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WITT-GROTHENDIECK RINGS AND π-HENSELIANITY

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Abstract

Consideramos neste artigo o subanel $\widehat{W}(\Sigma\dot{F}^2)$, do anel de Witt-Grothendieck de um corpo formalmente real F, gerado por elementos que são somas de quadrados. Nosso objetivo é demonstrar que se F contém um anel de valorização que tem a propriedade do levantamento para raízes quadradas de somas de quadrados, então $\widehat{W}(\Sigma\dot{F}^2)$ pode ser descrito como um quociente de um anel de grupo.

In this note we consider the subring $\widehat{W}(\Sigma\dot{F}^2)$, of the Witt-Grothendieck ring of a formally real field F, generated by sums of squares. Our goal is to show that if F admits a valuation ring which has the lift property relative to square roots of sums of squares, then $\widehat{W}(\Sigma\dot{F}^2)$ is a suitable quotient of a group ring.

1. Introduction

Let F be field of characteristic different from two. Following [L] we denote by $\widehat{W}(F)$ the Witt-Grothendieck ring of quadratic forms over F. Let $\Sigma\dot{F}^2$ be the subgroup of the multiplicative group $\dot{F} = F \setminus \{0\}$ consisting of all sums of squares. Set $\widehat{W}(\Sigma\dot{F}^2)$ for the subring of $\widehat{W}(F)$ generated by isometry classes of quadratic forms $\langle t_1, \ldots, t_n \rangle$ with $t_i \in \Sigma\dot{F}^2$, $i = 1, \ldots, n$. We shall also denote by F_π the pythagorean closure of F (F_π is the field which arises from F by iterating the process of adjoining the square roots of all sums of squares). A valuation ring A of F is called *π-henselian* if A extends uniquely to F_π. Equivalently, Hensel's Lemma applies to polynomials splitting over F_π.

The aim of this note is to establish a π-henselian analogue of a result of W. Scharlau concerning henselian fields of characteristic different from two ([Sch],

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Korollar 4.1.3). Scharlau's theorem extends an old result of T. A. Springer on local fields ([L], Chapter 6).

Main Theorem: *Let F be a formally real field which admits a π-henselian valuation ring A such that the residue field k of A is not formally real and* $char\, k \neq 2$. *Then there exists a subgroup T of $\Sigma\dot{F}^2/\dot{F}^2$ such that $\widehat{W}(\Sigma\dot{F}^2) \cong \widehat{W}(k)[T]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by $\{t\mathbb{H} - \mathbb{H} \mid t \in T$ and $\mathbb{H} = \langle 1, -1\rangle\}$.*

In next section we prove the Main Theorem and we deduce from it that $\widehat{W}(\Sigma\dot{F}^2)$ is isomorphic to the Witt-Grothendieck ring of a generalized formal power series field.

In section 3 we recall from [En] and [En2] how π-henselian valuation rings arise from totally positive rigid elements (see Definition 3.1). We deduce from this a partial converse of the Main Theorem where the link between π-henselian valuation rings and the Witt-Grothendieck ring of a generalized formal power series field will be completed (Theorem 3.6). This result may as well be regarded as a relative version for $\widehat{W}(\Sigma\dot{F}^2)$ of a result of Berman ([Ber], Theorem 1.1) on power series fields.

Conventions: Throughout the paper, unless otherwise explicitly stated, all valuation rings considered have non-formally real residue field of characteristic not 2.

In what follows all fields will have characteristic $\neq 2$ and for any field F, $\dot{F}$, $\dot{F}^2$, and $\Sigma\dot{F}^2$ will denote the multiplicative groups of nonzero elements, squares, and sums of squares, respectively. The quadratic closure of F and the pythagorean closure are denoted by $F(2)$ and F_π respectively. Write $Gal(F(2);F) = G_2(F)$ and $Gal(F_\pi;F) = G_\pi(F)$ for the Galois groups.

For $a_1, a_2, \ldots, a_n \in \dot{F}$, $\langle a_1, a_2, \ldots, a_n\rangle$ denotes the diagonal quadratic form $a_1X_1^2 + a_2X_2^2 + \cdots + a_nX_n^2$ over F and $D\langle a_1, a_2, \ldots, a_n\rangle$ stands for its nonzero values. If q_1 and q_2 are quadratic forms over F, $q_1 \simeq q_2$ means that q_1 and q_2 are isometric. Let $W(F)$ be the Witt ring of all isometry classes of anisotropic quadratic forms over F.

Observe that if F is not formally real then $\Sigma\dot{F}^2 = F$ and $F_\pi = F(2)$. Hence π-henselian valuation rings are the well-known 2-henselian valuation rings, $\widehat{W}(\Sigma\dot{F}^2) = \widehat{W}(F)$, and so our Main Theorem coincides with Scharlau's result ([Sch], Korollar 4.1.3). On the other side, if F is formally real, the restriction of the natural map $\widehat{W}(F) \longrightarrow W(F)$ to $\widehat{W}(\Sigma\dot{F}^2)$ is injective. Therefore $\widehat{W}(\Sigma\dot{F}^2)$ may be seen as a subring of $W(F)$.

