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# A NEW LOOK AT THE FEIT-THOMPSON ODD ORDER THEOREM

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*Dedicated to the memory of Professor Michio Suzuki (1926-1998)*

**Abstract**

The Feit-Thompson Theorem states that
Every finite group of odd order is solvable.

It was proved in 1963 after being conjectured more than 50 years earlier. It has many applications to classifying simple groups and to studying solvable groups.

In recent years, the entire proof has been revised and partially simplified by several authors in published and unpublished work. In particular, Feit and Thompson contributed an improvement in 1991. In this article, we discuss the background of the theorem, some ideas in the proof, recent revisions and simplifications, and a recent extension announced by Michio Suzuki.

## 1. Introduction

This article is devoted to an elementary discussion of the following

**Theorem (Walter Feit - John G. Thompson, 1963 [FT]).** *Every finite group of odd order is solvable.*

This result had been conjectured more than 50 years earlier. Its proof occupied an entire issue of the Pacific Journal of Mathematics, 255 pages long. For this work, the authors received the Cole Prize in Algebra of the American Mathematical Society in 1965. Both the theorem and its proof have many applications to general finite group theory and to classifying simple groups.

In recent years, the entire proof has been revised and partially simplified by several authors in published and unpublished work. In particular, Feit and

Thompson discovered an improvement in 1991. Thus, the theorem is attracting attention anew.

Although the proof is very long and complicated, we will discuss some aspects of the theorem that can be described in an elementary way. These aspects concern the historical background of the theorem, some ideas of the proof, and recent simplifications and revisions.

Our description of the proof is partially drawn from an account by Thompson [T1] and two accounts by Gorenstein ([Gor1], pp.450-461; [Gor2], pp.13-39). Since this article is based on talks to a wide audience, including graduate students, technical details are largely omitted. The reader familiar with finite group theory is encouraged to see these three accounts.

We would like to thank very warmly Said Sidki, Miguel Ferrero, the Fundação Universidade de Brasília, and the Escola de Álgebra for inviting us to Brazil and making the lectures and a very pleasant visit possible. We also thank Ronald Solomon for several improvements to this article.

Much in this article concerns the research of Michio Suzuki. Sadly, Professor Suzuki died about two months before the lectures were given. Over the years, he had given me much help and encouragement. Therefore, this article is dedicated to his memory, with deep gratitude and respect.

In this first part, we discuss the historical background of the theorem and some crucial properties of groups of odd order.

For convenience, all groups discussed in this article will be *finite*. We abbreviate the Feit-Thompson Theorem by FT.

## 1.1. Historical Background

A number of special cases of FT were proved about 1901 ([Bu2], p.503). For example, Burnside proved ([Bu1]) the case in which $|G|$ (the order of G) is less than 40000. In 1911, in the second edition of his book, he raised the question of whether FT is true ([Bu2], p.503). For this reason, it became known as "Burnside's Conjecture", although it may have been conjectured earlier.

The period from about 1890 to 1915 was one of intense activity in finite group theory relative to the total amount of activity in mathematics. This was followed by a comparatively less intense (but quite fruitful) period, which lasted until about 1955, and then a more intense period that has continued up to the present time. Paradoxically, one article in 1955 that may have stimulated interest in proving FT concerned groups of *even* order [BF]. In it, Richard Brauer and Kenneth Fowler used short, elementary proofs to obtain some remarkably strong results.

At this point, to understand the theorem, it may help to imagine temporarily that we are living in 1955 and that we are trying to prove FT. We use induction on $|G|$. Clearly, we may assume that $G$ is not abelian. If $G$ has a proper non-identity normal subgroup $N$, then, by induction, $N$, $G/N$, and hence $G$ are solvable. Thus, we may assume that

**$G$ is a non-abelian simple group**

Now we would like to study the proper subgroups of G. However, we know nothing about them except that they are solvable of odd order. (In fact, we really know that they do not exist.) Let us turn for guidance to a family of groups that really exist, namely, the non-abelian simple groups of even order. The analogue to $G$ is the smallest of these groups, the alternating group $A_5$ of degree 5 and order 60. Let us denote by $H^\#$ the set of all non-identity elements of a group $H$. It is easy to see that, for every $x \in {A_5}^\#$, the centralizer $C_{A_5}(x) = \{y \in A_5 | yx = xy\}$ is an *abelian* subgroup of $A_5$.

