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THE WEDDERBURN PRINCIPAL THEOREM, NILPOTENCY AND SOLVABILITY FOR FREUDENTHAL-KANTOR TRIPLE SYSTEMS

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Abstract

In this paper we deal with the Wedderburn principal theorem for Freudenthal-Kantor triple systems and investigate the relationship between nilpotency and solvability.

Introduction

The history of subject in this article started from Wedderburn for an associative algebra and from Levi for a Lie algebra. Since that time, several mathematicians have studied this subject for classes of nonassociative algebras.

We are interested to know how the Wedderburn principal theorem behaves in a Freudenthal-Kantor triple system $U(\varepsilon)$. Thus our purpose is to consider that theorem.

Using the result, we can show that the following are equivalent;

(i) $U(\varepsilon)$ is semisimple (that is, the radical of $U(\varepsilon)$ is zero)

(ii) $U(\varepsilon)$ is the direct sum of its simple ideals.

The second purpose of this paper is to establish the definition of nilpotency and solvability for triple systems and to discuss their relationships.

We shall be concerned with algebras and triple systems which are finite dimensional over a field Φ of characteristic 0. We shall mainly employ the notation and terminology in ([1],[2]).

1. Preliminaries

For $\varepsilon = \pm 1$, a triple system $U(\varepsilon)$ with triple product $< -, -, - >$ is called a *Freudenthal-Kantor triple system* if

$$< ab < cde >> = << abc > de > + \varepsilon < c < bad > e > + < cd < abe >>$$

$$K(< abc >, d) + K(c, < abd >) + K(a, K(c,d)b) = 0$$

where $K(a,b)c = < acb > - < bca >$.

Let $U(\varepsilon)$ be a Freudenthal-Kantor triple system. Then we can define a Lie triple system

$$T(\varepsilon) = U(\varepsilon) \oplus \overline{U(\varepsilon)}$$

($\overline{U(\varepsilon)}$ is a copy of $U(\varepsilon)$) with respect to the triple product defined by

$$\left[\begin{pmatrix} a. \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}\right]] =$$

$$\begin{pmatrix} L(a,d) - L(c,b) & K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

where, $L(a,b)c = < abc >$.

Then we can obtain the standard embedding Lie algebra associated with $T(\varepsilon)$. We denote it by $L(\varepsilon)$, that is,

$$L(\varepsilon) = D(T(\varepsilon), T(\varepsilon)) \oplus T(\varepsilon),$$

where $D(T(\varepsilon), T(\varepsilon))$ is the inner derivation algebra of $T(\varepsilon)$, (for details, to see [1],[2]).

Remark. All simple Lie algebras over the complex numbers are obtained by this construction. That is, from simple Freudenthal-Kantor triple systems, we can obtain simple Lie triple systems(symmetric space) and so Lie algebras (for example, to see [3],[4]).

Proposition 1 ([1]). *For a Freudenthal-Kantor triple system $U(\varepsilon)$, the Lie triple system $T(\varepsilon)$ associated with $U(\varepsilon)$, and the standard embedding Lie algebra $L(\varepsilon)$, we have*

(a) $R(T(\varepsilon)) = T(R(U(\varepsilon))$
(b) $R(T(\varepsilon)) = R(L(\varepsilon)) \cap T(\varepsilon)$,
(c) $R(L(\varepsilon)) = D(T(\varepsilon), R(T(\varepsilon)) \oplus R(T(\varepsilon))$,

where $R(T(\varepsilon))$ is the radical of $T(\varepsilon)$, $R(L(\varepsilon))$ is the radical of $L(\varepsilon)$ and $T(R(U(\varepsilon))$ is the Lie triple system associated with $R(U(\varepsilon))$. These radicals are solvable radicals.

Definition. *A pair (u, v) of elements of a Freudenthal-Kantor triple system $U(\varepsilon)$ is called left neutral if*

$$L(u, v) = Id.$$

Lemma 2. *Let $U(\varepsilon)$ be a Freudenthal-Kantor triple system with a left neutral pair (u, v) and let $L(\varepsilon)$ be the standard embedding Lie algebra. If A is any ideal of $L(\varepsilon)$, then A is decomposed into*

$$A \cap L_2(\varepsilon) \oplus A \cap L_1(\varepsilon) \oplus A \cap L_0(\varepsilon) \oplus A \cap L_1(\varepsilon) \oplus A \cap L_2(\varepsilon).$$

Furthermore, $A \cap L_1(\varepsilon)$ is an ideal of $U(\varepsilon)$, where $L_i(\varepsilon)$ is the eigen space for
$H := \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}$ corresponding to the eigenvalue i in $L(\varepsilon)(-2 \leq i \leq 2)$.
In fact, we get

$L_{-2}(\varepsilon)$ =the linear span of all $\begin{pmatrix} 0 & K(c, d) \\ 0 & 0 \end{pmatrix}$,
$L_{-1}(\varepsilon) = U(\varepsilon) \oplus (0)$,
$L_0(\varepsilon)$ = the linear span of all $\begin{pmatrix} L(a, b) & 0 \\ 0 & \varepsilon L(b, a) \end{pmatrix}$,
$L_1(\varepsilon) = (0) \oplus U(\varepsilon)$,

$L_2(\varepsilon)$=the linear span of all $\begin{pmatrix} 0 & 0 \\ -\varepsilon K(e,f) & 0 \end{pmatrix}$,

and $[H, L_i] = iL_i(-2 \leq i \leq 2)$.