Any unexplained property concerning quadratic forms can be found in [L], while any concerning valuation theory can be found in [E].

2. The subring $\widehat{W}(\Sigma\dot{F}^2)$

For every profinite group G let $H^i(G) = H^i(G, \mathbb{Z}/2\mathbb{Z})$ be the i-th cohomology group of G with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Since G operates trivially on $\mathbb{Z}/2\mathbb{Z}$, $H^1(G)$ is precisely the group of all continuous group homomorphisms from G to $\mathbb{Z}/2\mathbb{Z}$.

If $G = G_2(F)$, for some field F, then $H^1(G)$ and $H^2(G)$ can be studied by arithmetic objects within F itself. The natural isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$ induces an isomorphism $\dot{F}/\dot{F}^2 \longrightarrow H^1(G_\pi(F))$ where each class $t\dot{F}^2$ corresponds to χ_t defined by $\chi_t(\sigma) = \sigma(\sqrt{t})/\sqrt{t}$. Clearly kernel$\chi_t = G_\pi(F(\sqrt{t}))$. Observe that this isomorphism restricts naturally to $\Sigma\dot{F}^2/\dot{F}^2 \cong H^1(G_\pi(F))$. Indeed, if $F(\sqrt{t}) \subset F_\pi$, each order of F extends to $F(\sqrt{t})$. Thus t has to be totally positive in F.

It is well-known that $H^2(G) \cong Br(F)(2)$ = the subgroup of the Brauer group consisting of elements of order ≤ 2. Under this isomorphism $\chi_t \cup \chi_s$ corresponds to the Brauer class of the quaternion algebra (t, s), for every $t, r \in \dot{F}$ (see [Se], pp. 204-207). The isomorphism and Corollary 2.2 of [W2] imply that $H^2(G_\pi(F))$ may be seen as a subgroup of $Br(F)(2)$.

For a profinite group G, Scharlau defined the Witt-Grothendieck ring of G as the quotient ring $\widehat{W}(G) = \mathbb{Z}[H^1(G)]/Ke(G)$, where $Ke(G)$ is the ideal of the group-ring $\mathbb{Z}[H^1(G)]$ generated by the elements of the form $\chi_1 + \chi_2 - (\chi_3 + \chi_4)$ with $\chi_1, \chi_2, \chi_3, \chi_4 \in H^1(G)$ and $\chi_1\chi_2 = \chi_3\chi_4$ in $H^1(G)$ and $\chi_1 \cup$

$\chi_2 = \chi_3 \cup \chi_4$ in $H^2(G)$ ([Sch], Definition 1.2.1). Delzant ([D], or [Sch], Satz 1.3.1) stated that if G is the Galois group of the separable closure of a field F, then $\widehat{W}(G)$ is canonically isomorphic to $\widehat{W}(F)$. Actually, by ([Sch], Korollar 1.2.4) $\widehat{W}(F) \cong \widehat{W}(G_2(F))$. Ware has shown that $\widehat{W}(G_\pi(F)) \cong \widehat{W}(\Sigma\dot{F}^2)$ (see the proof of Theorem 2.5 in [W2]). Let us point out that the isomorphism $\widehat{W}(F) \cong \widehat{W}(G_2(F))$ (resp. $\widehat{W}(\Sigma\dot{F}^2) \cong \widehat{W}(G_\pi(F))$) is induced by $\langle t\rangle \longmapsto \chi_t$ ([Sch], Satz 1.3.1).

Next we record a result which will be crucial in the proof of the Main Theorem. Recall first that if N is a normal subgroup of a profinite group G, G/N has a natural action on $H^1(G)$. Namely, for every $g, h \in G$ if $\bar{h}$ is the class of h in G/N, $\chi^{\bar{h}}(g) = \chi(h^{-1}gh)$. As usual, *Inf* and *Res* denote respectively the maps inflation and restriction ([R], p. 131 and p. 134).

Proposition 2.1 ([Sch], Satz 4.1.1) *Let $1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$ be an exact sequence of profinite groups such that*

(a) N is abelian,

(b) the action of G/N on $H^1(N)$ is trivial,

(c) Inf: $H^2(G/N) \longrightarrow H^2(G)$ is injective,

(d) there exists $\alpha \in H^1(G/N)$ such that $\chi \cup \chi = \chi \cup Inf(\alpha)$ for every $\chi \in H^1(G)$.

If T is a subgroup of $H^1(G)$ for which $Res|_T : T \longrightarrow H^1(N)$ is an isomorphism, then $\widehat{W}(G) \cong \widehat{W}(G/N)[T]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by $\{t\langle 1\rangle + t\langle\alpha\rangle - \langle 1\rangle - \langle\alpha\rangle \mid t \in T\}$.

The above ring isomorphism is induced by the natural homomorphism $G \longrightarrow G/N$.

The conditions *(a)* to *(d)* are the motivation for the next three propositions.