This condition says that $A_5$ is a (CA)-*group*. Thus, we might hope that our group G is a (CA)-group.

The (CA)-groups were being extensively studied about this time. An article by Brauer,Suzuki,and G.E.Wall [BSW] in 1958 classified the non-abelian simple (CA)-groups of even order. In fact, I have been told that the result was obtained independently by some or all of the three authors as early as 1953 or 1954. However, our hope that $G$ might be a (CA)-group was dashed by Suzuki in the following result:

**Theorem (Suzuki, 1957 [Sz1]).** *Every (CA)-group of odd order is solvable.*

Thus, $A_5$ is not a suitable model for our group $G$. Let us turn instead to the second smallest nonabelian simple group of even order. This is the simple group of order 168, which is unique up to isomorphism. In fact, it is easy to see that it is *not* a (CA)-group, but satisfies a similar, weaker condition, namely, that the centralizer of any of its non-identity elements is *nilpotent*, i.e., is the direct product of its Sylow subgroups. Such a group is called a (CN)-*group*.

The non-abelian simple (CN)-groups of even order were classified by Suzuki ([Sz2]) in 1961. However, the hope that our odd order group $G$ might be a (CN)-group was frustrated one year earlier:

**Theorem (Feit - Marshall Hall - Thompson, 1960 [FHT]).** *Every (CN)-group of odd order is solvable.*

At this point, we notice a paradox. A group of even order contains subgroups of odd order, but a group of odd order cannot contain a subgroup of even order. Thus, in general, groups of odd order should be less complicated than groups of even order. However, for (CA)-groups, the Brauer-Suzuki-Wall Theorem was proved a few years earlier than Suzuki's Theorem. Similarly, the two theorems on (CN)-groups were proved at about the same time. The reason was that, although groups of even order are more complicated, one has the advantage of the Brauer-Fowler techniques when one studies them. It is my conjecture that this stimulated the desire for a similar "advantage" for groups of odd order and increased interest in Burnside's Conjecture.

FT was proved in 1963. It led to two types of consequences:

1) There were several articles that classified families of "small" simple groups by quoting FT and using its proof as a model.

2) There were several new results in FT about arbitrary finite groups that were developed into further results.

## 1.2. Special Properties of Odd Order Groups

What accounts for the vast difference between groups of odd order and groups of even order? The former must be solvable, while the latter can be extremely complicated, e.g., they can be nonabelian simple groups properly containing other nonabelian simple groups. We cannot really explain this, but we can describe some small, but significant, special properties of groups of odd order.

**Proposition 1.** *Suppose $G$ is a group and $|G| = p^2q$ for odd primes $p < q$. Then $G$ has a normal Sylow $q$-subgroup.*

**Counterexample for even order:** $G = A_4$, of order 12.

**Proof:** By Sylow's Theorem, the number of Sylow $q$-subgroups of $G$ has the form $1 + kq$ for some integer $k$, and is a divisor of $|G|$. Since $|G| = p^2q$, and $1 + kq$ is relatively prime to $q$, $1 + kq$ divides $p^2$. Hence, $1 + kq = 1$,$p$, or $p^2$.

If $1 + kq = 1$, then $G$ has a normal Sylow $q$-subgroup, as desired. An easy exercise eliminates the possibility that $1 + kq = p$ or $p^2$.

Henceforth, for every positive integer n, let us write $G_n$ to denote a cyclic group of order $n$, such as the roots of $x^n = 1$ in the complex field $\mathbb{C}$ let us denote by $\mathbb{C}^*$ the group of all nonzero complex numbers under multiplication.

**Proposition 2.** *Let $n$ be a positive integer. Consider the homomorphisms $f$ of $G_n$ into $\mathbb{C}^*$. Then $n$ is odd iff the only real-valued $f$ is the trivial homomorphism ($f(G_n) = \{1\}$).*

**Proof:** Exercise.

At the International Congress of Mathematicians in 1962, John Thompson gave the talk [T1], in which he described some important tools used in the proof of FT. Two, which we give below, are theorems whose proofs are beyond the scope of these lectures. (The second is about group characters: readers unfamiliar with group characters may ignore it and later references to them, as they are not essential to this article.)

