Matemática Contemporânea, Vol. 16, 137–186
http://doi.org/10.21711/231766361999/rmc169

ALGEBRAS WITH POLYNOMIAL IDENTITIES

Plamen Koshlukov*

Abstract

This survey represents a revised version of the notes for a minicourse with the same title delivered at the 15th Escola em Álgebra held at the beautiful town of Canela, RS. The minicourse was considered of medium level; it consisted of five lectures, approximately one section of the notes per lecture. The audience was mixed: (very) experienced researchers together with PhD students and young mathematicians. Thus the course was as self-contained as possible; it required only some knowledge of the representations of the symmetric and general linear groups. The title probably is somewhat misleading since the main topics discussed were the combinatorial methods in PI theory. The introduction is not exhausting, especially its historical parts; this certainly is not a shortage. Some important results were not included but it is impossible to include everything. The bibliography also is not complete but it must be helpful for those who become interested in the topic. Some of the proofs in this survey are omitted or substituted by hints, or by references. Of course we do not claim for originality, but the main goal of the survey is the dissemination of the PI theory. And as any work aiming at popularization it has to present the foundations of the theory, and open the door for future research.

It is my pleasure to thank the organizers of the 15th Escola em Álgebra, and especially Professor Miguel Ferrero for their very kind invitation to deliver this minicourse, for their constant encouragement, and for their (extremely successful) efforts to make an excellent scientific event. I am indebted to M. Ferrero, Yu. Bahturin, A. Giambruno, I. Hentzel, L. Peresi, I. Shestakov, R. Wiegandt, for their numerous and useful remarks that (I hope) improved the exposition. P. Brumatti helped me with his suggestions concerning the preliminary version of the notes. I would like to express my thanks to the Referee for his careful reading of the paper, and for his valuable suggestions. Thanks are due to FAPESP for their generous support.

*Partially supported by FAPESP, Grant # 1998/0962-5.

1. Introduction

The algebras that satisfy polynomial identities (so-called PI algebras) form an important class of algebras, and therefore they have been attracting for the last 50 years the attention of the algebraists. Let K be a field, and let $X = \{x_1, x_2, \ldots\}$ be a set of non-commuting variables. The free (associative) algebra $K(X)$ freely generated by X over K is the K–space with a basis the monomials $\{x_{i_1} \ldots x_{i_r} \mid r = 0, 1, 2, \ldots\}$. The multiplication in $K(X)$ is defined as $(x_{i_1} \ldots x_{i_r})(x_{j_1} \ldots x_{j_s}) = x_{i_1} \ldots x_{i_r} x_{j_1} \ldots x_{j_s}$; the elements of $K(X)$ are called polynomials. If A is any K–algebra and if a_1, $a_2 \ldots \in A$ then there exists a unique homomorphism $K(X) \to A$ such that $x_i \mapsto a_i$. The polynomial $f(x_1, \ldots, x_n) \in K(X)$ is a polynomial identity (PI) in A if f lies in the kernels of all homomorphisms $K(X) \to A$ i.e., $f(a_1, \ldots, a_n) = 0$ in A for all a_1, $\ldots$, $a_n \in A$.

The first research on PI algebras was initiated in 1922 by M. Dehn [6], motivated by Geometry. In 1936, W. Wagner ([47]) found identities for matrix and quaternion algebras. He showed that $(ab - ba)^2 c - c(ab - ba)^2 = 0$ for any matrices a, b, $c \in M_2(K)$—the algebra of the matrices 2×2 over K. That is, the polynomial $f(x_1, x_2, x_3) = (x_1 x_2 - x_2 x_1)^2 x_3 - x_3(x_1 x_2 - x_2 x_1)^2$ is a PI in $M_2(K)$.

The results of M. Dehn and W. Wagner had been "forgotten" for more than 10 years. The development of PI theory in its proper sense began with the research of N. Jacobson and I. Kaplansky in approximately 1947–48. We shall discuss some of the most important steps in this development.

Historically, there exist three principal directions in PI theory. The first, the classical one was inspired by an extremely important problem—describe all algebras (rings, groups, and so on). This problem, however, could not (and still cannot) be answered satisfactory. Therefore it turned out important to describe all algebras that satisfy certain conditions arising naturally. (We describe the life of the "ants" since it turned out complicated to describe life in general...) To summarize, the first direction deals with the following question. Suppose A

is an algebra that satisfies some polynomial identity, what can one say about the structure of A?

The second direction is more concrete; it studies the identities satisfied by a given algebra (for example simple), and the classes of algebras that satisfy these identities. The third is related to the second—it studies the structure of the ideals of identities. For more historical details see [20].

Our point of view will be related with the last two directions. On the other hand we do mention some principal facts concerning the first direction. The interested reader could find more facts and results in the first direction in [46]; let us mention that [46] is based on the notes for a mini-course with the same title given at the 12th Escola em Álgebra.

The survey is organized as follows. In this section we introduce the main "stars" and the relations among them. In Section 2, we give an exposition of results that lie at the borders of the directions mentioned above. Furthermore they show some of the classical applications of Combinatorics in PI theory. (Glance at the names!) In the third section we discuss a combinatorial method that is extremely important in PI theory, namely the representations of the symmetric and the general linear groups and the connection between them. In addition we show some examples of the usage of this method. The last two sections consist of applications: in the fourth we study the identities in $M_2(K)$ when K is a field of characteristic 0. Using Higman's theorem about partially well ordered sets we describe some important properties of the identities satisfied by well-known algebras. In the fifth chapter we deal with non-associative algebras satisfying polynomial identities and we discuss some problems of PI theory when the algebras are over a field of positive characteristic.

The exposition is self-contained with few exceptions. In Section 2 we consider tensor products of algebras; in the third section we suppose familiarity with the basics of the representations of finite groups over fields of characteristic 0, and with group algebras of finite groups. In Section 4, Razmyslov's theorem about the identities in $M_2(K)$ is stated without proof because of space restrictions. In Section 5, formally speaking, we do not require any knowledge of non-associative algebras (although it is desirable). It should be mentioned explicitly that the survey is not a complete introduction to PI theory; its main goal is to demonstrate to the reader and to convince her/him that PI theory is worth studying. The readers interested in the topic could use the references as a base of future research. Thus for example we do not consider at all one of the most important achievements in the structure theory of PI algebras namely the existence of central polynomials in matrix algebras. These are non-zero non-commutative polynomials in several variables that do not vanish on a given matrix algebra of order n but all their values are central. The existence of central polynomials was established independently by E. Formanek and by Yu. Razmyslov, we shall only refer to [12] and [36].

All vector spaces, modules, and algebras will be over a fixed field K. The algebras will be with 1. First we provide some examples of PI algebras. They show that PI algebras appear naturally, and that important types of algebras satisfy identities.

Example 1.1. 1. Let A be a commutative K–algebra. Then it is PI since it satisfies $x_1x_2 - x_2x_1$. Denote $x_1x_2 - x_2x_1$ as $[x_1, x_2]$, then the polynomial $[x_1, x_2]$ is a PI in A.

2. The algebra $M_2(K)$ satisfies the identity $[[x_1, x_2]^2, x_3]$. (Why?)

3. Let A be nil of bounded index (i.e., $a^n = 0$ for all $a \in A$ where n is fixed). Then A is PI since it satisfies x_1^n. (Here, and in the next example we suppose that the algebras are without 1.)

4. Let A be nilpotent i.e., $a_1 \dots a_n = 0$ for all $a_i \in A$ for fixed n. Then A satisfies the PI $x_1 \dots x_n$.

5. The algebra $A = T_n(K)$ of the upper triangular $n \times n$ matrices satisfies $[x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}]$ (Why?) and therefore it is PI.

6. The exterior (or Grassman) algebra A satisfies the PI $[[x_1, x_2], x_3]$.

Definition 1.2. *Let S_n be the symmetric group acting on the symbols $\{1, \dots, n\}$, and denote $s_n(x_1, \dots, x_n) = \sum_\sigma (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$ the* **standard polynomial**

where the summation is over all $\sigma \in S_n$, *and* $(-1)^\sigma$ *stands for the sign of the permutation* σ.

The degree $\deg f$ of the polynomial $f(x_1, \ldots, x_n) \in K(X)$ and the degree $\deg_{x_i} f$ of f with respect to x_i are defined in the usual manner. The n–tuple $(\deg_{x_1} f, \ldots, \deg_{x_n} f)$ is the multidegree of f. The polynomial f is homogeneous if all its monomials have the same multidegree. If f is homogeneous of multidegree $(1, \ldots, 1)$ then f is multilinear.

The polynomial $s_n(x_1, \ldots, x_n)$ is multilinear and skew-symmetric. In other words, if $x_i = x_j$ for some $i \neq j$ then $s_n(x_1, \ldots, x_i, \ldots, x_j \ldots, x_n) = 0$. If $f(x_1, \ldots, x_n, \ldots, x_k) \in K(X)$ is multilinear in $x_1, \ldots, x_n$ then f is n–Capelli if $x_i = x_j$ for some i and j, $1 \leq i < j \leq n$ implies $f = 0$. Denote as Cap_n the set of all n–Capelli polynomials in $K(X)$.

Example 1.3. Every finite dimensional algebra A, $\dim A = n$, satisfies $s_{n+1}(x_1, \ldots, x_{n+1})$. If $f \in Cap_{n+1}$ then f is a PI in A.

The examples provided show that the PI algebras can be thought of as certain generalization of the commutative algebras, and this generalization is rather natural. The PI algebras enjoy a lot of the properties of commutative algebras. On the other hand, the class of the PI algebras is much larger than that of the commutative algebras. In fact, the class of the commutative algebras can be defined as satisfying the identity $[x_1, x_2] = s_2(x_1, x_2)$. Of course, it is exaggeration to say that it is worth studying the commutative algebras using PI methods; the proper methods of commutative algebra are well developed and they function much better in this case.

Definition 1.4. *Let* A *be a* K*–algebra. The element* $a \in A$ *is called algebraic of degree* $\leq n$ *if* $a^n = \beta_1 a^{n-1} + \ldots + \beta_{n-1} a + \beta_n 1$ *where* $\beta_i \in K$. *The algebra* A *is algebraic if every* $a \in A$ *is algebraic;* A *is algebraic of degree* n *if all elements of* A *are algebraic of degrees* $\leq n$, *and* n *is the least possible.*

Example 1.5. If A is algebraic of degree n then A is PI; it satisfies the identities

$$f([x_1^n, x_2], [x_1^{n-1}, x_2], \ldots, [x_1, x_2], x_{n+1}, \ldots, x_k), \qquad f \in Cap_n.$$

If $a_1, a_2 \in A$, then $[a_1^n, a_2] = \sum_{i=1}^{n-1} \beta_{n-i}[a_1^i, a_2]$. Hence $b_i = [a_1^i, a_2]$, $i = 1, \ldots, n$ are linearly dependent and thus $f(b_1, \ldots, b_n, \ldots) = 0$ in A. One verifies that the statements above hold for algebras over any associative and commutative ring with unity; in this case one substitutes the word "algebraic" for "integral".

Remark 1.6. If A satisfies the identity s_n then it also satisfies s_m for all $m \geq n$; the same holds for the sets of polynomials Cap_n.

Definition 1.7. *1. The ideal $I \triangleleft K(X)$ is called T-ideal if $\varphi(I) \subseteq I$ for every endomorphism φ of the algebra $K(X)$. Denote it as $I \triangleleft_T K(X)$.*

2. Let A be an algebra. The set $T(A)$ of all PI in A is called the T-ideal of A; we denote it as $T(A) \triangleleft_T K(X)$. If $I \subseteq K(X)$ then denote as $\operatorname{var} I$ the class of all algebras that satisfy all PI in I. The class $\operatorname{var} I$ is called the variety of algebras defined by I. If $I = T(A)$ we write $\operatorname{var} I = \operatorname{var} A$. The quotient $K(X)/T(A) = F(A)$ is called the relatively free algebra in $\operatorname{var} A$.

Obviously $T(A) \triangleleft_T K(X)$ is an ideal in $K(X)$; if $A = 0$ then $T(A) = K(X)$; if $A = K(X)$ then $T(A) = 0$.

Now we recall the main properties and relations among the objects defined above. These are the contents of the next propositions. Their proofs are straightforward; they can be found, for example, in [33], Chapter 20, [34], [40].

Proposition 1.8. *a) Let $I \subseteq K(X)$. The variety $\operatorname{var} I$ is closed under taking subalgebras, homomorphic images, and direct products.*

b) Let V be a class of algebras, and let $I = I(V) = \{f \in K(X) \mid f$ is a PI in all $A \in V\}$. Then $I \triangleleft_T K(X)$ i.e., $\varphi(I) \subseteq I$ for every endomorphism φ of $K(X)$.

Let V be a class of algebras closed under taking subalgebras, homomorphic images, and direct products. Denote as $F(V)$ the quotient $K(X)/I(V)$ and as $\pi: K(X) \to F(V)$ the canonical projection.

Proposition 1.9. *a) The algebra $F(V)$ belongs to V.*

b) If $I(V) \neq K(X)$ then the map π is injective on X, and the image $\pi(X)$ generates $F(V)$.

c) If $A \in V$ and if $\varphi: \pi(X) \to A$ is a map then there exists an algebra homomorphism $\Phi: F(V) \to A$ such that $\Phi|_{\pi(X)} = \varphi$.

Corollary 1.10 *a) Let V be a class of algebras closed under taking subalgebras, homomorphic images, and direct products. Then V is a variety, and $V = var\,(F(V))$.*

b) The algebra $F(V)$ is relatively free in V.

c) If A is an algebra then $A \in V$ if and only if all finitely generated subalgebras of A belong to V.

Corollary 1.11. *Let $J \subseteq K(X)$ and suppose V is a class of algebras. Then the maps $J \mapsto var\,J$ and $V \mapsto I(V)$ invert the inclusions and furthermore:*

a) $I(var\,J) \supseteq J$; the equality holds if and only if $J \triangleleft_T K(X)$.

b) $var\,(I(V)) \supseteq V$ with equality if and only if V is a variety.

c) If $J \triangleleft_T K(X)$ then $J = T(K(X)/J)$.

Note that this Corollary is known as Birkhoff's theorem.