Proposition 2.2. *Let F be a field such that $\Sigma\dot{F}^2 = D\langle 1, r\rangle$ for some $r \in \Sigma\dot{F}^2$. Then, $(t, t) \cong (t, r)$, for every $t \in \Sigma\dot{F}^2$.*

Proof: Since $t \in D\langle 1, r\rangle$, $-r \in D\langle 1, -t\rangle$. Thus, by ([L], Proposition 1.3, p. 276), $\langle 1, -t, -r, tr\rangle \simeq 2\langle 1, -t\rangle \simeq \langle 1, -t, -t, 1\rangle$. Therefore Proposition 2.5, (p. 57), of [L] implies that $(t, t) \cong (t, r)$, as desired.

□

We now consider a formally real field F which admits a π-henselian valuation ring A with non-formally real residue field k, *char* $k \neq 2$. Let C be its unique extension to F_π and denote by $G^T(C; F)$ the inertia group of C over F. By ([En], Proposition 4.1) k_C is the quadratic closure of k. Therefore $G(k_C; k) = G_2(k)$. For the sake of completeness we shall record the description of $G_\pi(F)$ by means of $G^T(C; F)$ and $G_2(k)$.

Proposition 2.3. *Keeping above notation,*

(1) $G^T(C; F)$ is an abelian pro-2 group (see the comments at the beginning of §3 of [En2]).

(2) $G_\pi(F) \cong G^T(C; F) \rtimes G_2(k)$ ([En2], Proposition 3.2 (b)).

Proposition 2.4. *With the same notation of the last proposition,*

(1) $G_2(k)$ operates trivially on $H^1(G^T(C; F))$.

(2) Inf: $H^2(G_2(k)) \longrightarrow H^2(G_\pi(F))$ is injective.

Proof: We shall prove that $\mathrm{Res}\colon H^1(G_\pi(F)) \longrightarrow H^1(G^T(C; F))$ is a surjective map. Therefore the result follows from the "5-term exact sequence" ([R], Corollary 5.4 p. 177).

Let us denote by K the fix field of $G^T(C : F)$ (K is the inertia field of C over F). By ([En2], Proposition 3.6) the inclusion $F \subset K$ induces an isomorphism $\Sigma\dot{F}^2/(A^*\dot{F}^2 \cap \Sigma\dot{F}^2) \cong \Sigma\dot{K}^2/\dot{K}^2$. Thus, as remarked in the beginning of the section, for every $\chi \in H^1(G^T(C; F))$ there is $t \in \Sigma\dot{F}^2$ such that $\mathrm{Res}(\chi_t) = \chi$.

□

Proof of the Main Theorem: We shall prove that the conditions of Proposition 2.1 apply to $G_\pi(F)$, $G^T(C; F)$ and $G_2(k)$.

By the propositions 2.3 and 2.4 conditions *(a)* to *(b)* are satisfied.

Let $\varphi : A \longrightarrow k$ be the natural map. Since k is not formally real, there is $u \in A^* \cap \Sigma\dot{F}^2$ such that $\varphi(u) = -1$. In order to use Proposition 2.2 to verify *(d)* we shall show that $\Sigma\dot{F}^2 = D\langle 1, u\rangle$.

Let $t \in \Sigma\dot{F}^2$ and write $t = x_1 + \cdots + x_n$ where $x_1, \ldots, x_n \in \dot{F}^2$. We may assume that $n \geq 2$. Take $1 \leq i \leq n$ such that $v(x_i) \leq v(x_j)$, for every $1 \leq j \leq n$, where v is a valuation corresponding to A. Without loss of generality we may assume $i = 1$. Then $tx_1^{-1} = 1 + x_1^{-1}(x_2 + \cdots + x_n) \in A$ and also $x_1^{-1}(x_2 + \cdots + x_n) \in A$. If $x_1^{-1}(x_2 + \cdots + x_n) \in m$, the maximal ideal of A, then $tx^{-1} \in (1 + m) \cap \Sigma\dot{F}^2$ and so $tx^{-1} \in \dot{F}^2$, by ([En], Proposition 2.1). Since this contradicts $n \geq 2$, it follows that $x_1^{-1}(x_2 + \cdots + x_n) \in A^*$. We now consider two cases. If $tx_1^{-1} \in m$, then $\varphi(x_1^{-1}(x_2 + \cdots + x_n)) = -1 = \varphi(u)$. Hence $x_1^{-1}(x_2 + \cdots + x_n)u^{-1} \in (1 + m) \cap \Sigma\dot{F}^2$. Therefore, arguing as above, $x_1^{-1}(x_2 + \cdots + x_n)u^{-1} \in \dot{F}^2$ and $t \in D\langle 1, u\rangle$ as desired. If $tx_1^{-1} \in A^*$, for $a = \left((tx_1^{-1} + 1)/2\right)^2$ and $b = \left((tx_1^{-1} - 1)/2\right)^2$, $\varphi(tx_1^{-1}) = \varphi(a) - \varphi(b) = \varphi(a + ub)$. Once again $tx_1^{-1}(a + ub)^{-1} \in (1 + m) \cap \Sigma\dot{F}^2$ will imply $t \in D\langle 1, u\rangle$, proving the claim.

