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MONODROMY OF PROJECTIONS

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§0. Introduction

Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n and degree d, over the complex numbers. For each finite linear projection $\pi : X \to \mathbb{P}^n$ let $M(\pi)$ be the monodromy group of π, a subgroup of the symmetric group $\mathbb{S}_d$, defined up to conjugacy. Also, we denote by $M(X)$ the finite collection of all $M(\pi)$ for varying π. This is a discrete invariant associated to X.

The first two sections of this article contain preliminaries about the definition of $M(X)$. Some basic questions are posed in (2.8). The only new result is the calculation in (2.9) of $M(X)$ when X is a general plane curve.

In §3 we sketch Lazarsfeld's application to monodromy groups of linear series of the case in which X is a Grassmannian in its Plücker imbedding. This case was the main motivation for our work.

Section four is a report on an article by Zariski. In that article Zariski classifies the projections of a rational normal curve such that the monodromy group is a Frobenius group. The main result is stated in (4.22).

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§1. Coverings

Let X be a connected and locally path connected topological space and $f : Y \to X$ a covering (i.e. locally trivial with discrete fibers). Fix a point $x_0 \in X$ and let $F = f^{-1}(x_0)$. By lifting paths, the group $\pi = \pi_1(X, x_0)$ acts on F, giving rise to the monodromy (covariant) functor

$$\mu : \{\text{Coverings of } X\} \to \{\text{left } \pi - \text{sets}\}$$

The basic theorem of covering space theory (see [S] for example) may be formulated as saying that

$$\mu \ \text{ is an equivalence of categories.} \tag{1.1}$$

We consider the diagram

$$\pi \to \mathbb{S}(F) \hookleftarrow \text{Aut}(Y|X) \tag{1.2}$$

where $\mathbb{S}(F) = \text{Aut}(F)$ is the symmetric group of the set F, the left map is monodromy and the right map is restriction. We denote $M = M(Y|X) = \text{image}(\pi \to \mathbb{S}(F))$ the monodromy group of the covering and $D = D(Y|X) = \text{image}(\text{Aut}(Y|X) \hookrightarrow \mathbb{S}(F))$ the group of deck transformations.

Proposition 1.3. *$dm = md$ for all $d \in D$ and $m \in M$.*
In other words, $M \subset ZD$ and $D \subset ZM$, where Z denotes centralizer of a subgroup of the symmetric group.

Proof. It is straightforward: Suppose $d = \sigma|_F \in D$ for $\sigma \in \text{Aut}(Y|X)$ and $m \in M$ is induced by $\gamma \in \pi$. If $y_0 \in F$ is a point in the fiber, let Γ be a lift of γ such that $\Gamma(0) = y_0$. Then $dm(y_0) = \sigma(\Gamma(1))$. On the other hand, since $\Gamma' = \sigma \circ \Gamma$ is a lift of γ with initial point $\sigma(y_0)$, we have $md(y_0) = \Gamma'(1) = \sigma(\Gamma(1))$, as wanted.

Proposition 1.4. $D = ZM$.

Proof. If $\sigma \in \mathbb{S}(F)$ satisfies $\sigma m = m\sigma$ for all $m \in M$ then σ is a π-map and then, by (1.1), it belongs to D.

Remark 1.5. The equality $M = ZD$ does not hold in general. It holds if and only if the monodromy group M satisfies $M = ZZM$. Two cases where this happens are:
(i) $M = \mathbb{S}(F)$ (here $ZM = \{1\}$), and
(ii) if H is any group, consider the left and right regular representations of H

$$H \xrightarrow{L} \mathbb{S}(H) \xleftarrow{R} H$$

defined by $L(h)k = hk$ and $R(h)k = kh^{-1}$. It is easy to check that $ZL(H) = R(H)$, $ZR(H) = L(H)$ and hence $ZZL(H) = L(H)$.

We may combine (i) and (ii) as follows:

Let $n \in \mathbb{N}$ and H a group. Let M be the semi-direct product $\mathbb{S}_n \times H^n$, where $\mathbb{S}_n$ acts on H^n by permuting the factors (M is called the wreath product $\mathbb{S}_n \wr H$). We consider the action of M on the disjoint union $\{1, \ldots, n\} \times H$ of n copies of H such that $\mathbb{S}_n$ permutes the copies and H acts regularly on the right on each copy, that is,

$$(\sigma, h_1, \ldots, h_n).(j, h) = (\sigma(j), h.h_j^{-1}$$

for $\sigma \in \mathbb{S}_n$, $h, h_i \in H$ and $1 \leq j \leq n$. It is easily seen that the centralizer of M is the left regular action of H defined by $k.(j, h) = (j, k.h)$, $(h, k \in H)$, and the double centralizer of M coincides with M.