2. Classical combinatorial theorems

Here we consider fundamental results in the theory of PI algebras as well as applications. We begin with the identities satisfied by the algebras $M_n(K)$ of the matrices of order n. We follow the exposition in [33], Chapter 6, pp. 402–406. First we state some elementary properties of the standard polynomials.

Lemma 2.1. *If $s_n(x_1,\dots,x_n)$ is the standard polynomial then:*

a) $s_n(x_{\sigma(1)},\dots,x_{\sigma(n)}) = (-1)^\sigma s_n(x_1,\dots,x_n)$, $\sigma\in S_n$.

b) Assume $a_1,\dots,a_n\in A$ and $a_i=a_j$, for some i and j, $i\neq j$, then $s_n(a_1,\dots,a_n)=0$.

c) $s_n(x_1,\dots,x_n)=\sum_{i=1}^n(-1)^{i+1}x_i s_{n-1}(x_1,\dots,\hat{x}_i,\dots,x_n)=\sum_{i=1}^n(-1)^{n+i}$ $s_{n-1}(x_1,\dots,\hat{x}_i,\dots,x_n)x_i$, here $\hat{x}_i$ means that x_i is missing from the corresponding expression.

Lemma 2.2. *Let A be an algebra and suppose $P\subseteq A$ is a subset of A that spans A as a vector space over K. If $f(x_1,\dots,x_n)$ is a multilinear polynomial and if $f=0$ on the set P then f is a PI in A.*

Thus in order to verify whether some multilinear polynomial is a PI for an algebra A it is sufficient to check if it vanishes on some basis or even on some spanning set of A.

Lemma 2.3 (Staircase Lemma) *The algebra $M_n(K)$ satisfies no polynomial identity of degree $<2n$.*

Proof: We shall prove that $M_n(K)$ satisfies no multilinear identities of degrees $<2n$ and later we shall see that this is neither a restriction nor loss of generality. Let $f(x_1,\dots,x_m)=\sum_{\sigma\in S_m}\alpha_\sigma x_{\sigma(1)}\dots x_{\sigma(m)}$ where $m<2n$ and $\alpha_1\neq 0$. Then, if $m=2r$ one obtains $f(e_{11},e_{12},e_{22},\dots,e_{r,r+1})=\alpha_1 e_{1,r+1}\neq 0$; if $m=2r-1$ then $f(e_{11},e_{12},e_{22},\dots,e_{rr})=\alpha_1 e_{1r}\neq 0$. Here e_{ij} stands for the elementary matrix with 1 as (i,j)–th entry, and 0 as all other entries.

Since $\dim M_n(K)=n^2$ it is immediate that $M_n(K)$ satisfies the identity s_{n^2+1}. Which is the minimal degree m of s_m satisfied by $M_n(K)$? The answer is in the next theorem.

Theorem 2.4 (Amitsur and Levitzki) *The algebra $M_n(K)$ satisfies the identity $s_{2n}(x_1,\dots,x_{2n})$.*

Remark 2.5. The proof of this theorem in [33] was proposed by S. Rosset ([39]). It employs some elementary properties of Grassman algebras, and it is a pleasure to read it. We advise the reader to do so. There exist other proofs that use different methods and techniques.

Corollary 2.6. *If $V_n = var\, M_n(K)$ then $V_1 \subset V_2 \subset \dots$ where all inclusions are proper.*

Another important application of combinatorics in PI theory is the following result due to Nagata [32] and Higman [17]. It might be interesting to mention that the same result was obtained independently (and much earlier) in [11]; probably due to the WW2 it had remained unnoticed for more than 10 years. Here we give the proof of this theorem following [48], Chapter 6.1. Let A be an algebra and denote $I_n(A) = \{\sum_i \alpha_i a_i^n \mid \alpha_i \in K, a_i \in A\}$. Evidently $I_n(A)$ is a subspace of A.

Lemma 2.7. *If $J_n(A) = \{a \in A \mid (n!)^k a \in I_n(A) \text{ for some } k\}$ then $J_n(A) \lhd A$, and the quotient $A/J_n(A)$ has no elements of torsion $\leq n$.*

Proof: First we prove that the subspace $I_n(A)$ is a two-sided ideal in A. For $a_1, \dots, a_n \in A$ denote $\mathrm{sym}\,(a_1, \dots, a_n) = \sum_{\sigma \in S_n} a_{\sigma(1)} \dots a_{\sigma(n)}$. Then it can be checked that $\mathrm{sym}\,(a_1, \dots, a_n) \in I_n(A)$. Thus if $a,\ b \in A$ we have $a.\mathrm{sym}\,(b, a, \dots, a) = \mathrm{sym}\,(ab, a, \dots, a) \in I_n(A)$ and $(n-1)! \sum_{i=1}^n a_i \mathrm{sym}\,(b, a_1, \dots, \hat{a}_i, \dots a_n) \in I_n(A)$. When $a_1 = a$ and $a_2 = \dots = a_n = b$ this yields that

$$(n-1)!a.\mathrm{sym}\,(b, \dots, b) + (n-1)(n-1)!b.\mathrm{sym}\,(a, b, b, \dots, b) \in I_n(A)$$

and $(n!)^2 ab^n \in I_n(A)$. Analogously $(n!)^2 b^n a \in I_n(A)$.

Now evidently $J_n(A)$ is a subspace of A. If $a \in J_n(A)$ and $b \in A$ then $(n!)^k a \in I_n(A)$. Hence $(n!)^2((n!)^k a)b = (n!)^{k+2} ab \in I_n(A)$ and $ab \in J_n(A)$. Analogously $ba \in J_n(A)$. The assertion about the torsions is obvious.

Theorem 2.8 (Nagata, Higman, Dubnov, Ivanov) *If A is an associative* *algebra then* $A^{2^n-1} \subseteq J_n(A)$ *for all* n.

Proof: See, for example, [19], p. 274, or [48], Chapter 6.1, pp. 123–126.

The next corollary justifies the importance of this theorem.

Corollary 2.9. *If* A *is an associative algebra with no elements of torsion* $\leq n$, *and if* x^n *is a PI in* A *then* A *is nilpotent of class* $\leq 2^n - 1$.

Remark 2.10. It is known that $A^6 \subseteq J_3(A)$ for all algebras A. This means that the bound $2^n - 1$ is not the best possible. Its exact value $k(n)$ satisfies $n(n+1)/2 \leq k(n) \leq n^2$. See, for example, [4], pp. 123–126 for additional information and generalizations of the result just obtained.

Our next step is to show that the tensor product (over K) of two PI algebras is again PI algebra, a result due to A. Regev [37]. Here we consider a proof proposed by V. Latyshev [30].

First we recall some facts about multilinear identities. It is easy to verify that if A is a PI algebra and if f is an identity for A of degree $\deg f = d$ then A satisfies a multilinear identity of degree $\leq d$. (We shall discuss this in the next section.)

Now let A_1 and A_2 be two PI algebras with respective T-ideals $I' = T(A_1)$ and $I'' = T(A_2)$. In order to prove that $A = A_1 \otimes A_2$ is a PI algebra itself it is sufficient to show that the tensor product of the corresponding relatively free algebras $K(X)/I' \otimes K(X)/I''$ satisfies an identity.

Definition 2.11. *Denote as* P_n *the set of all multilinear polynomials in the variables* $x_1, \ldots, x_n$ *in* $K(X)$ *i.e.,* $P_n = \{f(x_1, \ldots, x_n) \in K(X) \mid f \text{ is multilinear}\}$. *If* $I \lhd_T K(X)$ *we denote* $I_n = I \cap P_n$ *and we call* $c_n(I) = \dim_K(P_n/I_n)$ *the* n*–th codimension of* I.

Remark 2.12. Clearly P_n is a vector space over K, and I_n is a subspace of P_n. The set $\{x_{\sigma(1)} \ldots x_{\sigma(n)} \mid \sigma \in S_n\}$ is a basis of P_n hence $\dim P_n = n!$, and

$c_n(I) \leq n!$. Note that the vector space I_n is too "large"; that is why we consider the quotient, and the codimensions. In fact this observation is the main point in the proof of the next proposition.

Proposition 2.13 (Regev). *Let I' and I'' be two T-ideals in the free associative algebra $K\langle X\rangle$ such that $c_n(I')c_n(I'') < n!$. Then the algebra $A = K\langle X\rangle/I' \otimes K\langle X\rangle/I''$ satisfies a multilinear identity of degree n.*

Proof: See for example [40], p. 240.

Remark 2.14. Consider the following assertion. "If $I \triangleleft_T K\langle X\rangle$ and if $K\langle X\rangle/I$ satisfies an identity of degree d then there exists $m(d)$ such that $c_n(I) \leq m(d)^n$ for all n." This assertion yields Regev's theorem since $n!$ grows "faster" than $m(d)^n$ for any real number $m(d)$.

Definition 2.15. *If $\sigma \in S_n$ we define $r(\sigma)$ as the length of the maximal "antichain" in σ i.e., $r(\sigma)$ stands for the largest k such that there exist $1 \leq i_1 < \cdots < i_k \leq n$ with $\sigma(i_1) > \cdots > \sigma(i_k)$.*

For example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 4 & 1 & 6 \end{pmatrix} \in S_6$ then $r(\sigma) = 4$: $\sigma(1) > \sigma(2) > \sigma(3) > \sigma(5)$.

If $\sigma \in S_n$ we define the Amitsur diagrams $T_1(\sigma)$ and $T_2(\sigma)$ as follows. Construct $T_1(\sigma) = (t_{ij})$ and $T_2(\sigma) = (u_{ij})$ by induction: $t_{11} = 1$ and $u_{11} = \sigma(t_{11})$. Suppose that $t_{1,j-1}$ has been defined, then set t_{1j} as the minimal k such that $t_{1,j-1} < k \leq n$ and $\sigma(k) > u_{1,j-1}$; then $u_{1j} = \sigma(t_{1j})$. If there does not exist such k we start filling the second row of $T_1(\sigma)$, see the example below.

Thus t_{21} will be the least $k \leq n$ that does not belong to the first row of $T_1(\sigma)$, and $u_{21} = \sigma(t_{21})$. Then continue with t_{22} etc. as on the first row (but disregarding the elements that have already appeared on the first row of $T_1(\sigma)$), then with the third row, and so on. Eventually $T_1(\sigma)$ and $T_2(\sigma)$ will have n elements each.

Example 2.16. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 5 & 3 & 2 & 6 & 7 \end{pmatrix} \in S_7$ then $T_1(\sigma) = \begin{pmatrix} 1\,3\,6\,7 \\ 2\,4 \\ 5 \end{pmatrix}$,
$T_2(\sigma) = \begin{pmatrix} 4\,5\,6\,7 \\ 1\,3 \\ 2 \end{pmatrix}$.

Theorem 2.17. *If $\sigma \in S_n$ then $r(\sigma)$ equals the number of the rows in $T_1(\sigma)$.*

Proof: Not so difficult combinatorial argument; see, e.g., [40], p. 241.

Lemma 2.18. *If $d \leq n$ then $|\{\sigma \in S_n \mid r(\sigma) < d\}| < (d-1)^{2n}$.*

Proof: See [40], Remark 6.1.11 on p. 242.

Theorem 2.19. *Suppose $I \lhd_T K(X)$ and let $K(X)/I$ satisfy a multilinear identity g of degree d. Then $c_n(I) \leq (d-1)^{2n}$.*

Proof: Denote as V the subspace of $P_n \subseteq K(X)$ spanned by the monomials $x_{\tau(1)} \ldots x_{\tau(n)}$ where $\tau \in S_n$ and $r(\tau) < d$; we will be done if $V + I_n = P_n$ for $I_n = I \cap P_n$ (Why?).

Choose $\sigma \in S_n$ and $h = x_{\sigma(1)} \ldots x_{\sigma(n)}$ such that h is the least monomial outside $V + I_n$ (the least with respect to the lexicographical order in P_n). Then $r(\sigma) > d$ and we can find $i_1 < \ldots < i_d$ with $\sigma(i_1) > \ldots > \sigma(i_d)$; then represent h as $h = h_1 x_{\sigma(i_1)} h_2 x_{\sigma(i_2)} \ldots x_{\sigma(i_d)} h_{d+1}$ where h_i stand for some monomials.

Consider the difference $h' = h - h_1 g(x_{\sigma(i_1)} h_2, x_{\sigma(i_2)} h_3, \ldots, x_{\sigma(i_d)} h_{d+1})$. The monomials of h' precede h. Then by induction we conclude that $h' \in V + I_n$. On the other hand $g \in I$ hence $h \in V + I_n$, a contradiction.

This theorem together with the assertions above proves the following theorem.

Theorem 2.20 (Regev). *If A_1 and A_2 are PI algebras over K then their tensor product $A_1 \otimes_K A_2$ is also a PI algebra.*

Note that this theorem states that the codimensions of a (non-zero) T–ideal cannot grow too "fast" i.e., its codimension growth cannot exceed the exponential. (This is not the case for Lie algebras, for example.) This means that in the case of T–ideals of associative algebras, the identities are "more" than the non-identities, at least asymptotically.

At the end of this section we consider one of the most interesting applications of Combinatorics to PI theory. This is a method introduced and developed by A. I. Shirshov. The method as well as the results obtained hold in a much more general setting than the one considered here. The reader could find the complete story in [48]. Although in a particular case, we shall demonstrate how this method works and what its major applications are. We use ideas from the exposition in [40], §4.2, and in [48], Chapter 5. The reader could find all missing details there as well.

Denote $\langle X\rangle$ the set of all monomials in X together with the element 1 (the empty monomial, it is of length 0), equipped with the usual multiplication in $\langle X\rangle$. The elements of $\langle X\rangle$ are called words in the alphabet X. If $w = x_{i_1}\dots x_{i_k}$ then k is the degree of the word w, and $\max(i_1,\dots,i_k)$ is the height of w. If w is a word of degree k and of height $\le t$ then w is called (k,t)–word. It is clear that the set of all words of heights $\le t$ is closed with respect to the multiplication. The word w_1 is called a subword of w if $w = w'w_1w''$ for some words w' and w''. For example, $x_1x_3x_2$ is a subword of $x_4x_1x_3x_2x_3$ and of $x_1x_3x_2x_4$.