2. Wedderburn principal theorem

Theorem 3 ([6]). *Let $U(\varepsilon)$ be a Freudenthal-Kantor triple system with a left neutral pair (u, v) and $R(U(\varepsilon))$ be the radical of $U(\varepsilon)$. Then there exists a triple subsystem S of $U(\varepsilon)$ such that $U(\varepsilon) = S \oplus R(U(\varepsilon))$ and $S \cong U(\varepsilon)/R(U(\varepsilon))$.*

Remark. If $\varepsilon = -1$ and $K(x, y) \equiv 0$ (identically zero), $U(\varepsilon)$ is said to be a Jordan triple system. Thus, the Jordan triple system is a special case of our triple systems.

Corollary. *Let U be a Jordan triple system with a left neutral pair. Then we have $U = S \oplus R(U)$.*

Theorem 4 ([1]). *Let $U(\varepsilon)$ be a Freudenthal-Kantor triple system with a left neutral pair. Then the following are equivalent;*

(1) $U(\varepsilon)$ is semisimple, (that is, $R(U(\varepsilon)) = 0$

(2) $U(\varepsilon)$ is the direct sum of its simple ideals.

3. Nilpotent, Solvable ideals for triple systems

Now we shall define the nil radical for Freudenthal-Kantor triple system.

Let A_1, A_2 be ideals of $U(\varepsilon)$. We put $A_3 = U(\varepsilon)$ and define

$$A_1 * A_2 := \sum_{\phi \in S_3} < A_{\phi(1)} A_{\phi(2)} A_{\phi(3)} >,$$

where S_3 is a symmetric group of degree 3.

Lemma 5. *If $U(\varepsilon)$ is a Freudenthal-Kantor triple system, and A_1, A_2 are ideals of $U(\varepsilon)$, then $A_1 * A_2$ is an idal of $U(\varepsilon)$.*

For every ideal A of $U(\varepsilon)$, we can define the subtriple systems $A^n(n \geq 0), A^{<n>}(n \geq 0)$ by

$$A^0 = A^{<0>} = A,$$

$$A^n := A^{n-1} * A, A^0 = A(n \geq 1),$$

$$A^{<n>} := A^{<n-1>} * A^{<n-1>}(n \geq 1).$$

That is,

$$A^1 = A^{<1>} =< AUA > + < AAU > + < UAA >$$

$$A^2 = < AUA^1 > + < AA^1U > + < A^1AU > + < A^1UA > + < UAA^1 > < UA^1A >,$$

$$A^{<2>} =< A^{<1>}UA^{<1>} > + < A^{<1>}A^{<1>}U > + < UA^{<1>}A^{<1>} > .$$

Remark. In any triple system, it seems that the definition of nilpotency and solvability of triple systems may be defined by the above. These are a slight modification in [5],[7].

Proposition 6. *Let A be any ideal of $U(\varepsilon)$. Then the subtriple system $A^n, A^{<n>}$ are ideals of $U(\varepsilon)$ for every integer n.*

An ideal A of $U(\varepsilon)$ is called nilpotent if there exists a positive integer n such that $A^n = (0)$. Similarly, A is called solvable if $A^{<n>} = 0$.

From [1] and [5], we recall the definitions of nilpotent radical and solvable radical. Since a Freudenthal-Kantor triple system is a finite dimensional vector space in this article, this implies that there is a unique maximal nilpotent ideal (resp.solvable idal), called the nil radical (resp.the solvable radical) which contains all other nilpotent ideals (resp.solvable ideals) of $U(\varepsilon)$. We denote it by $N_R(U(\varepsilon))$ (resp.$R(U(\varepsilon))$).

Theorem 7 ([5],[7]). *For a Freudenthal-Kantor triple system $U(\varepsilon)$ and the*

Lie triple system $T(\varepsilon)$ *associated with* $U(\varepsilon)$, *we have*

$$N_R(T(U(\varepsilon))) = T(N_R(U(\varepsilon))),$$

$$R(T(\varepsilon)) = T(R(U(\varepsilon))),$$

where $N_R(T(U(\varepsilon)))$ *is the nil radical of* $T(\varepsilon)$, $T(N_R(U(\varepsilon)))$ *is the Lie triple system* $T(\varepsilon)$, *and* $T(R(U(\varepsilon)))$ *is the Lie triple system associated with* $N_R(U(\varepsilon))$, $R(T(\varepsilon))$ *is the solvable radical of the Lie triple system associated with the solvable radical* $R(U(\varepsilon))$.

Theorem 8. **([5])** *For a Freudenthal-Kantor triple system* $U(\varepsilon)$, *we have*

$$R(U(\varepsilon))^1 \subset N_R(U(\varepsilon)).$$

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