Conversely, (after conversation with R. Steinberg)

Proposition 1.6. *A transitive permutation group which equals its double centralizer is isomorphic to a wreath product as in (1.5).*

Proof. First we make some general remarks.

(1.7) If F is a finite set and $M \subset \mathbb{S}(F)$ is a transitive permutation group, let M' denote the stabilizer of some $f \in F$. If $M/M' = \{m.M' | m \in M\}$ denotes the set of right cosets then M acts on M/M' by $n.(m.M') = nm.M'$ and the map $M/M' \to F$ sending $m \mapsto m(f)$ is an isomorphism of left M-sets.

If M is an abstract group and $M' \subset M$ a subgroup, the action of M on M/M' gives a homomorphism

$$\rho : M \to \mathbb{S}(M/M')$$

with kernel $M'' = \bigcap\{mM'm^{-1},\ m \in M\}$. The normalizer NM' of M' in M acts on M/M' by right multiplication, giving a homomorphism

$$\rho' : NM' \to \mathbb{S}(M/M')$$

It is easy to check that $\ker(\rho') = M'$ and $\text{image}(\rho') = Z\rho(M)$.

Let us remark that (1.2) may be redrawn as

$$M/M'' \hookrightarrow \mathbb{S}(M/M') \hookleftarrow NM'/M' \tag{1.8}$$

where the left arrow is left multiplication, and corresponds to monodromy, and the right arrow is right multiplication, and corresponds to deck transformations.

The action of NM'/M' on M/M', given by $mM'.n = mM'n^{-1} = mn^{-1}M'$ ($m \in M, n \in NM'$), preserves the partition of M/M' into right cosets $M/M' = \bigsqcup_{m \in M/NM'} m.NM'/M'$. Also, the action on each coset corresponds to the right regular action of the group NM'/M'. By (1.5), the centralizer of NM'/M' (that is, the double centralizer of M/M'') is the wreath product $\mathbb{S}(M/NM') \wr (NM'/M')$. In particular, if $M \subset \mathbb{S}(F)$ is a permutation group ($M'' = \{1\}$) such that $M = ZZM$ then M is a wreath product, as claimed.

(1.9) If X is path connected and $f : Y \to X$ is a covering with fiber $F = f^{-1}(x_0)$, then f corresponds to a homomorphism $\pi \to \mathbb{S}(F)$, which we may consider as consisting of two data: a homomorphism $m : \pi \to M$ of the fundamental group $\pi = \pi_1(X, x_0)$ into the group M, and a permutation representation $M \hookrightarrow \mathbb{S}(F)$. The associated Galois cover $\tilde{f} : \tilde{Y} \to X$ is, by definition, the covering corresponding to

$$\pi \xrightarrow{m} M \xrightarrow{L} \mathbb{S}(M)$$

(we only change the permutation representation, the new one being the left regular representation of the abstract monodromy group M). The Galois group $G(Y|X)$ is defined to be the group of deck transformations $D(\tilde{Y}|X)$ of the associated Galois covering. Notice that $G(Y|X)$ is abstractly isomorphic to the monodromy group $M(Y|X)$, but their natural actions are by right and left translations, respectively (see (1.8)).

A connected covering is said to be Galois when it is isomorphic to its associated Galois cover; or, equivalently, when D is transitive, or also, when $\mathrm{card}(D) = \mathrm{card}(M)$ (notice that since D acts with trivial stabilizers and M is transitive, the inequalities $\mathrm{card}(D) \leq \mathrm{card}(F) \leq \mathrm{card}(M)$ hold true).

As an example, any connected covering with abelian monodromy group is Galois; this follows from the fact that a transitive abelian group is regular, i.e. it is acting on itself by translations, see [W].

If f corresponds to the subgroup $M' \subset M$ as in (1.7) then the M-map $M \to M/M'$ gives a factorization of $\tilde{f}$ as $\tilde{Y} \to Y \xrightarrow{f} X$. Given a group homomorphism $\pi_1(X, x_0) \to M$ it induces a Galois cover $\tilde{Y} \to X$; intermediate covers $\tilde{Y} \to Y \to X$ correspond to subgroups of M.

§2. Projections

(2.1) If $f : X \to Y$ is a finite map of smooth complex algebraic varieties, we define the monodromy of f as the monodromy of the covering $f^{-1}(U) \to U$, where $U \subset Y$ is the complement of the branch locus, with its complex topology.