Definition 2.21. *For the words $w_1 = x_{i_1}\dots x_{i_k}\ne 1$ and $w_2 = x_{j_1}\dots x_{j_n}\ne 1$ we define $w_1 < w_2$ if there exists $r\le\min(k,n)$ such that $i_1 = j_1$, $i_2 = j_2$, …, $i_{r-1} = j_{r-1}$ but $i_r < j_r$.*

Remark 2.22. The order $<$ is partial; it is different from the lexicographical order: for example x_1x_2 and $x_1x_2x_3$ are incomparable with respect to the order $<$ while the lexicographical order is linear.

Lemma 2.23. *If $w_1 < w_2$ then $w_1w_3 < w_2w_4$ for all words w_3 and w_4.*

Definition 2.24. *The word w is called 1–initial if $w = x_1^u w'$ where $u \geq 0$ and w' does not contain x_1. The partition $w = w_1 \dots w_m$ is called factorization of w if every w_i is 1–initial. The factorization above is called minimal if m is the least possible.*

Lemma 2.25. *If w is a word then there exists a factorization of w. The minimal factorization of w is unique.*

Example 2.26. Let $w = 325111413251$ where we wrote i for x_i. Then $w = 3.2.5.1^2.14.1325.1$ is a factorization of w; the minimal is $325.1^3 4.1325.1$.

Definition 2.27. *The word w is called d–decomposable and $w = w_1 \dots w_d$ is called a d–decomposition of w if $w_{\sigma(1)} \dots w_{\sigma(d)} < w$ for every permutation $1 \neq \sigma \in S_d$.*

Lemma 2.28. *If $w = w_1 w'$ and $w' = w_2 \dots w_d$ is $(d-1)$–decomposable with $w_j < w_1$, $j = 2, \dots, d$ then $w = w_1 w_2 \dots w_d$ is a d–decomposition of w.*

Theorem 2.29. (Shirshov). *There exists a function $\beta(t, u, d)$ such that for every $k \geq \beta(t, u, d)$, any (k, t)–word has either (1) a subword of type w_0^u, or (2) a d–decomposable subword.*

Proof: Simultaneous induction on d and t. Obviously $\beta(t, u, 1) = 1$ since every word is 1–decomposable; also $\beta(1, u, d) = u$ since 1^u is a subword of every $(k, 1)$–word with $k \geq u$. Suppose there exist $\beta(t-1, u, d)$, and $\beta(t', u, d-1)$ for every $t' \in \mathbf{N}$. Define $t' = ut^{\beta(t-1,u,d)}$; $\beta(t, u, d) = (\beta(t-1, u, d) + u - 1)\beta(t', u, d-1)$. If w is a (k, t)–word that satisfies neither (1) nor (2) from the theorem we have to show that $k < \beta(t, u, d)$.

The word w admits a minimal factorization $w = w_1 \dots w_m$, $w_i = 1^{u_i}\tilde{w}_i$ where $\tilde{w}_i$ do not contain 1. This yields that $\tilde{w}_i$ are words in $\{2, \dots, t\}$. If $\deg \tilde{w}_i \geq \beta(t-1, u, d)$ for some i the proof would be complete. On the other hand, if $u_i \geq u$ for some i then (1) holds. In other words we can assume that

$\tilde{w}_i$ are of degrees $< \beta(t-1,u,d)$, and that $u_i < u$ for all i.

The word $w_i = 1^{u_i}\tilde{w}_i$ is called admissible if $\deg \tilde{w}_i < \beta(t-1,u,d)$ and $u_i < u$.

Now clearly $\deg w_i \le \beta(t-1,u,d)+u-1$ implies that $k \le (\beta(t-1,u,d)+u-1)m$. The number of the admissible words is less than t'. These words are ordered (linearly) with respect to the lexicographical order; hence there exists an order-preserving 1–1 map α from the set of all admissible words into the set $\{1,2,\ldots,t'\}$, $\alpha\colon v \mapsto \alpha(v)$. (The order in $\{1,\ldots,t'\}$ is the standard one.) We shall use v and $\alpha(v)$ as identical notations.

The word $w' = (\alpha(w_2),\ldots,\alpha(w_m))$ is a $(m-1,t')$–word. Suppose $m \ge \beta(t',u,d-1)$, then if (1) holds, the proof is completed. Hence without loss of generality we suppose that w' contains a subword $\alpha(w_i)\ldots\alpha(w_j)$, $2 \le i < j$, that is $(d-1)$–decomposable. Therefore the word $w_i \ldots w_j$ admits decomposition $w''_2 \ldots w''_d$ where every w''_p is product of words in $\{w_2,\ldots,w_m\}$. But the last words start with 1. Hence $\tilde{w}_{i-1} > w''_j$ for all j since $\tilde{w}_{i-1}$ cannot start with 1. Therefore $\tilde{w}_{i-1}w''_2 \ldots w''_d$ is a d–decomposable subword of w.

The last possibility to be considered is $m < \beta(t',u,d-1)$. This inequality leads to

$$k < (\beta(t-1,u,d)+u-1)\beta(t',u,d-1) = \beta(t,u,d),$$

and hence the proof of the theorem is completed.

Proposition 2.30. *Suppose that* $\deg w \ge d$ *and that* w *is not of the type* $w = (w')^j$ *for some* w' *and some* $j > 1$. *Then* w^{2d} *contains a* d*–decomposable subword.*

Proof: If $w = i_1 \ldots i_d w'$ we define $w_1 = w$, and $w_p = (i_p i_{p+1} \ldots i_d)w'(i_1 \ldots i_{p-1})$, $2 \le p \le d$. Obviously $w_p \ne w$ for every p and hence $w_1, \ldots, w_p$ are pairwise distinct. Hence there exists $\sigma \in S_d$ such that $w_{\sigma(1)} > \cdots > w_{\sigma(d)}$. On the other hand every w_p is a subword of w^2, and $w^2 = w'_p w_p w''_p$. Thus

$$w^{2d} = w'_{\sigma(1)}.w_{\sigma(1)}w''_{\sigma(1)}w'_{\sigma(2)}.\ \cdots\ .w_{\sigma(d-1)}w''_{\sigma(d-1)}w'_{\sigma(d)}.w_{\sigma(d)}w''_{\sigma(d)}.$$

The words $v_i = w_{\sigma(i)}w''_{\sigma(i)}w'_{\sigma(i+1)}$, $i = 1, \ldots, d-1$, and $v_d = w_{\sigma(d)}w''_{\sigma(d)}$ satisfy

$v_1 > \cdots > v_d$; therefore $v_1 \dots v_d$ is d–decomposable.

Corollary 2.31. (Shirshov's Theorem). *There exists a function $\beta(t,u,d)$ such that for each $u \geq 2d$ and each $k \geq \beta(t,u,d)$, every (k,t)–word contains either: (1) a subword w_0^u, $\deg w_0 < d$, or (2) a d–decomposable subword.*

Theorem 2.32. *Let A be a finitely generated PI algebra, $A = alg(a_1, \dots, a_t)$. Suppose every monomial in $a_1, \dots, a_t$ of degree $< d$ is algebraic over the field K. Then A is finite dimensional over K.*

Proof: Denote as B the set of all monomials of degrees $\leq \beta(t,u,d)$ where $u = \max(2d, g)$, and g is an upper bound of the algebraic degrees of the monomials of degrees $< d$; the set of these monomials is finite. We shall prove that B spans A as a K–vector space.

Let all monomials of degree $< k$ belong to $\ell(B)$, the span of the set B, and suppose $r = a_{i_1} \dots a_{i_k}$. Assume further that the monomials of degree k that precede r as (k,t)–words, also belong to $\ell(B)$. (Why this assumption does not lead to a loss of generality?)

The inequality $k < \beta(t,u,d)$ would yield the theorem. If $k \geq \beta(t,u,d)$ then r must satisfy either (1) or (2) from the last Corollary.

a) If (1) holds then r has as a subword $r_0^u = (a_{j_1} \dots a_{j_q})^u$, $r = r' r_0^u r''$. Thus $r_0^u = \sum_{i=0}^{n-1} \alpha_i r_0^i$, and we continue by induction on the degree.

b) If (2) holds then $r = r' r_1 \dots r_d r''$ is a decomposition. Consider the identity $f(x_1, \dots, x_d) = x_1 \dots x_d + \sum_{\sigma \neq 1} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(d)}$ in A, and write $r_1 \dots r_d$ as a sum of monomials that are less than $r_1 \dots r_d$, and the same for r; the monomials for r are less than r as (k,t)–words.

Remark 2.33. The last theorem provides an answer to the famous problem due to Kurosh in the case of PI algebras. This problem asks whether the algebraic algebras that are finitely generated are of finite dimension; the answer in general is negative. One can find this negative solution to Kurosh's problem due to E. Golod and I. Shafarevich in various books; we recommend [15], Chapter

8.

We would like to mention that Shirshov's theorem can also be used in the case of non-associative algebras, see [48] for such applications.

3. Polynomial identities and Young diagrams

In this section we introduce the principal "machinery" in the combinatorial approach to the study of polynomial identities satisfied by a given algebra.

Definition 3.1. *The polynomial $f \in K\langle X\rangle$ is called uniform if all monomials of f contain the same variables, possibly with different degrees.*

Obviously every polynomial $f \in K\langle X\rangle$ can be represented as $f = f_1 + \cdots + f_t$ where $f_1, \ldots, f_t$ are uniform polynomials and such that for every $j \neq k$ there exists x_i participating in all monomials of f_j but in none of the monomials of f_k (or vice versa). The polynomials $f_1, \ldots, f_t$ are the uniform components of f.

The next assertions (and their proofs) can be found in any text on PI algebras, see, for example, [33], Chapter 20.

Lemma 3.2. *If $I \lhd_T K\langle X\rangle$ and $0 \neq f \in I$ then the uniform components of f also belong to I. In other words, if f is an identity in an algebra A then its uniform components are, too.*

Theorem 3.3. *Suppose that $I \lhd_T K\langle X\rangle$, $0 \neq f \in K\langle X\rangle$, and $\deg f = n$. Then there exists a multilinear polynomial $g \neq 0$, $\deg g \leq n$ with $g \in I$.*

Proof: (Hint) Let $f = f(x_1, \ldots, x_m)$ and suppose $k = \max(\deg_i f \mid 1 \leq i \leq m)$. Denote as l the number of variables x_i such that $\deg_i f = k$. Here we write $\deg_i$ for the degree of f in x_i. In order to complete the proof induct on the pairs (k, l) ordered lexicographically. If $\deg_m f = k$ consider

$$h = f(x_1, \ldots, x_{m-1}, x_m + x_{m+1}) - f(x_1, \ldots, x_{m-1}, x_m) - f(x_1, \ldots, x_{m-1}, x_{m+1}) \in I,$$

and repeat the above procedure until the polynomial g obtained becomes multilinear.

Corollary 3.4. *If A satisfies a PI of degree n then it also satisfies multilinear one of degree $\leq n$.*

Remark 3.5. In fact, the last theorem proves more than "promised". Namely it shows that the polynomial g belongs to the T-ideal generated by f, and the same in the corollary.

Lemma 3.6. *Suppose that $|K| = \infty$ and that $I \triangleleft_T K(X)$. If $f = f_1 + \ldots + f_t \in I$ where $f_1, \ldots, f_t$ are multihomogeneos and of pairwise distinct multidegrees then $f_1, \ldots, f_t$ all belong to I.*

Proof: (Hint) Observe that $f(x_1, \ldots, a_i x_i, \ldots, x_n) = \sum_{j=1}^{t} a_i^{m(j)} f_j(x_1, \ldots, x_i, \ldots, x_n)$, and choose pairwise distinct scalars $a_1, \ldots, a_t$ in K. Then show that the system with variables $f_j(x_1, \ldots, x_i, \ldots, x_n)$ just obtained has at least one solution since its determinant is the Vandermonde determinant.

Theorem 3.7. *a) If $|K| = \infty$ then every T–ideal $I \triangleleft_T K(X)$ is generated by multihomogeneous polynomials.*

b) If $\operatorname{char} K = 0$ *then every T–ideal $I \triangleleft_T K(X)$ is generated by multilinear polynomials.*

Proof: b) Consider the multihomogeneous generators of I, and then repeat the procedure explained in the previous theorem, in order to obtain a multilinear polynomial starting from some multihomogeneous polynomial. Then it is easy to show that the multihomogeneous polynomial can be obtained by the multilinear one, too.

Definition 3.8. *The first procedure of the proof above is called linearization.*

The last theorem shows that when $\operatorname{char} K = 0$, without loss of generality one can restrict the consideration of the identities to the multilinear ones only.

Now consider the vector space P_n of all multilinear polynomials in $x_1, \ldots, x_n$. This space has the natural structure of S_n–module defined by $\sigma(x_{i_1} \ldots x_{i_n}) = x_{\sigma(i_1)} \ldots x_{\sigma(i_n)}$, $\sigma \in S_n$, $x_{i_1} \ldots x_{i_n} \in P_n$. All modules we consider will be left modules. In other words, the elements of S_n act as non-singular linear transformations on P_n. The reader that is not familiar with the theory of representations of groups could find the necessary information in any book concerning it. We suggest [5] as one of the most complete on this subject. If the reader decides that this book is quite encyclopaedic (it is indeed) we suggest [15] for the basics of this theory.

The group algebra KG of the group G has a basis consisting of the elements of G; the multiplication between the elements of the basis is defined as in G. A routine check shows that KG is an associative algebra. Every G–module (i.e., vector space where the elements of G act as non-singular linear operators) is a KG–module (i.e., vector space where the elements of KG act as operators), and vice versa. The G–module V is irreducible if it has only two submodules—0 and V; V is semisimple if it is a direct sum of irreducible submodules; equivalently, if every submodule has a direct sum complement. Maschke's theorem states that if G is a finite group and if K is a field such that $\operatorname{char} K$ does not divide $|G|$ then the G–module KG is semisimple. Obviously, submodules and quotients of semisimple modules are semisimple, too. Furthermore it is well-known that in this case every irreducible G–module V of finite dimension over K is isomorphic to a submodule of KG (that is, V is isomorphic to some minimal left ideal of KG, see [5].)

Lemma 3.9. *The S_n–modules KS_n and P_n are isomorphic.*

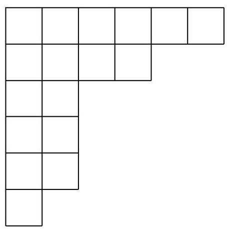
Proof: The map $\varphi: KS_n \to P_n$ defined by $\varphi(\sigma) = x_{\sigma(1)} \ldots x_{\sigma(n)}$, is the isomorphism we are looking for.