Once this is established, then Proposition 2.2 implies that $(t, t) \cong (t, u)$, for every $t \in \Sigma\dot{F}^2$. Thus, as we have seen in the beginning of the section, $\chi_t \cup \chi_t = \chi_t \cup \chi_u$ for every $\chi_t \in H^1(G_\pi(F))$. To complete the verification that condition *(d)* holds it remains to be seen that $\chi_u \in \mathrm{Inf}(H^1(G_2(k))$. To this end, assume first that $-1 \notin \dot{k}^2$ and take $L = F(\sqrt{u})$. Let B be the unique extension of A to L. It is pretty clear that $k(\sqrt{-1})$ is the residue field of B. Then L is unramified over F. Since *char* $k \neq 2$, by ([E], Theorem 20.21, p. 70), A is defectless in any finite subextension $F \subset E \subset F_\pi$. So, by ([E], Theorem 22.7, p. 182) it follows that $G^T(C; F) \subset G_\pi(L)$. Hence $G_2(k(\sqrt{-1})) = G_\pi(L)/G^T(C; F)$ which means that $\mathrm{Inf}(\chi_{-1}) = \chi_u$ and so *(d)* is true in this case. Consider now the case $-1 \in \dot{k}^2$. By ([En2], Lemma 5.6) we can choose $u = 1$. Since $\chi_1 = \chi_{-1}$ in $H^1(G_2(k))$, $\mathrm{Inf}(\chi_{-1}) = \chi_u$ again. Therefore $\alpha = \chi_{-1} \in H^1(G_2(k))$ verifies the condition *(d)*.

Finally, take a subgroup T of $\Sigma\dot{F}^2/\dot{F}^2$ such that $T \oplus (A^*\dot{F}^2 \cap \Sigma\dot{F}^2/\dot{F}^2) =$

$\Sigma\dot{F}^2/\dot{F}^2$. Then, by ([En2], Proposition 3.6), Res: $T_0 = \{\chi_t \mid t \in T\} \longrightarrow H^1(G^T(C;F))$ is an isomorphism. Therefore, Scharlau's result implies that $\widehat{W}(G_\pi(F)) \cong \widehat{W}(G_2(k))[T_0]/\mathfrak{a}_o$, where $\mathfrak{a}_o$ is the ideal generated by $\{\chi_t\langle 1\rangle + \chi_t\langle\chi_{-1}\rangle - \langle 1\rangle - \langle\chi_{-1}\rangle \mid t \in T\}$. Then, the statement follows immediately from the isomorphisms $\widehat{W}(\Sigma\dot{F}^2) \cong \widehat{W}(G_\pi(F))$ and $\widehat{W}(k) \cong \widehat{W}(G_2)$ previously described when one bears in mind that $\langle 1\rangle + \langle -1\rangle = \mathbb{H}$.

□

Observe that $\langle 1\rangle \in \widehat{W}(\Sigma\dot{F}^2)$ does not have finite additive order, while $W(k)$ is a torsion group, since k is not formally real. Therefore $\widehat{W}(k)$ cannot be replaced by $W(k)$ in the above result.

Corollary 2.5. *For a field F verifying the condition of the last theorem there are a non-formally real field k, char $k \neq 2$, and a totally ordered group Δ such that if $K = k((X))^\Delta$ is the field of generalized formal power series over k with respect to Δ, then $\widehat{W}(\Sigma\dot{F}^2) \cong \widehat{W}(K)$.*

Proof: As in the proof of the theorem let k be the residue field of A, Γ the value group, and v be the corresponding valuation. Let $\Delta = v(\Sigma\dot{F}^2)$ with the natural ordering. Observe that $\Sigma\dot{F}^2/(A^*\dot{F}^2 \cap \Sigma\dot{F}^2) \cong (\Sigma\dot{F}^2)(A^*\dot{F}^2)/A^*\dot{F}^2 \cong \Delta/2\Delta$ where the last isomorphism is induced by v.

It is well-known that the valuation ring $\mathcal{O}$ of $K = k((X))^\Delta$ corresponding to the Krull valuation w defined by $w\left(\sum_{\delta\in\Delta} a_\delta X^\delta\right) = \min\{\delta \in \Delta \mid a_\delta \neq 0\}$ is henselian, has value group Δ, and residue field k. Therefore the result follows by the previous theorem and ([Sch], Korollar 4.1.3).

□

3. Totally real rigid elements and π-henselianity

We start this section reviewing the results developed in [En] and [En2] for detecting the existence of π-henselian valuation rings of fields. We first recall two essential ingredients in this process.

Definition 3.1. *An element $t \in \Sigma\dot{F}^2$ is called rigid if $t \notin \dot{F}^2$ and $D\langle 1,t\rangle = \dot{F}^2 \cup t\dot{F}^2$.*
Let $B_\pi(F) = \{t \in \Sigma\dot{F}^2 \mid t$ is not rigid $\}$.

Next we improve Corollary 2.13 (2) of [En].