(2.2) Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n and degree d. Assume X is not contained in a hyperplane. For each linear subspace $L^{r-n-1} \subset \mathbb{P}^r$ disjoint from X we consider the projection $p_L : \mathbb{P}^r - L \to \mathbb{P}^n$ with center L, and its restriction to X

$$f_L : X \to \mathbb{P}^n$$

We denote $M_L(X) = M_L \subset \mathbb{S}_d$ the monodromy group of f_L. Notice that M_L is defined only up to conjugacy; it depends on the choice of a fiber F of f_L and a bijection $F \to \{1, \dots, d\}$.

Proposition 2.3. If X is smooth and connected, for general L the monodromy group $M_L(X)$ is the whole symmetric group $\mathbb{S}_d$.

Proof. By intersecting X with a general linear space, it is enough to consider the case where X is a curve. In fact, if $f_L : X \to \mathbb{P}^n$ is a projection as above, consider a general one dimensional linear subspace $\mathbb{P}^1 \cong P \subset \mathbb{P}^n$ and let $X' = f_L^{-1}(P) = L' \cap X$, where $L' = p_L^{-1}(P)$. By Bertini's Theorem (see [Ha]) X' is a smooth linear section of X of dimension one. We have a cartesian (pull-back) diagram

Since it is clear that the monodromy group M_L contains the monodromy group $M_{L'}$, it would be enough to see that $M_{L'}$ is the whole symmetric group $\mathbb{S}_d$, as claimed.

Suppose then that $X \subset \mathbb{P}^r$ is a curve.

(2.4) For further use, we now recall the fact that for a finite map of curves, the monodromy group is generated by the local monodromies.

More precisely, let $f : X \to Y$ be a finite map of degree d between smooth connected complete algebraic curves over the complex numbers. Suppose $y_1, \dots, y_w \in Y$ are the branch points of f. We have an equality of divisors on X

$$f^*(y_i) = \sum_{1 \le j \le d_i} e_{ij} x_{ij}$$

where $f^{-1}(y_i) = \{x_{i1}, \dots, x_{id_i}\}$ and $e_{ij} \in \mathbb{N}$ is the ramification index of f at x_{ij}; they satisfy $\sum_{1 \le j \le d_i} e_{ij} = d$ for $1 \le i \le w$. Let $Y' = Y - \{y_1, \dots, y_w\}$. Let us fix a point $y \in Y'$ and choose $\gamma_i \in \pi_1(Y', y)$ such that the index of rotation of γ_i around y_j is zero (resp. non-zero) for $j \ne i$ (resp. $j = i$). Then $\gamma_1, \dots, \gamma_w$ generate the kernel of $\pi_1(Y', y) \to \pi_1(Y, y)$.

Let us now assume $Y = \mathbb{P}^1$. Then the γ_i generate $\pi_1(Y', y)$. Also, $\pi_1(Y', y)$ is the free group generated by the γ_i with the relation $\prod_i \gamma_i = 1$ (but we will not need this last fact).

Denote $\sigma_i \in \mathbb{S}(F)$ the monodromy action of γ_i on $F = f^{-1}(y)$. We call σ_i the local monodromy around y_i. Since the γ_i generate $\pi_1(Y', y)$, the σ_i generate the monodromy group of f.

Also, we claim that σ_i is a product of disjoint cycles of orders e_{ij}, for $1 \le j \le d_i$. To see this it is convenient to choose the γ_i in the following explicit way: let $U_i \to D_i$ be a holomorphic coordinate chart around y_i, where $D_i \subset \mathbb{C}$ is an open disk in the complex plane. Let s_i be a small closed simple loop around y_i, image of a circle in D_i, and let t_i be a path joining y to a point of s_i; then take $\gamma_i = t_i^{-1} s_i t_i$. The covering

$$f|_{U_i} : f^{-1}(U_i - \{y_i\}) \to U_i - \{y_i\}$$

is a disjoint union of coverings isomorphic to $f_{ij} : D_i - \{0\} \to D_i - \{0\}$,

$f_{ij}(z) = z^{e_{ij}}$. The claim easily follows from this.

For example, if f has simple ramification (i.e. $d_i = 1$ for all i, and $e_{i1} = 2$) then the monodromy group is generated by transpositions. We refer to [F] for more details.