**Theorem 1([BG], Theorem 4.20, p.44)** *Suppose $G$ is a group of odd order and, for every prime $p$, $G$ has no subgroup isomorphic to the direct product $G_p \times G_p \times G_p$. Then the derived group (commutator subgroup) of $G$ is nilpotent.*

**Counterexamples for even order:** $S_4, A_5, S_5$.

**Theorem 2(Burnside; [Gor], p.133)** *Suppose $G$ is a finite group. Then $|G|$ is odd if and only if the only real-valued irreducible complex character of $G$ is the principal character (given by $f(G) = \{1\}$ ).*

The proof of Theorem 1 uses the ideas in the proof of Proposition 1. Theorem 2 generalizes Proposition 2.

A third tool mentioned by Thompson was numerical calculation regarding $p$-subgroups of the minimal counterexample $G$, where $p$ is a prime. In proving various properties by contradiction, the authors were able to obtain the inequality $p < 3$. Since $|G|$ is odd, this is impossible.

## 2. Outline of the proof

The proofs of the (CA)-Theorem and, especially, of the (CN)-Theorem give a miniature version of the proof of FT. Fortunately, part of the proof of the (CA)-Theorem is fairly elementary. Here we give some of the elementary part of the (CA)-theorem, as well as the general outline of the (CA)-Theorem. This leads to the outline of the proof of FT and to revisions and simplifications of FT.

### 2.1. (CA)-groups

We first introduce or recall some notation for an arbitrary group $G$. A *maximal* subgroup of $G$ is a maximal proper subgroup of $G$. An abelian subgroup $A$ of $G$ is a *maximal abelian* subgroup of $G$ if $A = B$ for every abelian subgroup $B$ of $G$ that contains $A$.

Henceforth in this section, assume that $G$ is a (CA)-group. Let $\mathcal{A}$ be the set of all maximal abelian subgroups of $G$. Define a relation $\sim$ on $G^{\#}$ by

$$x \sim y \quad if \quad xy = yx$$

**Proposition 1.** *(a) The relation $\sim$ is an equivalence relation on $G^{\#}$.*

*(b) For every equivalence class $C$ under $\sim$, the set $C \cup \{1\}$ is a maximal abelian subgroup of $G$. Moreover, every maximal abelian subgroup of $G$ arises in this way.*

*(c) Suppose $A \in \mathcal{A}$, $x \in A^{\#}$, and $1 < B \leq A$. Then*

$$C_G(x) = C_G(B) = A.$$

*(d) For $A_1, A_2$ distinct elements of $\mathcal{A}$, $A_1 \cap A_2 = 1$.*

*(e) Each element $A$ of $\mathcal{A}$ is a Hall subgroup of $G$, i.e., the order $|A|$ is relatively prime to the index $|G : A|$.*

**Proof:** Parts (a) through (d) are easy exercises.

For part (e), let $p$ be a prime divisor of $|A|$. Take $x$ in $A$ of order $p$, and let S be a Sylow $p$-subgroup of $G$ that contains $x$. Recall that the *center* of $S$ is defined by

$$Z(S) = C_S(S)$$

and, by an elementary property of finite $p$-groups, $Z(S) > 1$. By (c),

$$Z(S) \leq C_G(x) = A \quad and \quad S \leq C_G(Z(S)) = A$$

Therefore, $p$ does not divide $|G : A|$.

We now define an important concept for the proof of FT. Our definition is different from the usual one, but they are equivalent because of an important theorem of Frobenius.

**Definition.** *A finite group $H$ is a* Frobenius group *if it possesses a normal subgroup $K$ such that*

*(i) $1 < K < H$, and*

*(ii) $C_H(x) \leq K$ for every $x \in K^{\#}$.*

**Proposition 2.** *Let $A \in \mathcal{A}$ and $H = N_G(A)$. Then*

*either (i) $H = A$*

*or (ii) $H$ is a Frobenius group (with $K = A$).*

**Proof:** Exercise.

**Proposition 3.** *Suppose $G$ is a non-abelian simple (CA)-group. Then*

*(a) for each $A \in \mathcal{A}$, $N_G(A)$ is a Frobenius group, and*

*(b) $|G^{\#}| = \sum_{A \in \mathcal{A}} |A^{\#}|$.*

**Proof:** This, too, is an exercise. (For (a), one must use a theorem of Burnside to say that no non-identity Sylow subgroup $S$ can be contained in the center of its normalizer, $N_G(S)$. Part (b) does not require that $G$ be simple.)

