij} > 0$, $\lambda_{ii} = 0$ and $E_i = (c_{\alpha\beta})$ with $c_{ii} = 1$ and $c_{\alpha\beta} = 0$ if p or q is different from i, and $1 \leq i, j, \alpha, \beta \leq n$.

We define the Kähler form as usual by $\Omega(A, B) = \langle A, JB\rangle_{ds^2_\Lambda}$. If $d\Omega = 0$ we say that $(F(n), J, ds^2_\Lambda)$ is almost Kähler. If J is integrable we call $(F(n), J, ds^2_\Lambda)$ a Kähler manifold. In [23] the following theorem is proved:

Theorem 2.2. ([16]) If $(F(n), J, ds^2_{\Lambda=(\lambda_{ij})})$ is almost Kähler $\Leftrightarrow$ it is a Kähler manifold.

The case $\lambda_{ij} = 1 \forall 1 \leq i \neq j \leq n$ is called the Killing form metric or simply the Killing metric. It is known that $(F(n), J, \text{Killing metric})$ is not a Kähler manifold for $n \geq 3$ (nevertheless, there are an infinite number of metrics such that $(F(n), J, ds^2_\Lambda)$ is a Kähler manifold).

Definition 2.2. *A map $\phi : (M, J_1, g) \to (F(n), J, ds^2_\Lambda)$ is J-holomorphic if $d\phi \circ J_1 = J \circ d\phi$ (i.e., ϕ satisfies the Cauchy-Riemann equations).*

Lichnerowicz in [17] has proved a theorem showing that J-holomorphic maps are harmonic if g and ds^2_Λ are $(1, 2)$-symplectic, i.e. the $(1, 2)$-component of $d\Omega$ is zero, i.e. $d\Omega^{1,2} = 0$. (Notice that a Kähler metric is always (1,2)-symplectic). These metrics appear naturally in the theory of twistors. We can give an alternative proof of the following theorem of Gray-Wolf [16]. We use only tournament theory.

Theorem 2.3. *The Killing metric on $F(n)$ is (1,2)-symplectic if and only if $n \leq 3$.*

Sketch of the Proof: Using moving frames and Cartan structural equations (see [18] for all the details) we can prove that the number of 3-cycles in τ_J is $\binom{n}{3}$. However, this is impossible, because if $n > 3$, according to Gale's inequality [15] the number of 3-cycles in τ_J is less than or equal to $\frac{1}{24}(n^2-n)$ if n is odd or $\frac{1}{24}(n^3-4n)$ if n is even.

3. Harmonic maps on $F(n)$. Eells-Wood's maps

Let $\phi : M^2 \to F(n)$ be a smooth map from a Riemannian surface M^2 and $\widetilde{\phi} : M \to U(n)$ it lift map, i.e. $\phi = \Pi \circ \widetilde{\phi}$ where $\Pi : U(n) \to F(n)$ is the natural projection. Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{C}^n$, i.e. $e_j = (0, \dots, 1, \dots, 0)^t$. We denote by $\Pi_j : M \to gl(n, \mathbb{C})$ the matrix of the orthogonal projection onto $E_j = \{ae_j;\ a \in \mathbb{C}\}$ with respect to $e_1, \dots, e_n$. Then ϕ can be thought as $\phi = (\Pi_1, \dots, \Pi_n) : M^2 \to F(n)$. Then $\Pi_i \frac{\partial \Pi_j}{\partial z} := A_z^{ij}$ are the matrices associtated to the second fundamental forms A_{ji}, i.e.

$$A'_{ji}(e_1, ..., e_n) = (e_1, ..., e_n) A_z^{ij}$$

i.e. $A'_{ij}(e_i) = e_j A_z^{ij}$. According to Burstall [5], for $V \in \Gamma(\phi^* T(F(n))$ we set $q = \phi^*\beta(V)$ where $\phi^*\beta : \phi^* T(F(n)) \to M \times u(n)$ is the pull-back of the Maurer-Cartan forms. We define a variation of ϕ by: $\phi_t(x) := \Pi(\exp(-tq)\widetilde{\phi})$. Denote associated objects by $\Pi_j(t)$, $A_z^{ij}(t)$, ...