From now on, till the end of Section 4, we fix the *field K of characteristic 0*. In this case the description of the irreducible S_n–modules can be based on partitions and Young diagrams.

Definition 3.10. *Let $n \in \mathbf{N}$. A partition λ of n, $\lambda \vdash n$, is the sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-negative integers with $\lambda_1 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$.*

The set $[\lambda] = \{(i,j) \mid i,j \in \mathbf{N}, i \leq k, j \leq \lambda_i\}$ is called the Young diagram corresponding to λ. The elements (i,j) are the squares (or the cells) of $[\lambda]$. The graphic presentation of $[\lambda]$ in case $\lambda = (6,4,2,2,2,1) = (6,4,2^3,1) \vdash 17$ is the following (figure denoted by λ):

$[\lambda] =$



$D =$

3	15	17	11	14	10
12	6	13	16		
8	1				
5	9				
7	2				
4					

$D(\lambda)_\tau =$

$\tau(1)$	$\tau(r_1+1)$	$\ldots$	$\ldots$
$\tau(2)$	$\tau(r_1+2)$	$\ldots$	$\tau(n)$
$\vdots$	$\vdots$		
$\vdots$	$\tau(r_1+r_2)$		
$\tau(r_1)$			

The diagram $[\lambda]$ filled in some way with the numbers 1, $\ldots$, n, is called a λ–Young tableau D; see above, the figure denoted by D.

Let us recall that the algebra A is semisimple if it is semisimple considered as a left A–module. This means that for every left ideal L in A there exists another left ideal C such that $L \oplus C = A$. Since KS_n is semisimple we obtain that P_n is semisimple as S_n–module.

Now let $\lambda \vdash n$ be a partition of n, and denote $D = [\lambda]$ the corresponding Young tableau. We define an action of S_n on the set of the λ–tableaux D

as follows. If the entries of $D = [\lambda]$ are d_{ij} (i.e., d_{ij} is in the (i,j)–th cell of D), and $\sigma \in S_n$ then $\sigma(D)$ is the λ–tableau having entries $\sigma(d_{ij})$ in its (i,j)–th cells. Further, denote as $R(D) = \{\sigma \in S_n \mid \sigma(d_{ij}) = d_{ij'}\}$ and as $C(D) = \{\sigma \in S_n \mid \sigma(d_{ij}) = d_{i'j}\}$ the subgroups of S_n that preserve the sets of the elements on the rows, respectively on the columns of D. In other words the permutations of $R(D)$ preserve the first index of d_{ij}, and those of $C(D)$ preserve the second index of d_{ij}.

Theorem 3.11. *Let $\lambda \vdash n$ and let D_λ be a Young tableau corresponding to λ. Define $e(D_\lambda) = \sum (-1)^\tau \sigma\tau \in KS_n$ where the sum is over all $\sigma \in R(D_\lambda)$ and $\tau \in C(D_\lambda)$. Then $e(D_\lambda)$ is a scalar multiple of an idempotent in KS_n, $e(D_\lambda)^2 = \alpha e(D_\lambda)$, $\alpha \in K$. Furthermore:*

a) $KS_n.e(D_\lambda) = M(D_\lambda)$ is a minimal left ideal in KS_n. That is, $e(D_\lambda)$ generates an irreducible S_n–submodule $M(D_\lambda)$ of KS_n.

b) The modules $M(D_\lambda)$ and $M(D'_\mu)$ are isomorphic if and only if $\lambda = \mu$.

c) If $\lambda = \mu$ the isomorphisms $\varphi\colon M(D_\lambda) \to M(D'_\lambda)$ are defined by $\varphi(e(D_\lambda)) = k\sigma^{-1}e(D'_\lambda)$ where $0 \neq k \in K$, and $\sigma(d_{ij}) = d'_{ij}$ for all i and j.

d) If M is an irreducible submodule of KS_n then M can be generated by an element of the form $e_\lambda = \sum k(D_\lambda)\sigma_D^{-1}e(D_\lambda)$, $k(D_\lambda) \in K$, $\sigma_D \in S_n$, and the sum runs over all λ–tableaux D_λ.

This theorem is one of the basic facts in the theory of representations of S_n. The reader could find a proof of the theorem in [5], § 28. Using the isomorphism $\varphi\colon KS_n \to P_n$ we obtain the next corollary.

Corollary 3.12. *Every irreducible S_n–submodule in P_n is generated by a polynomial of the form $f = \sum k(D_\lambda)\varphi(\sigma_D^{-1}e(D_\lambda))$.*

Remark Let $f = f(x_1, \ldots, x_m) \in K(X)$ be homogeneous, $\deg_i f = m_i$. Then, by means of linearizations we can obtain, starting from f, a multilinear polynomial denoted by $\operatorname{lin} f$. It is easy to verify (Verify!) that the polynomials

f and $\operatorname{lin} f$ generate the same T–ideal in $K(X)$. Such polynomials are called equivalent (as identities). Hence every T–ideal $I \lhd_T K(X)$ is generated by its multilinear polynomials. In other words, I is generated as T–ideal by the union $\cup_{n\geq 0}(I \cap P_n)$. We denote $P_n(I) = I_n = I \cap P_n$.

This remark justifies our attention to the multilinear identities.

Denote as $K_m(X)$ the free K–algebra freely generated by the finite set $X_m = \{x_1, \ldots, x_m\}$, and denote as $A_m^{(n)}$ the subspace of $K_m(X)$ spanned by all homogeneous polynomials of degree n. The general linear group GL_m of order m acts in a natural way on $A_m^{(1)}$ (note that this vector space has a basis $x_1, \ldots, x_m$). Therefore $A_m^{(n)}$ is a left GL_m–module with respect to the action of GL_m defined as $g(\sum_i k_i x_{i_1} \ldots x_{i_n}) = \sum_i k_i g(x_{i_1}) \ldots g(x_{i_n})$, $k_i \in K$, $g \in GL_m$. Using the decomposition $K_m(X) = \oplus_{n\geq 0} A_m^{(n)}$ one defines an action of GL_m over $K_m(X)$. If N is some GL_m–submodule of $K_m(X)$ then $N = \oplus_{n\geq 0}(N \cap A_m^{(n)})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ be a partition of n, and suppose that the diagram $[\lambda]$ has $t = \lambda_1$ columns of respective lengths $r_1, \ldots, r_t$. If $\tau \in S_n$ is a permutation we fill the columns of $[\lambda]$ with the numbers $\tau(1), \ldots, \tau(n)$ starting with the first column, from top to bottom, after that we fill the second column, etc. See, for example $D(\lambda)_\tau$, the Young tableau obtained, at the last figure in Definition 3.10.

We define the polynomial $f_\tau(x_1, \ldots, x_m)$ as

$$\sum (-1)^{\sigma_1} \ldots (-1)^{\sigma_t} \ldots x_{\sigma_1(1)} \ldots x_{\sigma_2(1)} \ldots x_{\sigma_1(2)} \ldots x_{\sigma_2(2)} \ldots$$

where $x_{\sigma_1(1)}$ is at the $\tau(1)$–st position, $x_{\sigma_1(2)}$ is at the $\tau(2)$–nd position, …, $x_{\sigma_1(r_1)}$ at the $\tau(r_1)$–th position, …, $x_{\sigma_t(1)}$ at the $\tau(r_1 + \ldots + r_{t-1} + 1)$-st position, …, and $x_{\sigma_t(r_t)}$ stands at the $\tau(r_1 + \ldots + r_t)$–th position. Here $\sigma_i \in S_{r_i}$, $i = 1, \ldots, t = \lambda_1$.

For example, when $\tau = 1$ (the identical permutation) one obtains that

$$f_\tau = S_{r_1}(x_1, \ldots, x_{r_1}) S_{r_2}(x_1, \ldots, x_{r_2}) \ldots S_{r_t}(x_1, \ldots, x_{r_t}).$$

It is known that the finite dimensional GL_m–submodules in $K_m(X)$ are semisimple. The irreducible submodules in $K_m(X)$ can be described by Young tableaux having at most m rows.

Theorem 3.14. **(Weyl).** *[13] If $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ is a partition of n, $r \le m$, then the polynomial f_τ generates an irreducible GL_m–submodule N in $A_m^{(n)}$, denoted by $N_m(\lambda)$. Furthermore:*

a) Given a permutation $\rho \in S_n$, the mapping $f_\tau \mapsto \alpha f_\rho$ where $0 \ne \alpha \in K$, defines an isomorphism $(K.GL_m)f_\tau \cong (K.GL_m)f_\rho$. The submodules in $A_m^{(n)}$ that are isomorphic to $N_m(\lambda)$ are generated by polynomials of the form $f = \sum \alpha_\tau f_\tau$, $\tau \in S_n$.

b) If $n \le m$ then the irreducible GL_m–submodules in $A_m^{(n)}$ can be generated by their multilinear elements.

Corollary 3.15. *Suppose that $N = N_m(\lambda) \subseteq A_m^{(n)}$. Then the GL_m–module N is generated by a polynomial*

$$f = \Big(\prod_{i=1}^{t} S_{r_i}(x_1, \dots, x_{r_i})\Big) \sum \alpha_\sigma \sigma, \qquad \sigma \in S_n, \quad \alpha_\sigma \in K,$$

*where $r_1, \dots, r_t$ are the lengths of the columns of $[\lambda]$, and the (**right**) action of the group S_n on $A_m^{(n)}$ is defined via the rule $(x_{i_1} \dots x_{i_n})\sigma^{-1} = x_{i_{\sigma(1)}} \dots x_{i_{\sigma(n)}}$, $1 \le i_j \le m$.*

The irreducible representations of S_n and of GL_m can be described by means of partitions and Young diagrams. Hence there should exist some connection between these representations; and the same in the case of T–ideals. Let I be a T–ideal in $K(X)$. If $X = \{x_1, x_2, \dots\}$ is the set of the free generators of $K(X)$ then $K_m(X) \subseteq K(X)$. We have already observed that $I_n = I \cap P_n$ is a (semisimple) S_n–submodule of P_n. Analogously the GL_m–module $I \cap A_m^{(n)}$ is semisimple.

Let $\lambda \vdash n$ be a partition of n and let $\tau \in S_n$. Then $D(\lambda)_\tau$ is the respective λ–tableau; $M_n(\lambda)$ is the S_n–module that corresponds to $D(\lambda)_\tau$; it is generated by the polynomial $e_\tau = e_\tau(x_1, \dots, x_n) \in P_n$. We construct a polynomial $f \in A_m^{(n)}$ starting from e_τ by means of "symmetrization" (this is the opposite process to the linearization). In other words, we substitute the variables in e_τ whose indices belong to the i–th row of $[\lambda]$, by x_i for all i. The polynomial obtained in this way is a scalar multiple of the polynomial f_τ that generates the GL_m–module $N_m(\lambda)$ in $A_m^{(n)}$. On the other hand, since $N_m(\lambda)$ is irreducible, every non-zero element of $N_m(\lambda)$ generates the module.

Hence $M_n(\lambda) \subseteq N_m(\lambda)$. If $m \le n$ then the multilinear polynomials in $N_m(\lambda)$ form an S_n–submodule of P_n which is irreducible. Now let us gather these observations (cf. [2], [7]).

Theorem 3.16. *If $m \le n$ then the S_n–module $I \cap P_n$ and the GL_m–module $I \cap A_m^{(n)}$ have the same module structure: if $I \cap P_n = \sum a_\lambda M_n(\lambda)$ then $I \cap A_m^{(n)} = \sum a_\lambda N_m(\lambda)$ where a_λ stand for the multiplicities of the corresponding irreducible modules.*

Example 3.17. If $\lambda = (2,1) \vdash 3$ and $D = [\lambda] = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ then $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$, $R(D) = \{1,(23)\}$ and $C(D) = \{1,(12)\}$. Thus $e(D) = 1 - (12) + (23) - (132)$ and the generator of $M_3(D)$ is $f(x_1,x_2,x_3) = x_1x_2x_3 - x_2x_1x_3 + x_1x_3x_2 - x_3x_2x_1$. The polynomial $g(x,y) = yx^2 - xyx + yx^2 - xyx = 2(yx^2 - xyx)$ generates the module $N_2(D)$. Obviously the polynomial g is much simpler than f. It is often convenient to work with the generators of the GL_m–modules instead of the respective multilinear polynomials.

Remark 3.18. Let A be an algebra, $\dim A = t$ and let I be the T–ideal of A. Then $I \cap P_n$ contains all modules $M_n(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $k > t$. In particular A satisfies the identity $s_{t+1}(x_1, \ldots, x_{t+1})$.

Example 3.19. If A is a PI algebra and if $|K| = \infty$ then A satisfies some homogeneous polynomial identity $f(x_1, \ldots, x_m)$, $\deg_i f = n_i$, $i = 1, \ldots, m$. We denote $g = f(x_{11} + \ldots + x_{1n_1}, \ldots, x_{m1} + \ldots + x_{mn_m})$ hence g is also an identity in A. The component $h = \operatorname{lin} f$ of g that is linear in $x_{11}, \ldots, x_{1n_1}, \ldots, x_{m1}, \ldots, x_{mn_m}$ is an identity in A, too. It is called the complete linearization of f.

On the other hand, the polynomial obtained by h by means of the consecutive substitutions $x_{11} = \ldots = x_{1n_1} = x_1, \ldots, x_{m1} = \ldots = x_{mn_m} = x_m$, is a scalar multiple of f. More precisely, it equals $n_1! \ldots n_m! f(x_1, \ldots, x_m)$. For example, if $f = x^n$ then $g = (x_1 + \ldots + x_n)^n$ and $\operatorname{lin} f = \operatorname{sym}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)}$. The symmetrization of $\operatorname{lin} f$ is $n! x^n$. But a warning: When $\operatorname{char} K = p > 0$ and $p \leq n$ we have $n! = 0$ in K. This points out one of the difficulties in the case when $\operatorname{char} K > 0$.