## 2.2. Outline of Proof of special cases

Let us recall the (CA)-theorem:

**Theorem (Suzuki).** *Every (CA)-group $G$ of odd order is solvable.*

As we have mentioned, the proof of this theorem (and, even more, of the (CN)-theorem) are helpful in understanding the proof of FT. For this theorem, one uses induction on $|G|$ and easily reduces to the case in which $G$ is a non-abelian simple group. Then the proof is divided into two parts:

(1) *Local analysis.* (This includes the study of centralizers of elements and normalizers of subgroups of prime power order). In this part, Suzuki proves Propositions 1-3 and further results. It follows easily that each maximal sub-

group of $G$ has the form $N_G(A)$ for some $A$ in $\mathcal{A}$ (although this is not mentioned). The results of this section do not use essentially the fact that $|G|$ is odd; for example, they are valid for $A_5$.

(2) *Character theory.* (This refers to the study of the complex characters of $G$.) Here, Suzuki shows how the results of Part (1) lead to very strong information about the characters of $G$. In particular, he obtains "exceptional characters" arising from Frobenius groups as in the work of Brauer and Suzuki described in Section 4.5 of [Gor1]. Here, he needs to invoke the assumption that $|G|$ is odd to apply Burnside's result (Theorem 2 above). He obtains some inequalities from character theory that could be regarded as analogues of Proposition 3(b), and uses them to force a contradiction.

The (CN)-Theorem asserts

**Theorem (Feit-M.Hall-Thompson).** *Every (CN)-group of odd order is solvable.*

The proof is similar in outline to Suzuki's proof, but the much greater generality causes significant complications that the authors overcome. The first part uses local analysis. As in Suzuki's proof, they show that every maximal subgroup of the minimal counterexample is a Frobenius group. In the next part, they find a somewhat simpler character theory argument than Suzuki's and use it to obtain inequalities showing that $|G| \leq 70$, which easily yields a contradiction.

## 2.3. Outline of Proof of FT

The proof of FT was immensely more difficult than the proof of the previous special cases. A comparison of the lengths of the articles, although dramatic, only hints as to the difficulty: 10 pages for the (CA)-Theorem, 17 for the (CN)-Theorem, and 255 pages for FT. The most striking difference is that, in the previous cases, each maximal subgroup $M$ of the minimal counterexample $G$ is a solvable (CN)-group, while here $M$ is merely a solvable group of odd order. Thus, the whole group G is like a featureless block or cube delivered to a sculptor. Therefore, it is surprising that the general outline of the proof of FT, like that of the (CN)-Theorem, follows the outline of Suzuki's proof ([T1], p.296).

In the previous cases, it is fairly easy to see that $M$ is "close to" a Frobenius group; here, there are virtually no restrictions on $M$, which could have derived length one million, for example. On the other hand, very little was known about determining the characters of a group from subgroups other than Frobenius groups. Thus, the work had two main aspects:

To reduce the structure of a maximal subgroup $M$ by local analysis, from a random shape to some restricted shape.

To enlarge the scope of the character theory framework from the case where $M$ is a Frobenius group to a much more general case.

Amazingly, Feit and Thompson succeeded in this program. The local analysis was done mainly by Thompson and the character theory by Feit. Most of the proof of FT was worked out at a conference at the California Institute of Technology in Summer, 1960, and at an academic year program at the University of Chicago in 1960-61.

The published proof of FT is divided into six chapters. Chapters I-III consist mainly of preliminary results in local analysis and character theory, including many innovations. In Chapter IV, they begin the attack on the minimal counterexample $G$. They show, mainly by local analysis, that every maximal subgroup of $G$ is "close to" being a Frobenius group. In Chapter V, they use character theory to sharpen the results of Chapter IV and to obtain the exact structure of two maximal subgroups of $G$. Here, they draw heavily on Chapter III, in which they had extended the previously known character theory to cover subgroups like those in the conclusion of Chapter IV. Finally, in Chapter VI, they obtain a contradiction by using generators and relations, a technique not used in the (CA)- and (CN)-Theorems (although used in many previous and subsequent proofs classifying simple groups).