Proposition 3.2. *For every formally real field F, $B_\pi(F)$ is a subgroup of $\dot{F}$.*

Proof: If $B_\pi(F) = \Sigma\dot{F}^2$, there is nothing to prove. Assume $B_\pi(F) \neq \Sigma\dot{F}^2$. By ([Bos], Proposition 1.7), there exists modulo $\dot{F}^2$ exactly one $r \in \Sigma\dot{F}^2$ such that $\Sigma\dot{F}^2 = D\langle 1,r\rangle$ and also dr is rigid, for every rigid $d \in \Sigma\dot{F}^2$. Consequently, if rs is rigid for some $s \in B_\pi(F)$, then $r^2 s \in B_\pi(F)$ is also rigid, a contradiction. Hence $rB_\pi(F) \subset B_\pi(F)$. On the other side, since $\Sigma\dot{F}^2 = D\langle 1,r\rangle$, by ([En], Proposition 2.1), $B_\pi(F) = B_\pi(F) \cup rB_\pi(F)$ is a subgroup of $\dot{F}$.

□

If F is a non-formally real field ($F = \Sigma\dot{F}^2$), every rigid element t is *birigid* (when -t is also rigid ([CR], Corollary)). In this case $B_\pi(F)$ is the so-called set of basic elements and it is well-known that it is a subgroup of $\dot{F}$ ([W], Proposition 2.4). If F is formally real, a rigid element $t \in \Sigma\dot{F}^2$ is not birigid ([BCW], Proposition 1) and so $\Sigma\dot{F}^2$ is contained in the set of basic elements. Thus $B_\pi(F)$ is a proper subgroup of the group of basic elements.

The above proposition allows a more precise formulation of Theorem 2.8 of [En].

Proposition 3.3. *Let F be a formally real field such that $(\Sigma\dot{F}^2 : B_\pi(F)) \geq 2$ and $(\Sigma\dot{F}^2 : \dot{F}^2) = 4$ if $(\Sigma\dot{F}^2 : B_\pi(F)) = 2$. Then there exists a π-henselian valuation ring $\mathcal{O}$ of F such that its residue field k is a non-formally real field of characteristic not 2 verifying $(\dot{k} : B_\pi(k)) \leq 2$.*
Moreover,

(1) $\mathcal{O}$ can be chosen such that either $\dot{k} = \dot{k}^2$ or k does not admit any 2-henselian valuation ring.

(2) $\dot{k} = B_\pi(k)$ if and only if $B_\pi(F) = \mathcal{O}^*\dot{F}^2 \cap \Sigma\dot{F}^2$.

(3) If $\dot{k} \neq B_\pi(k)$, then $B_\pi(F) = \dot{F}^2$, $(\mathcal{O}^*\dot{F}^2 \cap \Sigma\dot{F}^2 : \dot{F}^2) = 2$, $\Sigma\dot{F}^2 = D\langle 1,1\rangle$, $-1 \in \dot{k}^2$ and $(\dot{k} : \dot{k}^2) = 2$.

Proof: In the case $(\Sigma\dot{F}^2 : B_\pi(F)) > 2$ the existence of $\mathcal{O}$ follows from ([En], Theorem 3.8) by putting $S = B_\pi(F)$.

In the other case, by ([En2], Proposition 5.4), there is a quadratic extension E of F such that $B_\pi(E) = \dot{E}^2$. Thus, applying the previous case to E we get a π-henselian valuation ring $\mathcal{O}'$ of E. By ([En2], Corollary 2.8) F also admits a π-henselian valuation ring.

(1) Following the notation of Corollary 2.6 of [En2] we choose $\mathcal{O} = A_{(2)}$ if $\mathcal{H}_2 \neq \emptyset$ and $\mathcal{O} = A_{(1)}$ otherwise. Hence the statement follows either from the properties of $A_{(2)}$ or $\mathcal{H}_2 = \emptyset$ and $A_{(1)}$.

(2) By ([En], Proposition 2.5) $B_\pi(F) \subset \mathcal{O}^*\dot{F}^2$. Hence $B_\pi(F) \subset \mathcal{O}^*\dot{F}^2 \cap \Sigma\dot{F}^2$ is always true ($B_\pi(F) \subset \Sigma\dot{F}^2$). Let φ the canonical projection $\mathcal{O} \longrightarrow k$. Since k is not formally real, $\varphi : \mathcal{O}^*\dot{F}^2 \cap \Sigma\dot{F}^2 \longrightarrow \dot{k}$ is a surjective map. By ([En], Corollary 2.15) $\varphi(\mathcal{O}^* \cap B_\pi(F)) = B_\pi(k)$. Therefore, an easy computation ends the proof of the equivalence.

(3) By the choice of $\mathcal{O}$, $\dot{k} \neq B_\pi(k)$ can only occur if $\dot{k} \neq \dot{k}^2$ and k does not admit any 2-henselian valuation ring. Hence, by ([AEJ], Theorem 2.16) $\dot{k}^2$ is *exceptional* ([AEJ], Definition 2.15). Since k is not formally real and $\dot{k} \neq \dot{k}^2$ ($char\, k \neq 2$), it follows that $B_\pi(k) = \dot{k}^2$ and $-1 \in \dot{k}^2$. Hence $(\dot{k} : \dot{k}^2) = 2$. On the other side, if m is the maximal ideal of $\mathcal{O}$, we know from ([En], Proposition 2.1) that $(1+m)\cap\Sigma\dot{F}^2 \subset \dot{F}^2$. Therefore φ induces an isomorphism from $(\mathcal{O}^* \cap \Sigma\dot{F}^2)/(\mathcal{O}^*)^2$ onto $\dot{k}/\dot{k}^2$. Thus $(\mathcal{O}^* \cap \Sigma\dot{F}^2 : (\mathcal{O}^*)^2) = 2$. Hence $((\mathcal{O}^* \cap \Sigma\dot{F}^2)\dot{F}^2 : \dot{F}^2) = 2$, as desired.