Now let us continue with the proof of (2.3). A general projection to $\mathbb{P}^2$ sends X birationally onto a nodal curve X', and projecting X' from a point which is not on a bitangent line or a flex line we obtain a linear projection $X \to \mathbb{P}^1$ with only simple ramification, and hence the monodromy group is generated by transpositions. The proof is completed by

Lemma 2.5. *If $M \subset \mathbb{S}_d$ is a transitive subgroup generated by transpositions then $M = \mathbb{S}_d$.*

Proof. Let T denote the set of transpositions which belong to M, so that T generates M. Since $T \neq \emptyset$, by renumbering we may assume $(12) \in T$. If $T.\{1,2\} = \{1,2\}$ then M would not be transitive (unless $d = 2$), so there exists $\tau = (ab) \in T$ with $a = 1$ or 2 and $b > 2$; by renumbering we may assume $(13) \in T$. Then, M contains $\mathbb{S}\{1,2,3\}$ = group generated by (12) and (13). If $d > 3$, since M is transitive $T.\{1,2,3\} \neq \{1,2,3\}$, T contains, say, (14); it follows that $M \supset \mathbb{S}\{1,2,3,4\}$. Continuing in this way we obtain the result.

Example 2.6. If $X \to \mathbb{P}^1$ is finite of degree $d > 2$ with simple branching then $\mathrm{Aut}(X|\mathbb{P}^1) = \{1\}$. In fact, by (2.5) the monodromy group is the full symmetric group and its centralizer is then trivial (see (1.4)).

Definition 2.7. *With notation as in (2.2), we let*

$$M(X) = \{M_L(X) : L \in \mathrm{Grass}(r-n-1, \mathbb{P}^r),\ L \cap X = \emptyset\}$$

(2.8) We remark that $M(X)$ is a collection of conjugacy classes of subgroups of a symmetric group; hence, it is a combinatorial invariant associated to the projective variety X. If $\mathcal{H}$ is a component of a Hilbert scheme, we define $M(\mathcal{H})$ as $M(X)$ for X a general point of $\mathcal{H}$. One may ask various questions about this invariant: is it trivial ? (i.e. are there $\mathcal{H}$'s for which $M(\mathcal{H}) \neq \{\mathbb{S}_d\}$), does it distinguish irreducible components of a Hilbert scheme ? (for example for curves in 3-space).

The following proposition answers such questions in a very modest case.

Proposition 2.9. If $X \subset \mathbb{P}^2$ is a general plane curve of degree d then $M(X) = \{\mathbb{S}_d\}$.

Proof. Let X be a plane curve of degree d without special bitangents or flex tangents (X is smooth, all bitangents are simply tangent at two points, all flex tangents have contact three at the flex, and no three bitangents or flex lines are concurrent). The general X satisfies these conditions. If $p \in \mathbb{P}^2 - X$ we claim that the monodromy M_p of the projection $f_p : X \to \mathbb{P}^1$ is the full symmetric group. To see this, let's first observe that if f_p is branched at the points $q_1, \ldots, q_w \in \mathbb{P}^1$ then M_p is generated by w permutations, but since the product of these w permutations is 1, M_p is actually generated by any $w - 1$ of them. We now distinguish a few cases:

(a) p belongs to at most one bitangent or flex line (or none).
(b) p is the intersection of two bitangents.
(c) p is the intersection of a bitangent and a flex line.
(d) p is the intersection of two flex lines.

In case (a) M_p is generated by transpositions, thus by (2.5) it is the full symmetric group. In cases (b) and (c) M_p is generated by (12)(34) and transpositions, and in case (d) by a 3-cycle and transpositions. The proof is finished by using the next two Lemmas.

Lemma 2.10. *If $M \subset \mathbb{S}_d$ is a transitive group generated by* (12)(34) *($d \geq 4$)*

and transpositions then $M = \mathbb{S}_d$.

Proof. Let $(12)(34) = \sigma$ and suppose $d \geq 5$ first. Since M is transitive, there is a transposition $\tau \in M$ that does not preserve the set $\{1,2,3,4\}$, and by renumbering we may assume $\tau = (15)$. Then, $(\sigma\tau)^3\sigma = (12) \in M$ and M is generated by transpositions. By (2.5), $M = \mathbb{S}_d$.
In case $d = 4$, since (σ) is not transitive there must be some transposition $\tau \in M$. If, for example, $(13) \in M$ (other cases are similar) then $(13)\sigma = (1234) \in M$ and then $M = \mathbb{S}_4$ since it contains a 4-cycle and a transposition.

Lemma 2.11. *If* $M \subset \mathbb{S}_d$ *is a transitive group generated by a 3-cycle and transpositions then* $M = \mathbb{S}_d$*, unless* $d = 3$ *and* M *is generated by the 3-cycle.*

Proof: If $d = 3$ and M contains a transposition then clearly $M = \mathbb{S}_3$, and if M does not contain any transpositions we are in the exceptional case. Considering $d \geq 4$, we may assume by renumbering that $\sigma = (123) \in M$. There exists a transposition $\tau \in M$ that does not preserve $\{1,2,3\}$, and we may assume $\tau = (14)$. Then, $\sigma\tau\sigma^{-1} = (24) \in M$ and $\sigma(24)\sigma^{-1} = (34) \in M$. Hence, $M \supset \mathbb{S}\{1,2,3,4\}$ and thus σ is a product of transpositions in M; then M is generated by transpositions and, by (2.5), is the full symmetric group.