4. T-ideals related to matrix algebras

In this section we shall consider some of the most important applications of the theory introduced in the previous section. Till the end of the section we fix the *field K of characteristic 0.*

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition, and let $D = [\lambda]$ be a Young tableau filled with the permutation $\{n_{ij}\}$ of $\{1, \ldots, n\}$, the number n_{ij} being situated at the (i, j)–th cell of D.

Definition 4.1. *The tableau D is called standard if $n_{ij} \leq n_{pq}$ for all $i \leq p$ and $j \leq q$. In other words, the entries of D increase along the rows from left to right, and along the columns from top to bottom.*

Example 4.2. If $\lambda = (3, 1, 1) \vdash 5$, the number of the standard tableaux is 6, as an easy calculation shows.

Consider the partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$; the number h_{ij} of the cells below the (i, j)–th (and on the same j–th column) and on the right of (i, j)–th (and on the same i–th row), including (i, j) is called the hook number of (i, j). In the example above $h_{11} = 5$, $h_{12} = 2$, $h_{13} = 1$, $h_{21} = 2$ and $h_{31} = 1$.

Theorem 4.3 (Hook formula). *[23] Let $\lambda \vdash n$ be a partition and let $M_n(\lambda)$ be an irreducible S_n–submodule in P_n. Then the dimension of $M_n(\lambda)$ equals $\dim M_n(\lambda) = n!/(\prod_{i,j} h_{ij})$ and it is equal to the number of the standard tableaux $[\lambda]$ filled with the numbers $\{1, \ldots, n\}$.*

Lemma 4.4. *Let $I \triangleleft_T K(X)$ be a T–ideal, and let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a fixed partition of n. If for all tableaux D corresponding to the partition λ, the irreducible modules $M_n(D)$ lie in the intersection $I \cap P_n = I_n$ then the polynomial $S_{r_1}(x_1, \dots, x_{r_1}) \dots S_{r_t}(x_1, \dots, x_{r_t})$ belongs to I where $t = \lambda_1$ and $r_1, \dots, r_t$ are the lengths of the respective columns of λ.*

Proof: Use the procedure of symmetrization.

And now it comes the turn of an important and interesting result due to A. Regev ([38]). Again we follow the exposition of this result given in [40], pp. 246–248.

Corollary 4.5. *Let $I \triangleleft_T K(X)$ be a T–ideal and let $c_n(I) = \dim(P_n/I \cap P_n)$ be the n–th codimension of I. If $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ is a partition such that $\dim M_n(\lambda) > c_n(I)$ then the intersection $I \cap P_n = I_n$ contains all S_n–modules that are isomorphic to $M_n(\lambda)$.*

Proof: If $D = [\lambda]$ and $M_n(D) \not\subseteq I \cap P_n$ then $M_n(D) \cap (I \cap P_n) = 0$ since $M_n(D)$ is irreducible. Therefore $\dim P_n = n! \geq \dim M_n(D) + \dim(I \cap P_n)$ and $c_n(I) + \dim(I \cap P_n) = n!$ that is, $c_n(I) \geq \dim M_n(D)$ which contradicts to the inequality of the corollary.

Proposition 4.6. *Suppose that $n = mk$ and that $\lambda = (m, \dots, m) \vdash n$ is the "rectangular" partition $k \times m$. Let $I \triangleleft_T K(X)$ be a T–ideal such that $c_n(I) \leq \alpha^n$ where $mk \geq 2e.\alpha(m+k)/2$, $e \approx 2,7172\dots$ is the base of the natural logarithms. Then $\dim M_n(\lambda) > c_n(I)$.*

Proof: Denote as h_{ij} the hook numbers of λ. Then, using the AM—GM inequality one obtains that

$$\Big(\prod_{i,j} h_{ij}\Big)^{1/n} \leq \Big(\sum_{i,j} h_{ij}\Big)/n = \sum_{i=1}^{k}\sum_{j=1}^{m}(i+j-1)/n = (k+m)/2.$$

Since $n(\log n - 1) + 1 = \int_1^n \log x \, dx < \sum_{p=1}^{n} \log p = \log(n!)$ we have the inequality

$(mk)^n < n!e^n$. On the other hand, $\alpha \leq (mk/e)(2/(m+k)) < \left(n!/\prod_{i,j} h_{ij}\right)^{1/n}$, and therefore $\dim M_n(\lambda) > \alpha^n \geq c_n(I)$.

Theorem 4.7 (Regev). *(see [38]) Let A be a PI algebra and let d stand for the degree of some multilinear identity satisfied by A. If m and k are positive integers such that $mk/(m+k) > (d-1)^2.e/2$ then the algebra A satisfies the identity $(S_k(x_1, \ldots, x_k))^m$.*

Proof: The polynomial $(S_k(x_1, \ldots, x_k))^m$ generates the GL_k–module $N_k(\lambda)$ where $\lambda = (m, \ldots, m) \vdash mk$. We have already proved (Regev's theorem about the tensor product in Section 2) that $c_n(I) < (d-1)^{2n}$. Then $\alpha = (d-1)^2$ in the last proposition implies the theorem.

Remark 4.8. The fact that every PI algebra A satisfies a power of the standard identity was first established by S. Amitsur, see for example [40], Theorem 1.6.46.

Definition 4.9. *Suppose that f and $g \in K\langle X\rangle$. The polynomial g is called a consequence of f (or the identity f implies the identity g, or the identity g follows from the identity f) if $g \in \langle f\rangle^T$ where $\langle f\rangle^T$ is the T–ideal generated by f. In other words, g is a consequence of f if in every algebra A where f is identity, g also is.*

The polynomials f and g are equivalent as identities if $f \in \langle g\rangle^T$ and $g \in \langle f\rangle^T$.

Example 4.10. 1. If f is homogeneous and if $|K| = \infty$ then $\operatorname{lin} f \in \langle f\rangle^T$; f and $\operatorname{lin} f$ are equivalent when $\operatorname{char} K = 0$.

2. If $m \geq n$ then $s_m \in \langle s_n\rangle^T$.

Definition 4.11. *The T–ideal $I \triangleleft_T K\langle X\rangle$ is called finitely based (abbreviated as f.b.) if I is generated as a T–ideal by some finite collection of polynomials $f_1, \ldots, f_n$; $I = \langle f_1, \ldots, f_n\rangle^T$. This means that the elements of I are conse-*

quences of a finite set of polynomials in I.

One of the most interesting and most difficult problems in the theory of algebras with polynomial identities is the famous Specht problem: whether every T–ideal is finitely based when $\operatorname{char} K = 0$. The problem was resolved positively by A. Kemer [24].

The solution to the Specht problem is very complicated; it depends heavily on a classification of the prime T–ideals as ideals of identities satisfied by certain known algebras (these are related to the matrix algebras and the Grassman algebra).

Definition 4.12. *The T-ideal $I \triangleleft_T K\langle X\rangle$ is called spechtian if it is finitely based and every T-ideal J, $I \subseteq J$, is finitely based, too. The algebra A is called spechtian if $T(A)$ is spechtian, and the variety var A is spechtian if the T–ideal $T(A)$ of A is.*

Exercise. If $I \triangleleft_T K\langle X\rangle$ is f.b. prove that I is spechtian if and only if every strictly ascending chain of T–ideals $I = I_1 \subset I_2 \subset \ldots$ is finite.

We cannot offer some general methods for establishing whether certain set of polynomials generates a fixed T–ideal, or whether some T–ideal is spechtian. One of the frequently used tools is that of the partially well ordered (in short, PWO) sets.

Definition 4.13. *The set Q equipped with the partial order $\leq$ is called partially well ordered if for every subset $R \subseteq Q$ there exists a finite subset $R_0 \subseteq R$ such that for each $r \in R$, $r_0 \leq r$ holds for some $r_0 \in R_0$.*

Lemma 4.14. *The following conditions are equivalent:*

(1) $(Q, \leq)$ is a PWO set.

(2) Every infinite sequence $\{q_1, q_2, \ldots\} \subseteq Q$ has an infinite increasing subsequence.

(3) Every infinite sequence $\{q_1, q_2, \ldots\} \subseteq Q$ *contains two elements* q_i *and* q_j *such that* $i < j$ *and* $q_i \leq q_j$.

(4) There exist neither infinite strictly descending sequences of elements of Q *nor infinite subsets of* Q *consisting of pairwise incomparable elements.*

Proof: It is similar to the elementary but important fact known from the courses in Calculus that every infinite sequence of real numbers contains an infinite and monotone subsequence. The reader can look at [16], Theorem 2.1, for the proof and further facts.

Lemma 4.15. *a) If* $(Q, \leq)$ *is PWO then every subset* $R \subseteq Q$ *of* Q *also is PWO (with respect to the same order as in* Q*).*

b) Let $(Q_1, \leq_1)$ *and* $(Q_2, \leq_2)$ *be two PWO sets. Define a partial order in the direct product* $Q = Q_1 \times Q_2$ *of* Q_1 *and* Q_2 *as follows. Set* $(q_1, q_2) \leq (q_1', q_2')$ *if* $q_1 \leq_1 q_1'$ *and* $q_2 \leq_2 q_2'$, q_1, $q_1' \in Q_1$, q_2, $q_2' \in Q_2$. *Then* $(Q, \leq)$ *is PWO.*

Definition 4.16. *If* $(Q, \leq)$ *is a partially ordered set we define a partial order in the set* $V(Q)$ *of all finite sequences of elements of* Q *as* $(q_1, \ldots, q_k) \leq (t_1, \ldots, t_m)$, q_i, $t_j \in Q$, *if* $k \leq m$ *and if there exists a map* $\varphi: \mathbf{N} \to \mathbf{N}$ *that is strictly increasing, and* $q_i \leq t_{\varphi(i)}$, $i = 1, \ldots, k$.

In other words $(q_1, \ldots, q_k) \leq (t_1, \ldots, t_m)$ *if the sequence* $(q_1, \ldots, q_k)$ *is bounded (term by term) by some subsequence of* $(t_1, \ldots, t_m)$.

Theorem 4.17 (Higman). *[16] If the set* Q *is PWO then* $V(Q)$ *also is.*

Proof: The proof is quite elementary i.e., it does not involve complicated and sophisticated facts but it is rather difficult and tricky as it sometimes happens with assertion that seem evident...

Corollary 4.18. *Let* $Y = \{y_1, \ldots, y_m\}$ *be a finite alphabet considered with the trivial order on it i.e.,* y_i *and* y_j *are incomparable when* $i \neq j$. *Then the set* $V(Y)$ *of all words in* Y *is PWO considered with respect to the order defined on*

$V(Y)$ as in Higman's theorem. This means that if p_1, p_2, ..., is a sequence of words then there must exist $i < j$ such that p_i is obtained by p_j after removing some of the letters in p_j.

Proof: Note that since the set Y is finite it is PWO with respect to the trivial order. The assertion follows easily from Higman's theorem. It should be noted that there exist direct proofs of this corollary (i.e., proofs that do not depend on Higman's theorem). We suggest that the reader try and find one.

Definition 4.19. *The sequence $i = (i_1, \ldots, i_k)$ is of type t if one can cut it in t consecutive parts, each of them ascending, and t is the least possible positive integer with this property.*

For example, $(173452698) = (17)(345)(269)(8)$ is of type 4.

Now let us consider the sequences s of type t with entries from the set $\{1, \ldots, k\}$. The matrix presentation of s is the following. Suppose $s = s^{(1)} \ldots s^{(t)}$ where $s^{(i)}$ are the consecutive increasing subsequences. We define the matrix $A(s) = (a_{ij})$ that corresponds to s as the $t \times k$ matrix with entries 0 and 1: $a_{ij} = 1$ if j belongs to $s^{(i)}$, and $a_{ij} = 0$ otherwise.

Example 4.20. If $s = (152346) = (15)(2346)$ then $A(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$

Definition 4.21. *Denote by R_{nt} the set of permutations $\sigma \in S_n$ such that the sequence $(\sigma(1), \ldots, \sigma(n))$ is of type t, and by R_t the union $R_t = \cup_{n \geq 1} R_{nt}$.*

Lemma 4.22. *Suppose that σ, $\tau \in R_{nt}$. If $\sigma \neq \tau$ then the matrices $A(\sigma)$ and $A(\tau)$ corresponding to σ and τ respectively, are different.*

Definition 4.23. *If $S = \cup_{n \geq 1} S_n$ then we define a partial order $\leq$ on S as follows. Suppose that $\sigma_i \in S_{n_i}$, $\sigma_j \in S_{n_j}$. Then $\sigma_i \leq \sigma_j$ if there exists an increasing 1-1 function $\varphi: \{1, \ldots, n_i\} \to \{1, \ldots, n_j\}$ such that the sequence $(\varphi(\sigma_i(1)), \ldots, \varphi(\sigma_i(n_i)))$ is a subsequence of $(\sigma_j(1), \ldots, \sigma_j(n_j))$. This* *means that* $\sigma_j(1)\dots\sigma_j(n_j) = p_1\varphi(\sigma_i(1))p_2\varphi(\sigma_i(2))\dots p_{n_i}\varphi(\sigma_i(n_i))p_{n_i+1}$ *for suitable words* $p_1,\dots,p_{n_i+1}$.

Theorem 4.24. *The set* $R_t \subset S$ *is PWO with respect to the order on* S *defined above.*

Proof: We apply Higman's theorem for the matrix presentations of the elements of R_t. The columns of the corresponding matrices are vectors of the type $(\varepsilon_1,\dots,\varepsilon_t)$ where $\varepsilon_i = 0$ or 1. The set E of all such vectors is finite and it has 2^t elements. Hence E with the trivial order is PWO. Then the corollary of Higman's theorem yields that the set $V(E)$ with respect to the order induced by the trivial order on E is PWO. The preceding lemma completes the proof since $\sigma \leq \tau$ if and only if $A(\sigma) \leq A(\tau)$. (Verify the last statement!)

The first non-trivial example of a spechtian T–ideal that we are going to consider, is the T–ideal L generated by the polynomial $l_n = [x_1,x_2][x_3,x_4]\dots[x_{2n-1}, x_{2n}]$ where $[x,y] = xy - yx$ is the usual commutator of x and y. The proofs of the statements that follow depend on the combinatorics on PWO sets, and especially on the fact that R_t is PWO.

Lemma 4.25. *The algebra* $T_n(K)$ *of the upper triangular matrices of order* n *satisfies the identity* l_n.