Now, Proposition 2.5 of [En] implies that $B_\pi(F) = \dot{F}^2$ and by ([En2], Lemma 5.6) $\Sigma\dot{F}^2 = D\langle 1,1\rangle$.

□

Observe that $\dot{k} = \dot{k}^2$ also implies $B_\pi(F) = \dot{F}^2$ ([En2], Proposition 2.5).

Next, let us organize some easy facts concerning $\widehat{W}(K)$. Following Lam ([L], Chapter 2, p. 34) we denote by $\widehat{I}(K)$ the kernel of the ring homomorphism "dimension" $dim : \widehat{W}(K) \longrightarrow \mathbb{Z}$. Let us write $\widehat{I}(\Sigma\dot{K}^2) = \widehat{I}(K) \cap \widehat{W}(\Sigma\dot{K}^2)$ and also $\widehat{W}(K)_t$ to denote the torsion subgroup of the additive group $\widehat{W}(K)$. Recall that if K is not formally real, then $\widehat{W}(\Sigma\dot{K}^2) = \widehat{W}(K)$ and $\widehat{I}(\Sigma\dot{K}^2) = \widehat{I}(K)$.

The next simple lemma gives a characterization of $\widehat{W}(\Sigma\dot{K}^2) \cap \widehat{W}(K)_t$.

Lemma 3.4. *For a field K of characteristic different from 2, $\widehat{I}(\Sigma\dot{K}^2) = \widehat{W}(\Sigma\dot{K}^2) \cap \widehat{W}(K)_t$.*
Consequently $\widehat{I}(K) = \widehat{W}(K)_t$, for a non-formally real field K.

Proof: Let $z = q_1 - q_2 \in \widehat{W}(\Sigma\dot{K}^2)$ and assume that there is $N \geq 1$ such that $Nz = 0$. By the construction of $\widehat{W}(K)$ this means that $Nq_1 \simeq Nq_2$. Therefore $dimq_1 = dimq_2$ and so $z \in \widehat{I}(\Sigma\dot{K}^2)$.

Take now $z = \langle a_1, \dots, a_n\rangle - \langle b_1, \dots, b_n\rangle \in \widehat{I}(\Sigma\dot{K}^2)$. Assume first $n = 1$. Since $a_1, b_1 \in \Sigma\dot{K}^2$, there is $M \geq 1$ such that $a_1 b_1^{-1} \in D(2^M\langle 1\rangle)$. Thus $a_1 b_1^{-1}(2^M\langle 1\rangle) \simeq (2^M\langle 1\rangle)$ and so $2^M\langle a_1\rangle \simeq 2^M\langle b_1\rangle$. Hence $z \in \widehat{W}(K)_t$ as desired. To complete the proof we proceed by induction on n.

□

We know from ([L], Proposition 1.2, p. 35) that $\widehat{I}(K)$ is additively generated by elements of the form $1 - \langle a\rangle$, $a \in \dot{K}$. The same argument shows that $\widehat{I}(\Sigma\dot{K}^2)$ is additively generated by $1 - \langle a\rangle$, $a \in \Sigma\dot{K}^2$. Hence the ideals $\widehat{I}^2(\Sigma\dot{K}^2)$, $\widehat{I}^3(\Sigma\dot{K}^2)$ are additively generated by the expressions $(1 - \langle a\rangle)(1 - \langle b\rangle)$ and $(1 - \langle a\rangle)(1 - \langle b\rangle)(1 - \langle c\rangle)$, $a, b, c \in \Sigma\dot{K}^2$ respectively.

The next property is well-known and we produce it here to highlight that its restriction to $\Sigma\dot{K}^2$ holds.

Lemma 3.5. *Let K be a field of characteristic $\neq 2$. The map $g(a\dot{K}^2) = (1 - \langle a\rangle) + \widehat{I}^2(\Sigma\dot{K}^2)$ is an isomorphism from $\Sigma\dot{K}^2/\dot{K}^2$ onto $\widehat{I}(\Sigma\dot{K}^2)/\widehat{I}^2(\Sigma\dot{K}^2)$.*