Then with respect to $ds^2_{\Lambda=(\lambda_{ij})}$ the energy of ϕ_t is defined by:

$$E(\phi_t) := \int_M \sum \lambda_{ij} |A_z^{ij}(t)|^2_{v_g}.$$

After some calculations (see [18] or [19] for details) we obtain the Euler-Lagrange equations for the energy functional:

Propostition 3.1. *$\phi : (M, g) \to (F(n), ds^2_\Lambda)$ is harmonic if and only if*

$$Re\left(\frac{\partial}{\partial \bar{z}} A_z^\Lambda\right) = 0 \Leftrightarrow \frac{\partial}{\partial x} A_x^\Lambda + \frac{\partial}{\partial y} A_y^\Lambda = 0,$$

where $A_x^\Lambda := \sum \lambda_{ij} \Pi_i \frac{\partial \Pi_j}{\partial x}$, $A_y^\Lambda := \sum \lambda_{ij} \Pi_i \frac{\partial \Pi_j}{\partial y}$.

On the other hand, given a holomorphic and nondegenarate map $h : M \to \mathbb{CP}^{n-1}$ we can lift it locally in $\mathbb{C}^n$, i.e., for every $p \in M$ we can find a neighborhood of p such that $u : U \to \mathbb{C}^n$ satisfies $h(z) = [(u_0(z), ..., u_{n-1}(z))]$.

Now we define the k-th associate curve of h (or u) denoted by $\mathcal{O}_k$ as: $\mathcal{O}_k : M \to G_{k+1}(\mathbb{C}^n)$; $z \mapsto u(z) \wedge u'(z) \wedge ... \wedge u^{(k)}_{(z)}$.

We can see that $\mathcal{O}_k$ is well defined. Hence, we consider $h_k : M \to \mathbb{CP}^{n-1}$ as $h_k(z) = \mathcal{O}_k^{\perp}(z) \cap \mathcal{O}_{k+1}(z)$, $0 \le k \le n-1$. We have the following theorem due to Eells, Wood, Din, Zakarewski, Glaser and Stora.

Theorem 3.1. ([14]) *For each $k \in [0, n-1] \cap I\!N$, $h_k : M \to \mathbb{CP}^{n-1}$ is harmonic. Furthermore, given $\phi : (\mathbb{CP}^1, g) \to (\mathbb{CP}^{n-1}, \text{Fubini-Study metric})$ harmonic, then there are unique k and h like above such that $\phi = h_k$.*

Then this theorem above provides us with a natural set of maps $\psi : M \to F(n)$, namely: $\psi(z) = (h_0(z), \ldots, h_{n-1}(z))$ for an arbitrary map $h : M \to \mathbb{CP}^{n-1}$ holomorphic and full (a map $\phi : M \to \mathbb{CP}^{n-1}$ is said to be full if $\phi(M)$ is not contained in any proper subspace of $\mathbb{C}^n$). We call any arbitrary map in this collection an Eells-Wood's map.

Definition 3.1. *An arbitrary map $\phi : M^2 \to F(n)$ is said to be equi-harmonic if $\phi : (M, g) \to (F(n), ds^2_\Lambda)$ is harmonic for any left invariant Borel type metric ds^2_Λ.*

Theorem 3.2. ([20]) *If $\psi = (h_0, ..., h_{n-1}) : M^2 \to F(n)$ is an Eells-Wood's map then ψ is equi-harmonic.*

4. Equi-harmonic tori

In this section we extend Uhlenbeck's separation of variables method as described in [24].