Definition 4.26. *Let* f *be a multilinear polynomial and let* $J = \langle f\rangle^T$ *be the T-ideal generated by* f. *If* I *is a T-ideal such that* $J \subseteq I$ *and if* $g \in I \cap P_k$ *is multilinear of degree* k *we denote by* $L_n(g,I,k)$ *the set of all polynomials* $h \in I \cap P_k$ *such that* $m(h) \leq m(g)$ *and* $m(h)$ *is of type* $\leq n$. *Here we denote as* $m(h)$ *the monomial of* h *having the largest order with respect to the usual lexicographical order. We also identify the type of the monomial* $x_{\sigma(1)}\dots x_{\sigma(k)}$ *with the type of the permutation* $(\sigma(1),\dots,\sigma(k))$.

Then the polynomial f *is called an* n*-polynomial if for every T-ideal* I, $J \subseteq I$, *every* k *and* $g \in I \cap I_k$ *the inclusion* $g \in J \cap P_k + L_n(g,I,k)$ *holds.*

Lemma 4.27. *The polynomial* l_n *is a* $2n$*-polynomial.*

Proof: See [31], or [40], pp. 250–252.

Theorem 4.28. (Latyshev). *([31]) If the T–ideal* $I \lhd_T K(X)$ *contains some* $2n$*-polynomial then* I *is finitely based.*

Corollary 4.29. *The T–ideal* $\langle [x_1, x_2][x_3, x_4] \ldots [x_{2n-1}, x_{2n}] \rangle^T$ *is spechtian.*

Proof: The generator of this T–ideal is the $2n$–polynomial l_n.

Remark 4.30. The last assertion is some rather particular case of Kemer's theorem already mentioned. On the other hand it is worth mentioning that Kemer's theorem does not find bases of the identities in some fixed algebra. For example, the identities satisfied by the matrix algebras $M_n(K)$ over a field K of characteristic 0, are f.b. A basis of these identities is known only in the "second simplest" case $n = 2$ ($n = 1$ being trivial).

In 1973 Yu. P. Razmyslov obtained that the T–ideal $T(M_2(K))$ can be generated by nine identities when $\operatorname{char} K = 0$, and in 1974 he established the validity of the Specht property for this T–ideal. The proofs of these statements are rather complicated. The reader could look for the complete version in [36].

In 1981 V. Drensky proved that the polynomials which we have already net in Sections 1 and 2, $[[x_1, x_2]^2, x_3]$ and $s_4(x_1, x_2, x_3, x_4)$ form a basis of the T–ideal $T(M_2(K))$ when $\operatorname{char} K = 0$, and that these polynomials are independent as identities. This means that neither the first follows from the second nor the second is a consequence of the first, see [8].

Exercise. Prove that the last two polynomials are identities in $M_2(K)$.

Definition 4.31. *If* $g_1, \ldots, g_n \in K(X)$ *we define the (long) commutators by* *induction:*

$$[g_1, g_2] = g_1 g_2 - g_2 g_1, \ [g_1, g_2, g_3] = [[g_1, g_2], g_3],$$
$$[g_1, \ldots, g_{n-1}, g_n] = [[g_1, \ldots, g_{n-1}], g_n], \ n \geq 3.$$

Denote as $B(X)$ the subalgebra of $K(X)$ generated by all commutators in the set X. The elements of $B(X)$ are called commutator (or proper) polynomials.

Example 4.32. The polynomial $[x_2, x_1, x_1]$ is proper; the polynomial $[x_1 x_2, x_3]$ is not.

Remark 4.33. Let A be an algebra and let $a_1, \ldots, a_m \in A$. If for some i the element a_i is central (i.e., belongs to the centre of A) then $[a_1, \ldots, a_m] = 0$. This observation leads to the following criterion. Let $f(x_1, \ldots, x_n) \in K(X)$ be a polynomial, then $f \in B(X)$ if and only if $\partial f / \partial x_i = 0$ for every $i = 1, \ldots, n$. Here $\partial / \partial x_i$ stands for the usual partial derivation; in the non-commutative case it is defined as $\partial x_j / \partial x_i = 0$ if $i \neq j$ and 1 if $i = j$, if w is a monomial, $w = uv$ then $\partial(uv)/\partial x_i = (\partial u / \partial x_i)v + u(\partial v / \partial x_i)$, and then it is extended by linearity to all polynomials. (Prove that the above definition is correct i.e., it does not depend on the representation $w = uv$!)

The "only if" part is evident, we suggest that the reader try and prove the "if" part of the criterion. Some hint for this can be found in the proof of the next theorem.

Theorem 4.34. (W. Specht). *Suppose that K is an infinite field and that A is a K–algebra with unit element $1 \in A$. Then the T–ideal $I = T(A)$ of A is generated as a T-ideal by its proper polynomials. In other words, I is generated by the set $I \cap B(X)$.*

Proof: The free Lie algebra $L(X)$ freely generated by the set X over K, has as its universal enveloping algebra the free associative algebra $K(X)$. Then the well-known Poincaré–Birkhoff–Witt theorem yields that there exists a basis of the vector space $K(X)$ that consists of the polynomials $x_1^{r_1} \ldots x_p^{r_p} v_1^{t_1} \ldots v_q^{t_q}$ where p, q, r_i and t_i are non-negative integers, and the polynomials $\{v_i\}$ together with

x_1, x_2, ... form a basis of $L(X)$. Note that one can choose all v_i as commutators and that the products $v_1^{t_1} \dots v_q^{t_q}$ form a basis of $B(X)$.

If $f(x_1, \dots, x_n) \in I$ is homogeneous then $f = \sum_r x_1^{r_1} \dots x_n^{r_n} f_r(x_1, \dots, x_n)$ where the sum is over all n-tuples $r = (r_1, \dots, r_n)$, $f_r \in B(X)$, and all polynomials are homogeneous. Suppose that $\alpha \in K$ and that $a_1, \dots, a_n \in A$; we apply the last remark, and thus obtain that $f_r(a_1 + \alpha.1, a_2, \dots, a_n) = f_r(a_1, \dots, a_n)$. Therefore $f(a_1 + \alpha.1, a_2, \dots, a_n) = \sum_r (a_1 + \alpha.1)^{r_1} a_2^{r_2} \dots a_n^{r_n} f_r(a_1, \dots, a_n)$. Now we choose the largest r_1 such that $f_r(a_1, \dots, a_n) \neq 0$; using the properties of Vandermonde determinant we obtain that $\sum a_2^{r_2} \dots a_n^{r_n} f_r(a_1, \dots, a_n) = 0$. Hence we can complete the proof by induction.

Corollary 4.35. *If* $\operatorname{char} K = 0$ *then every* T*–ideal* $I \triangleleft_T K(X)$ *is generated by its proper multilinear polynomials.*

Definition 4.36. *Denote as* $\mathcal{M}$ *the variety* $\operatorname{var} M_2(K)$ *where* $\operatorname{char} K = 0$*; as* $F(\mathcal{M}) = K(X)/T(\mathcal{M})$ *the relatively free algebra in this variety, and as* $Q_n = P_n \cap B/(P_n \cap B \cap T(\mathcal{M}))$ *the* S_n*-module consisting of all proper multilinear polynomials of degree* n *in* $F(\mathcal{M})$.

Lemma 4.37. *a) The* S_n*-module* Q_n *is semisimple.*

b) The irreducible submodules in Q_n *correspond to Young diagrams with at most 3 rows.*

Proof: The first statement holds since P_n is semisimple. Now we prove the second. Let $g(x_1, \dots, x_n)$ be a proper multilinear polynomial which generates an irreducible S_n–module isomorphic to $M_n(\lambda)$, $\lambda = (\lambda_1, \dots, \lambda_k)$. If $\lambda_k \neq 0$ then the symmetrization of g is $f(x_1, \dots, x_k) = \sum_{\sigma \in S_k} \alpha_\sigma f_\sigma(x_1, \dots, x_k)$ where f_σ generate irreducible GL_k–modules. The linearization $\operatorname{lin} f_\sigma$ of f_σ is a multilinear polynomial that is skew-symmetric in some k variables, say in $x_{i_1}, \dots, x_{i_k}$.

Now in order to verify whether $g(x_1, \dots, x_n)$ is an identity in $M_2(K)$ it is sufficient to substitute the variables $x_1, \dots, x_n$ only for the matrices $e_{12} =$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_{11} - e_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ since these matrices together with $e = e_{11} + e_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of $M_2(K)$, and the last matrix is central, i.e., it makes g vanish. Hence, if $k \geq 4$ then $g_\sigma = 0$ on e_{12}, e_{21}, $e_{11} - e_{22}$, and $g \in T(\mathcal{M})$.

Corollary 4.38. *If $0 \neq f \in F(\mathcal{M})$ then f is equivalent as an identity to some finite set of polynomials in ≤ 3 variables.*

Theorem 4.39. *([7]) The S_n-module Q_n, $n > 1$, is a direct sum of irreducible submodules that are pairwise non-isomorphic, and that correspond to the partitions $\lambda = (p + q + r, p + q, p) \vdash n$ with $p + q \neq 0$, and if $q = r = 0$, with $p > 1$.*

Proof: We shall need the following assertion. If $f(x_1, x_2, x_3) \in K(X)$ is homogeneous of degree d_i in x_i and if $d_i \equiv \varepsilon_i \pmod 2$, $\varepsilon_i = 0$ or 1 then

$$f(e_{11} - e_{22}, e_{12} + e_{21}, e_{12} - e_{21}) = \varepsilon(e_{11} - e_{22})^{\varepsilon_1}(e_{12} + e_{21})^{\varepsilon_2}(e_{12} - e_{21})^{\varepsilon_3}, \qquad \varepsilon \in K.$$

This assertion can be proved by induction on $d_1 + d_2 + d_3$ starting with the monomials in $K(X)$.

Now it is sufficient to consider only the polynomials in 3 variables. Denote as $K[\xi_{ij}, \eta_{ij}, \zeta_{ij}]$ the ring of the commutative polynomials in ξ_{ij}, η_{ij}, ζ_{ij}, $i, j = 1$, 2, and as Ω the algebraic closure of the field of fractions of this ring. The polynomial $h(x_1, x_2, x_3) \in K_3(X)$ is an identity in $M_2(K)$ if and only if $f(\xi, \eta, \zeta) = 0$ for some matrices ξ, η, $\zeta \in M_2(\Omega)$ whose entries are algebraically independent, and in addition, $\operatorname{tr}\xi = \operatorname{tr}\eta = tr\zeta = 0$. Let us choose $\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & -\xi_{11} \end{pmatrix}$, $\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & -\eta_{11} \end{pmatrix}$, $\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & -\zeta_{11} \end{pmatrix}$. These matrices are non-singular (invertible) and diagonalizable.

Claim 1. There exists $\lambda \in M_2(\Omega)$ such that $\lambda^{-1}\xi\lambda = \alpha(e_{11} - e_{22})$, $\lambda^{-1}\eta\lambda = \beta_{11}(e_{11} - e_{22}) + \beta_{12}e_{12} + \beta_{21}e_{21}$, $\lambda^{-1}\zeta\lambda = \gamma_{11}(e_{11} - e_{22}) + \gamma_{12}e_{12} + \gamma_{21}e_{21}$ where α, β_{ij} and γ_{ij} are algebraically independent.

Claim 2. There exists $\mu \in M_2(\Omega)$ such that $\mu^{-1}(e_{11}-e_{22}) = e_{11}-e_{22}$ and $\mu^{-1}(\beta_{12}e_{12}+\beta_{21}e_{21})\mu = \beta(e_{12}+e_{21})$, and β is algebraically independent with the elements of Claim 1.

Claim 3. If $\lambda\mu = \nu$ then $\nu^{-1}\xi\nu = \alpha(e_{11}-e_{22})$, $\nu^{-1}\eta\nu = \beta_1(e_{11}-e_{22})+\beta(e_{12}+e_{21})$, $\nu^{-1}\zeta\nu = \gamma_1(e_{11}-e_{22})+\gamma_2(e_{12}+e_{21})+\gamma(e_{12}-e_{21})$ where α, β, β_1, γ, γ_1, γ_2 are algebraically independent.

The proofs of these three claims are left to the reader as exercises in Linear Algebra.

Now let $f(x_1,x_2,x_3) = \sum_{\sigma\in S_n} k_\sigma f_\sigma(x_1,x_2,x_3)$ be a generator of $N_3(\lambda) \subseteq (K_3(X)\cap B(X))$. Then clearly

$$\begin{aligned}\nu^{-1}f_\sigma(\xi,\eta,\zeta)\nu &= f_\sigma(\alpha(e_{11}-e_{22}),\beta(e_{12}+e_{21}),\gamma(e_{12}-e_{21}))\\ &= \alpha^{d_1}\beta^{d_2}\gamma^{d_3}\varepsilon(e_{11}-e_{22})^{\varepsilon_1}(e_{12}+e_{21})^{\varepsilon_2}(e_{12}-e_{21})^{\varepsilon_3}.\end{aligned}$$

This shows that if f_σ and f_τ are two polynomials in the decomposition of the polynomial f above then $\varepsilon_\tau f_\sigma - \varepsilon_\sigma f_\tau$ is an identity in $M_2(K)$. This means that the irreducible and isomorphic modules in $P_n\cap B$ are "glued" together in Q_n, and that Q_n is a direct sum of non-isomorphic irreducible S_n–submodules.

Let $f \in K_3(X)$ be a polynomial that is not an identity in $M_2(K)$. If f generates the irreducible GL_3–submodule corresponding to the partition $(p+q+r,p+q,p)\vdash n$ then $\deg_{x_1} f = p+q+r$, $\deg_{x_2} f = p+q$, $\deg_{x_3} f = p$.