Proof: The map g is clearly well-defined. It is a homomorphism since $(1-\langle a\rangle) + (1 - \langle b\rangle) - (1 - \langle ab\rangle) = (1 - \langle a\rangle)(1 - \langle b\rangle) \in \widehat{I}^2(\Sigma\dot{K}^2)$. As $\widehat{I}(\Sigma\dot{K}^2)$ is generated by $1 - \langle a\rangle$, $a \in \Sigma\dot{K}^2$, g is surjective. To see the injectivity take $a \in \Sigma\dot{K}^2$ such that $1 - \langle a\rangle \in \widehat{I}^2(\Sigma\dot{K}^2)$. Let $z = \sum_{i=1}^{n}(1 - \langle s_i\rangle)(1 - \langle t_i\rangle) \in \widehat{I}^2(\Sigma\dot{K}^2)$

verifying $1 - \langle a \rangle = z$ in $\widehat{W}(\Sigma \dot{K}^2)$. Then $1 - \langle a \rangle = \sum_{i=1}^n \langle 1, s_i t i \rangle - (\sum_{i=1}^n \langle s_i, t_i \rangle)$, in $\widehat{W}(\Sigma \dot{K}^2)$. Thus $\langle a \rangle + \sum_{i=1}^n \langle 1, s_i t i \rangle \simeq \langle 1 \rangle + \sum_{i=1}^n \langle s_i, t_i \rangle$. Taking determinant, we have $a s_1 t_1 \cdots s_n t_n \dot{K}^2 = s_1 t_1 \cdots s_n t_n \dot{K}^2$. Therefore $a \in \dot{K}^2$.

□

The next result will provide the converse of the Main Theorem under the limitations imposed by Proposition 3.3.

Theorem 3.6. *Let F be a formally real field such that $\widehat{W}(\Sigma \dot{F}^2) \cong \widehat{W}(k)[T]/\mathfrak{a}$, (ring isomorphism) where k is not formally real and char $k \neq 2$, T is an abelian group of exponent 2, and $\mathfrak{a}$ is the ideal generated by $\{t\mathbb{H} - \mathbb{H} \mid t \in T$ and $\mathbb{H} = \langle 1, -1 \rangle\}$.*
Furthermore, we assume that $(\dot{k} : \dot{k}^2) = \mid T \mid$ if $\mid T \mid \leq 2$.
Then F admits a π-henselian valuation ring A verifying the conditions of the Main Theorem.

Proof: We first consider the case $\mid T \mid = 1$ and $\dot{k} = \dot{k}^2$. Then $\widehat{W}(\Sigma \dot{F}^2) \cong \widehat{W}(k) \cong \mathbb{Z}$. Hence $\widehat{W}(\Sigma \dot{F}^2)$ is torsion free, and so $\widehat{I}(\Sigma \dot{F}^2) = \{0\}$. On the other hand, for $r \in \Sigma \dot{F}^2$, $\langle r \rangle - \langle 1 \rangle \in \widehat{I}(\Sigma \dot{F}^2)$. Hence $\langle r \rangle \simeq \langle 1 \rangle$, and so $r \in \dot{F}^2$. Thus F is pythagorean, which trivially implies that every valuation ring A of F is π-henselian.

We assume now $\mid T \mid \geq 2$. Let Δ be a totally ordered abelian group such that $\Delta / 2\Delta \cong T$. Take now $K = k((X))^\Delta$. As in Corollary 2.5, $\widehat{W}(K) \cong \widehat{W}(k)[T]/\mathfrak{a}$ and so $\widehat{W}(K) \cong \widehat{W}(\Sigma \dot{F}^2)$. Recall that the valuation ring $\mathcal{O} = k[[X]]^\Delta$ is henselian with residue field k and value group Δ. Hence $\dot{K}/\dot{K}^2 \cong \dot{k}/\dot{k}^2 \times \Delta/2\Delta \cong \dot{k}/\dot{k}^2 \times T$. For further reference let us denote by w the valuation corresponding to $\mathcal{O}$.

Next, denote by θ the isomorphism $\widehat{W}(K) \longrightarrow \widehat{W}(\Sigma \dot{F}^2)$. By Lemma 3.4 $\theta(\widehat{I}(K)) = \widehat{I}(\Sigma \dot{F}^2)$ (observe that K is not formally real). So θ induces the following isomorphisms

$$\theta_1 : \widehat{I}(K)/\widehat{I}^2(K) \longrightarrow \widehat{I}(\Sigma \dot{F}^2)/\widehat{I}^2(\Sigma \dot{F}^2)$$

$$\theta_2 : \widehat{I}^2(K)/\widehat{I}^3(K) \longrightarrow \widehat{I}^2(\Sigma \dot{F}^2)/\widehat{I}^3(\Sigma \dot{F}^2)$$

From θ_1 and the previous lemma we get an isomorphism $\Theta : \dot{K}/\dot{K}^2 \longrightarrow \Sigma\dot{F}^2/\dot{F}^2$ where $\Theta(a\dot{K}^2) = s\dot{F}^2$ if $\theta_1((1-\langle a\rangle) + \widehat{I}^2(K)) = (1-\langle s\rangle) + \widehat{I}^2(\Sigma\dot{F}^2)$.

For every $x \in K$ such that $w(x) \notin 2\Delta$ take $s \in \Sigma\dot{F}^2$ verifying $\Theta(x\dot{K}^2) = s\dot{F}^2$.