For convenience, we refer to the local analysis (Chapter IV and its necessary preliminaries) as Part (1); the character theory (Chapter V and its necessary preliminaries) as Part (2); and Chapter VI as Part (3).

## 2.4. Revision of the proof

In 1962, Thompson wrote ([T1], p.299) about FT "It is certain that these techniques can be refined to give a neater proof...[of FT] than the present one." This expectation has been borne out, partially by recent work, including work by Feit and Thompson themselves.

The first dramatic improvement in the proof of FT was achieved by Helmut Bender in 1970 [Be]. It concerned an important result in Part (1) of the proof: a subgroup $H$ of $G$ is contained in a *unique* maximal subgroup of $G$ if $H$ contains the direct product $G_p \times G_p \times G_p$ of three copies of some cyclic group of prime order $p$. This result is called the Uniqueness Theorem. Its original proof occupies roughly the first half of the attack on $G$ in Chapter IV of [FT], about 50 pages. By a new technique, now called "the Bender method", Bender reduced the proof to 10 pages.

In 1991, Feit and Thompson obtained a further improvement in Part (1) that eliminated a difficult point in the proof ([BG], pp.121,122,133,157-166). This and Bender's improvement were included in a revision of Part (1) by Bender and myself (with the assistance of Walter Carlip) published in 1994 [BG].

In 1976, David Sibley obtained [Sib1] a major advance in character theory that was applicable to FT. In 1988, Sibley used this advance (and an earlier simplification by Everett Dade [D]) to revise most of Part (2) of FT in unpublished work [Sib2]. Now Thomas Peterfalvi has revised all of Part (2) in work that is likely to be published in 1999 [P2]. His revision partially draws upon Sibley's work.

In 1984, Peterfalvi published [P1] a revision of Part (3) of FT that halved its length. His proof is included in [BG] (Appendix C).

These revisions relied partly on advances concerning solvable groups or groups of odd order that were developed from innovations in [FT]. Most of the techniques that they replaced did not become obsolete, but were needed in proofs of later theorems classifying families of simple groups.

In my opinion, these revisions have greatly reduced the technical difficulty of the proof of FT. Bender's and Sibley's work make unnecessary many arguments for special cases in the original proof. However, the revisions have not substantially reduced the length of the proof, partly because they were not written for journals, and include more details than the original.

I would estimate that Part (1) takes up about 120 pages in the original and about 155 in the main body of [BG]. (This is partly because we include the proofs of some results that are quoted without proof in [FT].) Similarly, I estimate that Peterfalvi has reduced Part (2) from 110 to 90 pages. For Peterfalvi's proof as in [BG], Part (3) is reduced from 17 to 8 pages. In my mind, this suggests the following:

**Open problem:** *Find a key idea in the original proof or a radical new idea to shorten the proof of FT.*

For example, Bender's proof of the Uniqueness Theorem reduces it from about 50 pages to 15 in [BG]. A similar reduction of the entire proof would reduce it from 255 pages to about 75. I would like to think of the current revisions as a middle stage in the understanding of FT that may inspire some reader to find a better way.

During the lectures, Dr. A. Solecki asked how much background is needed for reading these revisions. More generally, one can ask about the earlier special cases of FT as well. In brief, the answers are:

small for the (CA)-Theorem
medium for the (CN)-Theorem
very large for FT.

One can read Suzuki's 10 page proof of the (CA)-Theorem with only a standard first-year graduate course in algebra and an elementary introduction to character theory. The same background, together with parts of [F1], is sufficient to read the proof of the (CN)-Theorem in [F1], Section 27.

For FT, the background is much more extensive. The prerequisites for [BG] are described in detail in Appendix A of the book. The main text of [BG] has been presented in several seminars and classes ([BG]. p. xi). In [P2], the author suggests some additional reading in character theory. However, Part (3) requires no more background than Suzuki's (CA)-Theorem.

The (CN)-Theorem includes the (CA)-Theorem. For someone who plans to read FT, it is not strictly necessary to read the (CN)-Theorem. The local analysis part of the (CN)-Theorem is subsumed in Part 1 of FT, while the final contradiction is obtained as part of [P2]. Nevertheless, I would strongly recommend reading the (CN)-Theorem first, because it is an important special case of FT and because its proof gives an excellent preview of the outline and some main themes of FT.