Definition 4.1. *An f-structure on $F(n)$ is a section $\mathcal{F}$ of End$(T(F(n)))$ such that $\mathcal{F}^3+\mathcal{F}=0$. This concept is due to Yano.*

The set of $U(n)$-invariant f-structures on $F(n)$ is naturally identified with the set of T-equivariant ($T=\underbrace{U(1)\times\cdots\times U(1)}_{n\,\text{times}}$) endomorphisms $\mathcal{F}$ such that $\mathcal{F}^3+\mathcal{F}=0$. Therefore, an $U(n)$-invariant almost complex structure J on $F(n)$ results from a T-invariant f-structure.

We notice that an invariant almost complex structure J on $F(n)$ is a special case of f-structure. Suppose that $\phi:\mathbb{R}^2\to F(n)$ is defined by

$$\phi=\Pi\circ\widetilde{\phi},\quad \widetilde{\phi}(x,y)=e^{Ax+By},$$

where A, $B\in u(n)$, $[A,B]=0$. Then $\widetilde{\phi}(x,y)=e^{By}\cdot e^{Ax}$ and

$$\frac{\partial\widetilde{\phi}}{\partial x}=\widetilde{\phi}A,\qquad \frac{\partial\widetilde{\phi}^*}{\partial x}=-A\widetilde{\phi}^*.$$

Therefore

$$\frac{\partial\Pi_i}{\partial x}=\frac{\partial}{\partial x}(\widetilde{\phi}E_i\widetilde{\phi}^*)=\widetilde{\phi}[A,E_i]\widetilde{\phi}^*.$$

So,

$$A_x^{ji}=\Pi_j\frac{\partial\Pi_i}{\partial x}=\widetilde{\phi}E_j[A,E_i]\widetilde{\phi}^*=\widetilde{\phi}E_jAE_i\widetilde{\phi}^*.$$

Similarly we have $X=\frac{1}{2}(A-\sqrt{-1}\,B)$. But

$$\frac{\partial}{\partial x}(A_x^{ij})=\widetilde{\phi}[A,E_iAE_j]\widetilde{\phi}^*\quad\text{and}\quad\frac{\partial}{\partial y}(A_y^{ij})=\widetilde{\phi}[B,E_iBE_j]\widetilde{\phi}^*.$$

Using Proposition 3.1 we have:

Proposition 4.1. *Suppose that $\phi:\mathbb{R}^2\to F(n)$ above defined is doubly periodic. Then ϕ is harmonic with respect to ds^2_Λ if and only if $[A,\sum\lambda_{ij}E_iAE_j]+[B,\sum\lambda_{ij}E_iBE_j]=0$.*

We will construct a class of non-f-holomorphic tori for any f-structures on $F(n)$ which are equi-harmonic, thus showing that the converse of Black's theorem is not true.

Theorem 4.1. **([19] or [20])** *Let* $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k \in \mathbb{Q} - \{0\}$ *and*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_j = \begin{pmatrix} \alpha_j X & 0 \\ 0 & \beta_j X \end{pmatrix}, \; B_j = \begin{pmatrix} \beta_j X & 0 \\ 0 & \alpha_j X \end{pmatrix}, \quad j = 1, ..., k \leq \frac{n}{4},$$

$$A = \sqrt{-1} \begin{pmatrix} A_1 & \dots & 0 \\ & & \\ \vdots & A_{k_0} & \vdots \\ & & \\ 0\dots & 0 & \dots 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & \dots & 0 \\ & & \\ \vdots & B_{k_0} & \vdots \\ & & \\ 0\dots & 0 & \dots 0 \end{pmatrix}.$$

Then:

1) $\phi(x, y) = \Pi e^{(Ax+By)}$ *has double periods;*

2) $\phi : T^2 \to F(n)$ *is equi-harmonic;*

3) ϕ *is not* f*-holomorphic with respect to any* f*-structure* $\mathcal{F}$ *on* $F(n)$.

Sketch of the Proof: The proof is similar to the particular case discussed in [21] where the f-structure is in fact an almost complex structure; see [19] for more details.

5. Stability of Eells-Wood's maps

Now we can compute the second variation of the energy; see [22] for the details of this long computation.