Claim 4. The polynomials $f_{pqr}(x_1,x_2,x_3) \in B(X)$ generate irreducible GL_3–modules that correspond to the partitions $(p+q+r,p+q,p)$ and these polynomials are not identities in $M_2(K)$ where:

a) If $q=0$ and $r\equiv 1 \pmod 2$,

$f_{p0r} = \sum(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}(\operatorname{ad} x_1)^r(\operatorname{ad} x_{\sigma(3)})s_3^{p-1}(\operatorname{ad} x_1,\operatorname{ad} x_2,\operatorname{ad} x_3)$.

b) If $q>0$ and $r\equiv 1 \pmod 2$,

$f_{pqr} = s_2(x_1,x_2)(\operatorname{ad} x_1)^r(\operatorname{ad} s_2(x_1,x_2))^{q-1}s_3^p(\operatorname{ad} x_1,\operatorname{ad} x_2,\operatorname{ad} x_3)$.

c) If $r \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$,

$$f_{pqr} = \sum(-1)^\sigma s_2(x_1,x_2)(\operatorname{ad} x_1)^r(\operatorname{ad} x_{\sigma(1)})(\operatorname{ad} s_2(x_1,x_2))^{q-3}(\operatorname{ad}[s_2(x_1,x_2), x_{\sigma(2)}])s_3^p(\operatorname{ad} x_1, \operatorname{ad} x_2, \operatorname{ad} x_3)$$

d) If $q = r = 0$, $p > 1$ (and $n > 3$),

$f_{p00} = \sum(-1)^\sigma x_{\sigma(1)}(s_3(\operatorname{ad} x_1, \operatorname{ad} x_2, \operatorname{ad} x_3))^{p-1} x_{\sigma(2)} x_{\sigma(3)}$.

e) If $r \equiv 0 \pmod 2$, $r > 0$, $q = 0$,

$f_{p0r} = \sum(-1)^\sigma x_{\sigma(1)}(\operatorname{ad} x_1)^r(s_3(\operatorname{ad} x_1, \operatorname{ad} x_2, \operatorname{ad} x_3))^{p-1} x_{\sigma(2)} x_{\sigma(3)}$.

f) If $q \equiv r \equiv 0 \pmod 2$, $q > 0$,

$f_{pqr} = f_{p,q-1,r}(x_1,x_2,x_3)s_2(x_1,x_2)$.

Here $\operatorname{ad} y$ is the linear operator in the vector space $K(X)$ defined as $x(\operatorname{ad} y) = [x,y]$.

In order to verify that the polynomials (a), ..., (f), are not identities in $M_2(K)$ one can choose the matrices $a = -\sqrt{-1}(e_{11} - e_{22})/2$, $b = \sqrt{-1}(e_{12} + e_{21})/2$, $c = (e_{12} - e_{21})/2 \in M_2(K)$. Using the relations $ab = -ba = c/2$, $bc = -cb = a/2$, $ca = -ac = b/2$ it is easy to obtain $[a,b,b] = -a$, $[c,b,b] = -c$, $as_3(\operatorname{ad} a, \operatorname{ad} b, \operatorname{ad} c) = -2a$. The reader could verify these relations. Hence $f_{pqr}(a,b,c) \neq 0$ in $M_2(K)$, in all cases (a), ..., (f).

Now, in order to complete the proof of the theorem it is sufficient to observe that only the polynomial $s_3(x_1,x_2,x_3)$ corresponds to $(1,1,1)$ but this polynomial is not proper (Why?).

Remark 4.40. Using Razmyslov's theorem as well as the description of the structure of Q_n as an S_n-module one can easily show that the T–ideal $T(M_2(K))$ is spechtian in case $\operatorname{char} K = 0$. The reader could find the proof in [7], or in the last section where we provide a hint.

5. Non-associative algebras. Other applications

In this last section we consider Lie and Jordan algebras with identities. It must be obvious to the reader that it is an impossible task to develop this theory in one section. Hence we are going to offer a short exposition; we hope that nevertheless the interested reader could continue the study, using the references, too.

Let $X = \{x_1, x_2, \ldots\}$ be a set and denote as $V(X)$ the set of all non-associative words in the alphabet X i.e., $V(X)$ consists of all words in X with all possible dispositions of the brackets. Define a multiplication in the K–space $F(X)$ having as a basis the elements (the monomials) of $V(X)$ as follows: $(\sum_i \alpha_i u_i)(\sum_j \beta_j v_j) = \sum_{i,j} \alpha_i \beta_j (u_i v_j)$. Here α_i, $\beta_j \in K$, u_i, $v_j \in V(X)$ and $(u_i v_j) = (u_i)(v_j)$ is the concatenation of the words u_i and v_j preserving the dispositions of the brackets in u_i and in v_j. Then the multiplication in $F(X)$ is distributive and $\alpha(uv) = (\alpha u)v = u(\alpha v)$, $1.u = u$ where u, $v \in V(X)$, $\alpha \in K$.

Definition 5.1. *The K-space A equipped with an operation called multiplication that obeys the laws above is called an algebra (linear, or non-associative) over K. The algebra $F(X)$ is called the absolutely free algebra. The elements of $V(X)$ are called monomials; those of $F(X)$ are called polynomials.*

Clearly "non-associative" in the last definition should read "not necessarily associative"—every associative algebra is an algebra according to this definition.

The notions of subalgebra, ideal, homomorphism, etc., are defined naturally, as in the case of associative algebras. The algebra $F(X)$ is free in the sense that every K–algebra A can be obtained as a homomorphic image of $F(X)$ for a suitable set X.

Exercise. If $\varphi: X \to A$ is a map prove that there exists a unique homomorphism $\Phi: F(X) \to A$ such that $\Phi|_X = \varphi$.

Definition 5.2. *If A is an algebra, the polynomial f is called a polynomial identity (as in the associative case, we shall abbreviate it PI) in A if $f \in \ker\varphi$ for every homomorphism $\varphi: K(X) \to A$. This means that $f(a_1, \ldots, a_n) = 0$ for every $a_i \in A$.*

Clearly the set $T(A)$ of all identities in A is an ideal in $F(X)$ that is closed with respect to the endomorphisms of $F(X)$. It is called the T–ideal of A. As in the associative case we define the variety $var\, A$ generated by A, the relatively free algebra in $var\, A$, consequences of an identity, etc.

In the same manner as for associative algebras it can be proved that if $|K| = \infty$ then every polynomial $f \in F(X)$ is equivalent as an identity to the collection of its homogeneous components and if $\operatorname{char} K = 0$, that f and $\operatorname{lin} f$ are equivalent.

We shall consider two of the most important classes of non-associative algebras, namely Lie and Jordan algebras. These two classes form varieties, as we shall prove soon.

Definition 5.3. *Suppose K is a field of characteristic not 2.*

a) Let L be a non-associative algebra with multiplication denoted as $[a, b] \in L$, a, $b \in L$. The algebra L is called Lie algebra if $[a, b] = -[b, a]$ and $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all a, b, $c \in L$.

b) Let J be a non-associative algebra with multiplication $a \circ b \in J$, a, $b \in J$. Then J is a Jordan algebra if $a \circ b = b \circ a$ and $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$ for all a, $b \in J$.

Remark 5.4. 1. The classes of Lie and Jordan algebras are defined by means of identities. Hence they are varieties. The free algebras in these varieties are the free Lie algebra $L(X)$ and the free Jordan algebra $J(X)$.

2. If $\operatorname{char} K = 2$ one substitutes in the definition of Lie algebras, the first identity for $[a, a] = 0$. In this case the definition of a Jordan algebra has to be modified, too. But this is much more complicated, see [22].

Example 5.5. 1. The real vector space $\mathbf{R}^3$ of dimension 3, with the usual vector product, is a Lie algebra.

2. The space $M_n(K)$ of the matrices $n \times n$ is a Lie algebra with respect to the multiplication $[a, b] = ab - ba$. The subspace $sl_n(K)$ of the traceless matrices

is a Lie subalgebra of $M_n(K)$.

Exercise. If A is an associative algebra prove that the space $A^- = A$ equipped with the multiplication $[a, b] = ab - ba$ becomes a Lie algebra. It is called the adjoint Lie algebra of A.

Remark 5.6. It is a consequence of the well-known theorem due to Poincaré, Birkhoff and Witt that every Lie algebra over a field is a subalgebra of the Lie algebra A^- for some suitable associative algebra A.

Example 5.7. The space $M_n(K)$ with the multiplication $a \circ b = (1/2)(ab + ba)$ (we suppose that $\operatorname{char} K \neq 2$) is a Jordan algebra. Its subspace $H_n(K)$ of the symmetric matrices is a Jordan subalgebra of $M_n(K)$.

Exercise. Let A be an associative algebra ($\operatorname{char} K \neq 2$). Prove that the space $A^+ = A$ equipped with the multiplication $a \circ b = (1/2)(ab + ba)$ becomes a Jordan algebra.

Remark 5.8. The connection between associative and Jordan algebras is not that close as in the Lie algebra case. The Jordan algebras of type A^+ and their subalgebras are called special; otherwise they are exceptional. An example due to P. Cohn ([3]) shows that there exist special Jordan algebras having exceptional homomorphic images. That is, the special Jordan algebras do not form a variety.

Example 5.9. Let $\operatorname{char} K \neq 2$ and let V be a K-space equipped with a bilinear and symmetric form $(\ ,\ )$. The multiplication $(\alpha + u) \circ (\beta + v) = (\alpha\beta + (u, v)) + (\alpha v + \beta u)$, α, $\beta \in K$, u, $v \in V$, defines on $G = K \oplus V$ a structure of Jordan algebra. It is the algebra of the form $(\ ,\ )$. The algebra G is special and if the form is non-degenerate it is simple. If C is the Clifford algebra of the space V then $G \subset C^+$. See for example, [21], or [22], or [48] for the precise definitions and the basic properties of G and C.

Remark 5.10. Consider the Lie algebra $K(X)^-$; the subalgebra generated by X is isomorphic to the free Lie algebra $L(X)^-$. The same construction applied to $K(X)^+$ yields the free special Jordan algebra $SJ(X)$.

Every special Jordan algebra is homomorphic image of $SJ(X)$. If $|X| \geq 3$ then there exist homomorphic images of $SJ(X)$ that are exceptional ([3]).

The theory of Lie and Jordan algebras with identities has been developed to an adequate level. The reader could find information about the "state of art" in the monographs [1], [22], [36], [48].

We shall restrict our attention to the identities in $sl_2(K)$ and in G (as in Example 5.9) and, for a while, only to the case char $K = 0$. In 1974, Yu. Razmyslov (see, for example, [36]) obtained that the identities in $sl_2(K)$ admit a finite basis if char $K = 0$. Furthermore he established the Specht property for them.

Definition 5.11. *Denote $\mathcal{N} = var\, sl_2(K)$, and let $F(\mathcal{N}) = L(X)/T(\mathcal{N})$ be the free algebra in $\mathcal{N}$.*

As in the case of the identities in $M_2(K)$ we describe the structure of $F(\mathcal{N})$ (see the previous section). We have already shown that $L(X) \subseteq K(X)$. Denote as L_n the intersection $L(X) \cap P_n$. Clearly L_n is an S_n–submodule of P_n; it is generated by the multilinear Lie polynomials.

Exercise. Prove that $\dim L_n = (n-1)!$ (Hint: First prove that every "composite" commutator $[u_1, u_2, \ldots]$ where u_i are commutators in X can be presented as a linear combination of commutators $[x_{i_1}, x_{i_2}, \ldots]$. Then $\{[x_1, x_{i_2}, \ldots, x_{i_n}] \mid \{i_2, \ldots, i_n\} = \{2, \ldots, n\}\}$ form a basis of L_n.)

Lemma 5.12. *If $Q_n^* = L_n/(L_n \cap T(\mathcal{N}))$ then the S_n-module Q_n^* is semisimple. The irreducible submodules in Q_n^* correspond to Young diagrams having 2 or 3 rows.*

Proof: The proof is the same as in the case Q_n, see the previous section. The polynomial x^n does not belong to $B(X)$, and hence $\lambda = (n)$ can participate neither in the decomposition of Q_n, nor of Q_n^*.

Corollary 5.13. *If $0 \neq f \in F(\mathcal{N})$ then f is equivalent to a finite set of polynomials in ≤ 3 variables.*

Theorem 5.14. *([7]) If $n > 1$ then the S_n-module Q_n^* can be decomposed as a direct sum of irreducible non-isomorphic submodules that correspond to the partitions $\lambda = (p+q+r, p+q, p) \vdash n$ with $p+q > 0$, and $q \equiv 1 \pmod 2$, or $r \equiv 1 \pmod 2$.*

Proof: The proof is a "subset" of that in the case $M_2(K)$ and Q_n. The only fact that has to be mentioned is that when $q \equiv 1 \pmod 2$ or $r \equiv 1 \pmod 2$ the polynomial f_{pqr} belongs to L_n.

Remark 5.15. The theorems describing the structure of Q_n and of Q_n^* show that there cannot exist infinite strictly descending chains of subvarieties of $\mathcal{M}$ and in $\mathcal{N}$, see the next exercise.

Exercise. Let $0 \neq f_{pqr}(x_1, x_2, x_3) \in F(\mathcal{N})$ be a standard generator of the irreducible GL_3–module corresponding to $\lambda = (p+q+r, p+q, p) \vdash n$.

a) Prove that $h'(x_1, x_2, x_3) = f_{pqr}(x_1, x_2, x_3)(s_3(\operatorname{ad} x_1, \operatorname{ad} x_2, \operatorname{ad} x_3))^2 \neq 0$ in $F(\mathcal{N})$, and that h' is equivalent as an identity to $f_{p+2,q,r}$. In other words, the identity f_{pqr} implies $f_{p+2,q,r}$ in $F(\mathcal{N})$.

b) Prove that $h''(x_1, x_2, x_3) = f_{pqr}(x_1, x_2, x_3) \sum (-1)^\sigma (-1)^\tau (\operatorname{ad} x_{\sigma(1)})(\operatorname{ad} x_{\tau(1)}) (\operatorname{ad} x_{\sigma(2)})(\operatorname{ad} x_{\tau(2)}) \neq 0$ in $F(\mathcal{N})$ and it is equivalent to $f_{p,q+2,r}$.

c) Let h_1, h_2, h_3 be the linear in u_1, u_2, u_3 components of the polynomials $f_{pqr}(x_1 + u_1, x_2, x_3)$, $f_{pqr}(x_1, x_2 + u_2, x_3)$, $f_{pqr}(x_1, x_2, x_3 + u_3)$, respectively, and set $h(x_1, x_2, x_3, u_1, u_2, u_3) = h_1 + h_2 + h_3$.