## 3. The final contradiction and Suzuki's new result

### 3.1. Introduction

In Section 2, we gave an outline of the proof of FT, roughly as follows:

(1)(Local analysis) Show that all the maximal subgroups of the minimal counterexample $G$ are "close to" Frobenius groups.

(2)(Character theory) Sharpen the results of (1).

(3)(Generators and relations) Obtain a contradiction.

We also described Part (1) for Suzuki's (CA)-Theorem. Here, we describe some aspects of Part (2) for the (CA)-Theorem, Parts (2) and (3) for FT, and a new, unpublished result of Suzuki.

Thompson describes Suzuki's (CA)-theorem as "a marvel of cunning"([T2], pp.10-11) that "removed one of the major stumbling blocks" in the proof of FT ([T1], p.296). He is referring mainly to Part (2) of the proof. Here, one is dealing with Frobenius groups of the form $N_G(A)$, where $A$ is a maximal abelian subgroup of $G$. Suzuki shows that

(a) for each irreducible character $\chi$ of $N_G(A)$ not containing $A$ in its kernel, $\chi$ corresponds to an "exceptional" character $\chi'$ of $G$, and

(b) each non-principal irreducible character of $G$ has the form $\chi'$, for some $A$ unique up to conjugacy in $G$, and some unique $\chi$.

He uses these facts to obtain a series of inequalities that could be regarded as far-reaching generalizations of Proposition 3(b) in Section 2.1 above. These inequalities yield a contradiction.

Part (2) of FT is inspired by Part (2) of the (CA)-Theorem and vastly generalizes it.

Thompson writes ([T2], pp.10-11),

In order to have a genuinely satisfying proof of the odd order theorem, it is necessary, it seems to me, not to assume this [Suzuki's] theorem. Once one accepts this theorem as a step in a general proof, one seems irresistibly drawn along the path which was followed. To my colleagues who have grumbled about the tortuous proofs in the classification of simple groups, I have a ready answer: find another proof of Suzuki's theorem.

### 3.2. Intermediate Results in Parts (2) and (3) of FT

In Part (2) of FT, Feit and Thompson sharpen the results of Part (1) to obtain very precise information about the minimal counterexample $G$. In particular, they obtain distinct odd primes $p$ and $q$ and maximal subgroups $S$ and $T$ for which they prove the following:

(A) Let $\mathbb{F}$ be the finite field of order $p^q$ and $u = (p^q - 1)/(p-1)$. Then $q$ and $u$ are relatively prime to $p-1$ and $S$ is isomorphic to the group of all permutations of $\mathbb{F}$ of the form

$$x \mapsto ax^\sigma + b \qquad\qquad (x \in \mathbb{F}),$$

where $a, b \in \mathbb{F}$ and $a^u = 1$, and $\sigma$ lies in the Galois group of $\mathbb{F}$ over the prime field $\mathbb{F}_p = GF(p)$.

It is easy to describe the structure of $S$. It has normal subgroups $E$ and $EU$ as follows:

$E$ corresponds to the group of all translations $x \mapsto x + b$ (thus, $E$ is isomorphic to the additive group of $\mathbb{F}$);

$U$ corresponds to the group of all field multiplications of the form $x \to ax$, where $a^u = 1$ (thus, $U$ is a cyclic group of order $u$);

$S/EU$ is a cyclic group of order $q$.

In addition, $N_G(U)$ contains a cyclic subgroup $W$ of order $pq$ such that

$$N_G(W) = W \quad \text{and} \quad W \cap C_G(U) = 1.$$

(B) $T$ has the same structure as $S$, with $p$ and $q$ interchanged, and $S \cap T$ is cyclic of order $pq$ (and thus isomorphic to $W$).

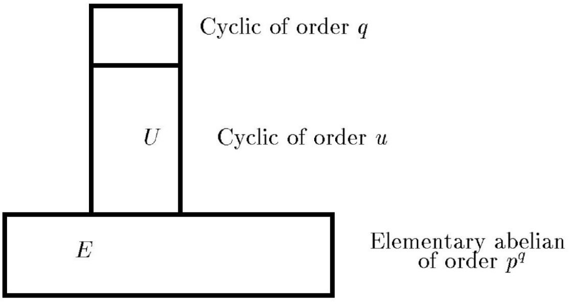
(C) For every maximal subgroup $M$ of $G$,

(i) $M$ is conjugate to $S$ in $G$, or

(ii) $M$ is conjugate to $T$ in $G$, or

(iii) $M$ is a Frobenius group.