Proposition 5.1. *Let* $\phi = (\Pi_1, ..., \Pi_n) : (M^2, g) \to (F(n), ds^2_{\Lambda=(\lambda_{ij})})$ *be a harmonic map. Then*

$$\frac{d^2}{dt^2} E(\phi_t)|_{t=0} = I^{\phi}_{\Lambda}(q) = 4Re \int_M \langle q A^{\Lambda}_z, \frac{\partial q}{\partial \bar{z}} \rangle v_g + 2Re \sum_{ij} \lambda_{ij} \int_M \langle \Pi_i \frac{\partial q}{\partial z} \Pi_j, \frac{\partial q}{\partial z} \rangle v_g,$$

where $q : M^2 \to u(n)$ is an arbitrary variation.

We will now prove a very useful proposition.

Definition 5.1. *$\Lambda' = (\lambda'_{ij})$ is said to be a perturbation of $\Lambda = (\lambda_{ij})$ associated to a map $\phi = (\Pi_1, \dots, \Pi_n) : M^2 \to F(n)$ when:*

(i) $\lambda'_{ij} = \lambda_{ij}$ if $(i,j) \neq (i_1, j_1)$, (j_1, i_1), $\dots$, (i_r, j_r) and (j_r, i_r).

(ii) $\lambda'_{i_k j_k} = \lambda_{i_k j_k} + \epsilon_k$ for $1 \leq k \leq r$.

(iii) $A_z^{i_1 j_1} = A_z^{j_1 i_1} = \cdots = A_z^{i_r j_r} = A_z^{j_r i_r} = 0$, where ds^2_Λ and $ds^2_{\Lambda'}$ are Borel type metrics.

Proposition 5.2. *Let $\phi : (M^2, g) \to (F(n), ds^2_\Lambda)$ be an equi-harmonic map. Then:*

$$I^\phi_{\Lambda'}(q) = I^\phi_\Lambda(q) + \int_M \left\{ \varepsilon_1 \left(\left| \Pi_{i_1} \frac{\partial q}{\partial z} \Pi_{j_1} \right|^2 + \left| \Pi_{j_1} \frac{\partial q}{\partial z} \Pi_{i_1} \right|^2 \right) + \cdots + \right.$$
$$\left. 2\varepsilon_r \left(\left| \Pi_{i_r} \tfrac{\partial q}{\partial z} \Pi_{j_r} \right|^2 + \left| \Pi_{j_r} \tfrac{\partial q}{\partial z} \Pi_{i_r} \right|^2 \right) \right\} v_g$$

where Λ' is a perturbed Λ-matrix associated to ϕ.

Proof: $I^\phi_{\Lambda'}(q) = 2\mathrm{Re}\left\{ \int_{M^2} \sum \lambda'_{ij} |\Pi_i \frac{\partial q}{\partial z} \Pi_j|^2 v_g \right\} + 2\mathrm{Re}\left\{ \int_M \langle [A_z^{\Lambda'}, q], \frac{\partial q}{\partial z} \rangle v_g \right\}$. But we notice that $A_z^{\Lambda'} = A_z^\Lambda$ since $A_z^{i_1 j_1} = \cdots = A_z^{j_r i_r} = 0$; therefore:

$$\begin{aligned} I^\phi_{\Lambda'}(q) &= 2\sum_{k=1}^r \varepsilon_k \int_{M^2} \langle \Pi_{i_k} \frac{\partial q}{\partial z} \Pi_{j_k} + \Pi_{j_k} \frac{\partial q}{\partial z} \Pi_{i_k}, \frac{\partial q}{\partial z} \rangle v_g + \\ &\quad + 2\mathrm{Re}\left\{ \int_M \langle \sum \lambda_{ij} \Pi_i \frac{\partial q}{\partial z} \Pi_j + [A_z^\Lambda, q], \frac{\partial q}{\partial z} \rangle v_g \right\} = \\ &= I^\phi_\Lambda(q) + 2\int_M \sum_{k=1}^r \varepsilon_k \left(\left| \Pi_{i_k} \frac{\partial q}{\partial z} \Pi_{j_k} \right|^2 + \left| \Pi_{j_k} \frac{\partial q}{\partial z} \Pi_{i_k} \right|^2 \right) v_g \end{aligned}$$