We claim that s is rigid (Definition 3.1). Observe that $a \in D\langle 1, s\rangle$, if and only if $a\langle 1,s\rangle \simeq \langle 1,s\rangle$ ([L], Corollary 1.7, p. 279), if and only if $(1-\langle a\rangle)(1+\langle s\rangle) = 0$ in $\widehat{W}(\Sigma\dot{F}^2)$. As $(1-\langle a\rangle)(1+\langle s\rangle) = (1-\langle a\rangle)^2 + (1-\langle a\rangle)(1-\langle s\rangle) \in \widehat{I}^2(\Sigma\dot{F}^2)$, if $y \in K$ verifies $\Theta(y\dot{K}^2) = a\dot{F}^2$, then $\theta_2\big((1-\langle y\rangle)(1+\langle x\rangle) + \widehat{I}^3(K)\big) = \big((1-\langle a\rangle)(1+\langle s\rangle) + \widehat{I}^3(\Sigma\dot{F}^2)\big) = 0$. Thus $(1-\langle y\rangle)(1+\langle x\rangle) \in \widehat{I}^3(K)$ which implies $(1-\langle y\rangle)(1+\langle x\rangle) \in I^3(K)$ in $W(K)$. On the other side, since $(1-\langle y\rangle)(1+\langle x\rangle) = (1-\langle y\rangle)(1-\langle -x\rangle)$ in $W(K)$ we have that $(1-\langle y\rangle)(1-\langle -x\rangle)$ is hyperbolic by ([L], Corollary 3.4, p. 290). Hence ([L], Theorem 2.7, p. 58) implies that $y \in D\langle 1, x\rangle$. Since $\mathcal{O}$ is henselian and $w(x) \notin 2\Delta$, it follows from ([En], Proposition 2.5 (1)) that x is rigid. Therefore $y \in \dot{K}^2 \cup x\dot{K}^2$ and so $a \in \dot{F}^2 \cup s\dot{F}^2$. Hence s is rigid as claimed.

Observe now that Θ induces an isomorphism $\dot{k}/\dot{k}^2 \times \Delta/2\Delta \longrightarrow \Sigma\dot{F}^2/\dot{F}^2$. Therefore we may assume that $\Theta(\Delta/2\Delta)$ is a subgroup of $\Sigma\dot{F}^2/\dot{F}^2$. By the claim above $\Theta(\Delta/2\Delta) \cap B_\pi(F) = \{1\}$. Hence $\mid \Sigma\dot{F}^2/B_\pi(F) \mid \geq \mid \Delta/2\Delta \mid$ and so the conditions of Proposition 3.3 are verified and the result is proved.

□

Remark. *(1)* If F is pythagorean, by ([En], Proposition 4.1), k_A is quadratically closed for every valuation ring A such that k_A is not formally real of characteristic not 2. Hence $\widehat{W}(\Sigma\dot{F}^2) \cong \mathbb{Z} \cong \widehat{W}(k_A)$. Consequently, the condition $\widehat{W}(\Sigma\dot{F}^2) \cong \mathbb{Z}$ characterizes pythagorean fields.
(2) Let k_A and Γ_A be the residue field and the value group of the valuation ring A of the last theorem. We may conclude that $\dot{k}_A/\dot{k}_A^2 \cong \dot{k}/\dot{k}^2$ and $\Gamma_A/2\Gamma_A \cong \Delta/2\Delta$, but we cannot state any further relationship between k_A and k, Γ_A and Δ.
(3) Disregarding one exceptional case, the valuation ring A of our Main Theorem can be chosen such that $B_\pi(F) = A^*\dot{F}^2 \cap \Sigma\dot{F}^2$, by Proposition 3.3. Hence $T \cong \Sigma\dot{F}^2/B_\pi(F)$ and $\dot{k}/\dot{k}^2 \cong B_\pi(F)/\dot{F}^2$. Actually, if $\widehat{W}(B_\pi(F))$ denotes the

subring of $\widehat{W}(\Sigma\dot{F}^2)$ generated by $\langle b\rangle$, $b \in B_\pi(F)$ and φ is the canonical projection $A \longrightarrow k$, then φ induces an isomorphism $\widehat{W}(B_\pi(F)) \cong \widehat{W}(k)$, where $\langle b_1, \dots, b_n\rangle \longmapsto \langle\varphi(b_1), \dots, \varphi(b_n)\rangle$.
(4) Let $u \in \Sigma\dot{F}^2$ be the element in the proof of the Main Theorem which corresponds to $-1 \in k$. Since $\Sigma\dot{F}^2 = D\langle 1, u\rangle$ it follows that $\langle a\rangle\langle 1, u\rangle \simeq \langle 1, u\rangle$, for every $a \in \Sigma\dot{F}^2$. Thus $\widehat{W}(\Sigma\dot{F}^2)\langle 1, u\rangle = \mathbb{Z}\langle 1, u\rangle$ and this ideal corresponds to the ideal $\mathbb{Z}\mathbb{H}$ by the isomorphism $\widehat{W}(\Sigma\dot{F}^2) \cong \widehat{W}(k)[T]/\mathfrak{a}$. Therefore $\widehat{W}(\Sigma\dot{F}^2)/\mathbb{Z}\langle 1, u\rangle \cong W(k)[T]$.

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