We may describe the structure of $S$ by a diagram:




In Part (3) of FT, the authors obtain a contradiction. By the symmetry between $p$ and $q$ above, they can assume that $p > q$. They take the subgroup $E_o$ of $E$ corresponding to the translations $x \mapsto x + b$ for $b$ ranging over $\mathbb{F}_p$, and they consider the set

$$D = \{a \in \mathbb{F} | a^u = (2-a)^u = 1\}.$$

Using elementary character theory, they show that $D$ contains at least 2 elements. (Note that $1 \in D$.) This is a property of finite fields, independent of the proof of FT. They also show that $EU = UE_oU$. Then, by arguments with generators and relations, they eventually obtain that $p \leq q$, a contradiction.

Part (3) is the shortest and easiest part of FT to read (only 8 pages in Appendix C of [BG]). It does not require familiarity with the results and methods of Parts (1) and (2), except elementary character theory for one lemma. It was the first part of FT that I read completely (after reading Gorenstein's enthusiastic description in [Gor1], pp.459-461). I strongly recommend reading Gorenstein's description and then Part (3) itself.

### 3.3. Suzuki's New Result

In [Sz3], Suzuki announced some unpublished work in which he obtains much of the main results of Part (1) and (2) of FT under a more general hypothesis that is actually satisfied by many familiar simple groups. Thus, the beautiful ideas of Feit and Thompson can be applied to groups that really exist, instead of vanishing with the final contradiction.

To state Suzuki's main results, we define the *prime graph* $\Gamma(G)$ of an arbitrary finite group $G$. The vertices of the graph are the elements of $\pi(G)$, the set of all prime divisors of $|G|$. Two primes $p$ and $q$ form an *edge* $\{p, q\}$ if $p \neq q$ and if $G$ contains an element of order $p$ and an element of order $q$ that centralize each other. A subset $\Delta$ of $\pi(G)$ is a *clique* if $\{p, q\}$ is an edge for every distinct $p, q$ in $\Delta$.

A non-identity proper subgroup $H$ of $G$ is *isolated* if $C_G(x) \leq H$ for every $x$ in $H^{\#}$.

Now consider the following conditions:

**Hypothesis C.** *(i) $G$ is a finite group*

*(ii) $\Delta$ is a non-empty connected component of $\Gamma(G)$*

*(iii) $\Delta$ is a proper subset of $\pi(G)$ and $2 \notin \Delta$.*

We have two results:

**Theorem 1. (Suzuki [Sz3]; announced 1997, unpublished)** *Assume Hypothesis C. Suppose $G$ is simple. Then $\Delta$ is a clique. Moreover,*

*either (a) $G$ contains a nilpotent Hall $\Delta$-subgroup that is isolated in $G$*

*or (b) for some distinct primes $p$ and $q$, $\Delta = \{p, q\}$ and $G$ possesses a cyclic subgroup $W$ of order $pq$ such that $N_G(W) = W$.*

As an example for Theorem 1, one may take any integer $n$ greater than 1 and let

$$G = L_2(q) = SL(2, q) \quad \text{for} \quad q = 2^n$$

Then $G$ has a cyclic subgroup $G_1$ of order $q + 1$, and one can take $\Delta = \pi(G_1)$.

**Theorem 2.** *Assume Hypothesis C and the classification of finite simple groups. Then*

*(a) (Gruenberg, Kegel, J. S. Williams [Sz3]) $\Delta$ is a clique, and*

*(b) (J. S. Williams [W]) if $G$ is simple, then $G$ contains a nilpotent Hall*

*$\Delta$-subgroup isolated in $G$.*

Suzuki says that Theorem 2(a) has not been stated before [Sz3], but follows easily from work of Gruenberg and Kegel [GK] and Williams. Note that it does not require $G$ to be simple. Although Theorem 1 follows from Theorem 2, its proof does not require assuming the classification of finite simple groups, but only approximately the same background as FT. (For proving FT, one allows $\Delta = \pi(G)$ at first, but eventually eliminates this possibility.)

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