Theorem 5.1. *Let $\Psi = (h_0, ..., h_{n-1}) : (M^2, g) \to (F(n), ds^2_{\Lambda'})$ be an Eells-Wood map where Λ' is the perturbed Λ-matrix associated to ψ; $\varepsilon_1, \varepsilon_2, \dots , \varepsilon_r$ are* *non-negative real numbers and $I^{\psi}_{\Lambda}(q) \geq 0$ for any variation q. Then ψ is stable.*

Proof:

$$I^{\psi}_{\Lambda'}(q) = I^{\psi}_{\Lambda}(q) + 2\int_M \left\{ \varepsilon_1 \left(\left| \Pi_1 \frac{\partial q}{\partial z} \Pi_3 \right|^2 + \left| \Pi_3 \frac{\partial q}{\partial z} \Pi_1 \right|^2 \right) + \cdots + \right.$$
$$\left. + \varepsilon_r \left(\left| \Pi_1 \tfrac{\partial q}{\partial z} \Pi_n \right|^2 + \left| \Pi_n \tfrac{\partial q}{\partial z} \Pi_1 \right|^2 \right) \right\} v_g \geq 0.$$

Borel in [3] described precisely the set of invariant Kähler metrics on $F(n)$, which are, up to permutation, given by:

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 & \lambda_1 + \lambda_2 & \ldots & \lambda_1 + \ldots + \lambda_{n-1} & \\ & 0 & \lambda_2 & \lambda_2 + \lambda_3 \ldots & & \\ & & 0 & & & \\ & * & & & & \\ & & * & & & \\ & * & & & 0 & \lambda_{n-1} \\ & & & & & 0 \end{pmatrix}$$

Therefore, associating Lichnerowicz's theorem [17] with theorem 5.1 we will see that the perturbation Λ' of a Kähler metric produces unstable Eells-Wood's maps $\psi : (M^2, g) \to (F(n), ds^2_{\Lambda'})$. In fact, consider now $\Lambda' = (\lambda'_{ij})$ the following perturbation of $\Lambda = (\Lambda_{ij})$: $\lambda'_{12} = \lambda_{12}$, $\lambda'_{23} = \lambda_{23}$, $\ldots$, $\lambda'_{(n-1)n} = \lambda_{(n-1)n}$, $\lambda_{13} = \lambda_{12} + \lambda_{23} - \varepsilon_1 = \lambda_1 + \lambda_2 - \varepsilon_1$, $\ldots$, $\lambda'_{1n} = \lambda_{12} + \ldots + \lambda_{(n-1)n} - \varepsilon_\ell = \lambda_1 + \ldots + \lambda_{n-1} - \varepsilon_\ell$. We recall that the Lichnerowicz theorem may be seen as stating that $I^{\Psi}_{\Lambda}(q) \geq 0$ if ds^2_{Λ} is a Kähler metric. According to [20] we can prove:

Theorem 5.2. *Let $\psi : (M^2, g) \to (F(n), ds^2_{\Lambda'})$ be a full Eells-Wood's map where Λ' is the perturbation of Λ right above defined. Then ψ is not stable.*

Corollary 5.1. ([20]) *Let $\psi = (h_0, \ldots, h_{n-1}) : M^2 \to F(n)$ be a full Eells-Wood's map, where $F(n)$ is equipped with the Killing form metric. Then ψ is not stable.*

Proof: Just apply theorem 5.2 for $\lambda_{12} = \ldots = \lambda_{(n-1)n} = 1$, $\varepsilon_1 = 1$, $\ldots$, $\varepsilon_\ell = n - 2$.

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