Prove that $h'''(x_1, x_2, x_3) = h(x_1, x_2, x_3, -[x_1, x_1, x_1], -[x_2, x_1, x_1], -[x_3, x_1, x_1]) \neq 0$ in $F(\mathcal{N})$ and it is equivalent to $f_{p,q,r+2}$.

d) If g_1, $g_2, \ldots \in F(\mathcal{N})$, $g_i \notin \langle T(\mathcal{N}), g_1, \ldots, g_{i-1} \rangle^T$, $i = 1, 2, \ldots$ generate irreducible GL_3–modules use Higman's theorem and a), b), c), of this exercise in order to obtain a contradiction to the existence of such a chain.

It is worth mentioning that in [18] it was proved that when L is a Lie algebra of finite dimension over K, $\operatorname{char} K = 0$, then the T–ideal of L is spechtian. The methods used in this paper are similar to those of [24] in the case of associative algebras.

The structure of the identities satisfied by the Jordan algebra G is similar to that of the identities in $M_2(K)$. We are going to discuss only the most important points in its description without proofs (these are technically rather complicated, and they require theories outside the scope of the course). The reader could look at the papers cited in the references.

In [44] it was obtained that the T–ideal of G is generated by one identity, and when $\dim V = k < \infty$ we have to add one more identity. In [9] a description of the relatively free algebra $F(var\, G)$ in $var\, G$ as a GL_m–module was provided. As we already mentioned the algebra G is special. Using this it can be proved that $F(var\, G)$ is special, too. (In fact this variety consists of special algebras, see [41].)

Denote $SJ_m = K_m(X) \cap SJ(X)$ and $PJ_n = P_n \cap SJ(X)$; therefore SJ_m and PJ_n are GL_m– and S_n–modules, resp. In [9] it was proved that $BJ_m(G) = (B_m \cap SJ(X))/(T(G) \cap (B_m \cap SJ(X))) \cong \oplus_\lambda N_m(\lambda)$ where $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, $k \leq m$, $\lambda_2 \neq 0$, and at most one of λ_i is odd. In [26] it was proved that there do not exist infinite and strictly decreasing chains of subvarieties of $var\, G$. The complete description of the subvarieties in $var\, G$ can be found in [10]. The proof of the latter fact uses such a criterion. If f_λ and f_μ generate irreducible GL_m submodules of $BJ_m(G)$ then f_μ is a consequence of f_λ if and only if $[\lambda] \subseteq [\mu]$. This criterion reduces the problem to Higman's theorem on PWO sets.

Remark 5.16. In [42] it was obtained that every finitely generated Jordan algebra over a field of characteristic 0 is spechtian. Let us mention that the algebra G is not finitely generated if $\dim V = \infty$.

Finally we consider aspects of the PI theory when the characteristic of the field is positive, $\operatorname{char} K = p > 0$. This theory is being developed accompanied by difficulties of various type. It seems to us that the principal of them are due to the fact that the multilinear identities in general cannot determine the T–ideals. Another "stumbling block" is that it is not possible, in general, to decompose the GL_m– and the S_n–modules (i.e., Maschke's theorem does not hold); some of the modules that are irreducible in characteristic 0 could turn out reducible (but not semisimple) in positive characteristic.

Therefore the representations of GL_m and of S_n are to be substituted for other methods. Some of them include the theory of invariants and the weak identities.

The weak identities were introduced by Yu. P. Razmyslov (see [36]) in his study of the identities in $M_2(K)$ and in $sl_2(K)$. Let A be an associative algebra and let V be a subspace of A such that $alg(V) = A$ i.e., V generates A as an algebra.

Definition 5.17. *The polynomial $0 \neq f(x_1, \ldots, x_n) \in K(X)$ is a weak (polynomial) identity, abbreviated WPI in the pair (A, V) if $f(v_1, \ldots, v_n) = 0$ for all $v_i \in V$.*

Depending on the properties of A and V one can define various rules for consequences of a WPI. Clearly the set $T(A, V)$ of all WPI in (A, V) is an ideal in $K(X)$ that is closed with respect to linear substitutions of the variables.

Definition 5.18. *Let $\emptyset \neq \Omega \subseteq K(X)$. The polynomial $g \in K(X)$ is called Ω-consequence of $f \in K(X)$ if $g \in \langle f \rangle^{\Omega}$ where $\langle f \rangle^{\Omega}$ is the ideal in $K(X)$ generated by $\{f(\omega_1, \ldots, \omega_n) \mid \omega_i \in \Omega\}$.*

Example 5.19. a) When $\Omega = K(X)$, and $A = V$, one obtains the polynomial identities in A together with the usual rules for consequences.

b) If $\Omega = L(X)$, and if V is a Lie subalgebra of A^-, $A = alg(V)$ then the weak Lie identities are obtained.

c) Let $\Omega = SJ(X)$, V be a Jordan subalgebra of A^+ and let $A = alg(V)$. Then we obtain the weak Jordan identities.

d) When $\Omega = \ell(X)$ is the vector space spanned by X, and $A = alg(V)$, we obtain the "weakest" identities, the so-called GL-identities.

Remark 5.20. In the examples b) and c), the consequences of the polynomial $f \in K\langle X\rangle$ are obtained by means of substitutions $f(\omega_1, \ldots, \omega_n)$ where $\omega_i \in L(X)$, resp. $\omega_i \in SJ(X)$. In case d) we have that $\omega_i = \sum_j \alpha_{ij}x_j$, $\alpha_{ij} \in K$.

The description of the WPI in $(M_2(K), sl_2(K))$, and in (C, G) and (C, V) was one of the important steps in obtaining the Specht property for $var\, sl_2(K)$ and $var\, G$ when $\operatorname{char} K = 0$. In [43] and [44], using weak identities, finite bases of the identities in $sl_2(K)$ and in G were found when $|K| = \infty$ and $\operatorname{char} K \neq 2$. The WPI in $(M_2(K), sl_2(K))$ are finitely based, too [27], and the same holds for (C, V) [28].

Concerning the invariants of the classical groups we would like to mention the articles [4] where this theory was developed without depending on the characteristic of the field, and [27], [28], [43], [44] where one could find applications. Using invariants of the symplectic group as well as Higman's theorem it was proved in [29] that the multilinear identities for $(M_2(K), sl_2(K))$ are Spechtian when $\operatorname{char} K = 2$, and that the Specht property fails for all weak identities in $(M_2(K), sl_2(K))$.

Identities in other classes of algebras also have been objects of profound studies, see, e.g. [48]. Of course the main interest is attracted by the identities in associative algebras. But nevertheless we would like to mention some of the most important results about non-associative algebras satisfying identities.

In 1970, M. R. Vaughan-Lee ([45]) showed that the Lie algebra $M_2(K)^-$ where $|K| = \infty$ and $\operatorname{char} K = 2$, is not spechtian. It is still an open problem whether the identities in $M_2(K)$ under the same restrictions are finitely based.

On the other hand there are descriptions of the identities of minimal degrees satisfied by some important classes of non-associative algebras, see, for example, [14], [35]. Such descriptions are very important due to their applications, see [44].

Considering identities in associative algebras, we only mention a result due to A. Kemer ([25]). It states that every associative algebra A over a field of characteristic $p > 0$ satisfies the identity $\mathrm{sym}_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)}$ for some n, and the identity $s_m(x_1, \ldots, x_m)$ for some m. Recently it was announced by A. Grishin that there exists non-spechtian variety of associative algebras over a field of characteristic 2.

At this point we would like to put the final stop. We hope that the notes will be useful and that they can serve as a base (**only**) for future and more profound studies on algebras satisfying polynomial identities.

References

[1] Bakhturin, Y. A., *Identities in Lie algebras*, VNU Science Press, Utrecht, 1987.

[2] Berele, A., *Homogeneous polynomial identities*, Israel J. Math. 42, (1982), 258–272.

[3] Cohn, P., *On homomorphic images of special Jordan algebras*, Canadian J. Math. 6, (1954), 253–264.

[4] Concini, C. de and Procesi, C., *A characteristic free approach to invariant theory*, Adv. Math. 21, (1976), 330–354.

[5] Curtis, C. and Reiner, I., *The representation theory of finite groups and associative algebras*, Wiley–Interscience, 1962.

[6] Dehn, M., *Über die Grundlagen der projektiven Geometrie und algemeine Zahlsysteme*, Math. Annalen 85, (1922), 184–194.

[**7**] Drensky, V., *Representations of the symmetric group and varieties of linear algebras*, Math. USSR Sbornik 43, (1981), 85–101.

[**8**] Drensky, V., *A minimal basis for the identities of a second order matrix algebra over a field of characteristic 0*, Algebra and Logic 20, (1981), 188–194.

[**9**] Drensky, V., *Polynomial identities for the Jordan algebra of a symmetric bilinear form*, J. Algebra 108, (1987), 66–87.

[**10**] Drensky, V. and Koshlukov, P., *Polynomial identities for Jordan algebras of degree two*, J. Indian Math. Soc. 55, (1990), 1–30.

[**11**] Dubnov, J. and Ivanov, V., *Sur l'abaissement du degré des polynômes affineurs*, C. R. Acad. Sci. USSR 41, (1943), 95–98.

[**12**] Formanek, E., *The polynomial identities and invariants of $n \times n$ matrices*, CBMS Regional Conf. Series Math. 78, AMS, Providence, RI, 1991.

[**13**] Green, J. A., *Polynomial representations of GL_m*, Lect. Notes Math. 830, Springer, 1980.

[**14**] Hentzel, I. R. and Peresi, L. A., *Identities of Cayley-Dickson algebras*, J. Algebra 188, (1997), 292–309.

[**15**] Herstein, I., *Noncommutative rings*, Carus Math. Monographs 15, MAA, Washington, DC, 1968.

[**16**] Higman, G., *Ordering by divisibility in abstract algebras*, Proc. London Math. Soc. 2, (1952), 326–336.

[**17**] Higman, G., *On a conjecture of Nagata*, Math. Proc. Cambridge Phil. Soc. 52, (1956), 1–4.

[**18**] Il'tyakov, A. V., *On finite basis of identities of Lie algebra representations*, Nova J. Algebra Geom. 1, No. 3, (1992), 207–259.

[19] Jacobson, N., *Structure of rings*, AMS Colloq. Publ. 37, AMS, 1964.

[20] Jacobson, N., *PI algebras, Ring theory*, Proceedings Oklahoma Conf., Marcel Dekker 3, (1974), 1–30.

[21] Jacobson, N., *Lie algebras*, Tracts in Pure and Appl. Math. 10, Wiley–Interscience, 1979.

[22] Jacobson, N., *Structure and representations of Jordan algebras*, AMS Colloq. Publ. 39, AMS, 1968.

[23] James, G. and Kerber, A., *The representation theory of the symmetric group*, Encyclopaedia Math. and its Applications 16, Addison–Wesley, 1981.

[24] Kemer, A. R., *Ideals of identities in associative algebras*, Transl. Math. Monographs 87, AMS, 1991.

[25] Kemer, A. R., *The standard identity in characteristic p. A conjecture of I. B. Volichenko*, Israel J. Math. 81, (1993), 343–355.

[26] Koshlukov, P., *Polynomial identities for a family of simple Jordan algebras*, Comm. Algebra 16, (1988), 1325–1371.

[27] Koshlukov, P., *Weak polynomial identities for the matrix algebra of order two*, J. Algebra 188, (1997), 610–625.

[28] Koshlukov, P., *Finitely based ideals of weak polynomial identities*, Commun. Algebra 26, No. 10(1998), 3335–3359.

[29] Koshlukov, P., *Ideals of identities of representations of nilpotent Lie algebras*, (1998), submitted.

[30] Latyshev, V., *On Regev's theorem on identities in a tensor product of PI algebras*, Uspekhi Math. Nauk 27, (1972), 213–214 (in Russian).

[31] Latyshev, V., *Partially ordered sets and nonmatrix identities of associative algebras*, Algebra and Logic 15, (1976), 34–44.

[32] Nagata, M., *On the nilpotency of nil algebras*, J. Math. Soc. Japan 4, (1952), 296–301.

[33] Peirce, R., *Associative algebras*, Grad. Texts Math. 88, Springer, 1982.

[34] Procesi, C., *Rings with polynomial identities*, Pure Appl. Math. 17, M. Dekker, New York, 1973.

[35] Racine, M., *Minimal identities of octonion algebras*, J. Algebra 115, (1989), 251–260.

[36] Razmyslov, Y. P., *Identities of algebras and their representations*, Transl. Math. Monographs 138, AMS, 1994.

[37] Regev, A., *Existence of identities in $A \otimes B$*, Israel J. Math. 11, (1972), 131–152.

[38] Regev, A., *The representations of S_n and explicit identities for PI algebras*, J. Algebra 51, (1978), 25–40.

[39] Rosset, S., *A short proof of the Amitsur-Levitzki theorem*, Israel J. Math 23, (1976), 187–188.

[40] Rowen, L. H., *Polynomial identities in Ring theory*, Academic Press, NY, 1979.

[41] Sverchkov, S., *Varieties of special algebras*, Comm. Algebra 16, (1988), 1877–1919.

[42] Vais, A. and Zelmanov, E., *Kemer's theorem for finitely generated Jordan algebras*, Izvestiya VUZ 6, (1989), 42-51 (in Russian). English transl.: Soviet Math. 33, No. 6 (1989), 38-47.

[43] Vasilovskii, S., *Basis of identities of a three-dimensional simple Lie algebra over an infinite field*, Algebra and Logic 28, No. 5 (1989), 355–368.

[44] Vasilovskii, S., *A finite basis for polynomial identities of the Jordan algebras of bilinear form*, Siberian Adv. Math. 1, No. 4(1991), 1–43.

[45] Vaughan-Lee, M. R., *Varieties of Lie algebras*, Quart. J. Math. Oxford 21 (2) (1970), 297–308.

[46] Vonessen, N., *Rings with polynomial identities: an elementary introduction*, Mat. Contemp. 7, (1994), 199–231.

[47] Wagner, W., *Über die Grundlagen der projektiven Geometrie und algemeine Zahlensysteme*, Math. Annalen 113, (1937), 528–564.

[48] Zhevlakov, K. A., Slin'ko, A. M., Shestakov, I. P. and Shirshov, A. I., *Rings that are nearly associative*, Pure Appl. Math. 104, Academic Press, 1982.

IMECC—UNICAMP
Cx. P. 6065
13083-970, Campinas, SP
E-mail plamen@ime.unicamp.br