Remark 2.12. The exceptional case in (2.11) corresponds to a cubic with three concurrent flex lines; this curve is a triple cyclic cover of $\mathbb{P}^1$ and hence has an automorphism of order three with fixed points, its j-invariant is then zero. Then, this case does not occur for a general cubic.

§3. Monodromy of linear series with $\rho = 0$

(3.1) Let X be a smooth projective variety of dimension n, and E a vector bundle on X of rank r. For each linear subspace $V \subset H^0(X, E)$ of dimension

r, consider the homomorphism

$$e_V : V \otimes \mathcal{O}_X \to E$$

restriction of the canonical homomorphism $H^0(X, E) \otimes \mathcal{O}_X \to E$ to $V \otimes \mathcal{O}_X$. If V is such that $\det(e_V) \in H^0(X, \wedge^r E)$ is non-zero then we denote by D_V the divisor of zeros of $\det(e_V)$, that is, D_V is the first degeneracy locus of e_V. Assuming that for general V it is true that $\det(e_V) \neq 0$, we have a rational map

$$\delta : \mathrm{Grass}(r, H^0(X, E)) \to \mathbb{P}H^0(X, \wedge^r E)$$

(3.2) We remark that δ factorizes as

$$\mathrm{Grass}(r, H^0(X, E)) \xrightarrow{\wp} \mathbb{P} \wedge^r H^0(X, E) \xrightarrow{\gamma} \mathbb{P}H^0(X, \wedge^r E)$$

where $\wp$ is the Plücker imbedding, sending V into $\wedge^r V$, and γ is the canonical linear map. Therefore, when γ is surjective, δ may be thought as a linear projection of a Grassmanian in its Plücker imbedding. Let us also notice that for δ to be finite one needs the numerical condition

$$h^0(X, \wedge^r E) - 1 = r(h^0(X, E) - r)$$

.

(3.3) Now let $f : X \to B$ be a family of smooth complete algebraic curves of genus g. Fix $r, d \in \mathbb{N}$ and let $\sigma_f : W^r_d(f) \to B$ denote the family of linear series of dimension r and degree d on the fibers of f. Suppose $\rho = g-(r+1)(g-d+r) = 0$ and σ_f is finite. Eisenbud and Harris considered in [EH] the question of what is the monodromy group of σ_f when f is a family of curves of genus g with general moduli.

(3.4) To prove, for instance, that such a monodromy group is the full symmetric group, it is enough to do it on a particular family $f : X \to B$ as above. Lazarsfeld proposed in [L] to consider a complete linear system $|C|$ of curves on a general K3 surface X. He showed that the problem reduces to determining the monodromy of a certain finite projection of a Grassmannian as in (3.2), with E a rank two bundle on X such that $\wedge^2 E = \mathcal{O}_X(C)$. The idea is that $\delta : \text{Grass}(2, H^0(X,E)) \to |C|$ is isomorphic to the family of linear series in the members of $|C|$. We refer to [L] for details. This construction motivated our work on monodromy of projections, but to our knowledge the case of Grassmanianns remains open. Specifically we ask, as a particular case of (2.8):

(3.5) If $G \subset \mathbb{P}^N$ is a Grassmannian in its Plücker embedding, what are the monodromy groups of the finite linear projections of G ?

§4. Zariski's article

In this section we give a report on [Z]. That article partially answers the questions posed in (2.8), in the particular case where $X \subset \mathbb{P}^d$ is a rational normal curve of degree d. By composing with the Veronese embeding, the question becomes:

(4.1) which groups occur as monodromy groups of degree d maps

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

We refer to [GT] for current work on this and related questions.

Let $f : X \to Y$ be a degree d map of Riemann surfaces, branched at $y_1, \ldots, y_w \in Y$, so that $f^*(y_i) = \sum_j d_{ij} x_{ij}$ with $\sum_j d_{ij} = d$. We have the Riemann-Hurwitz formula

$$2g(X) - 2 = d(2g(Y) - 2) + \sum_{i,j}(d_{ij} - 1) \tag{4.2}$$

If f is Galois then $d_{ij} = d_i$ (all j) and (4.2) reads

$$2g(X) - 2 = d(2g(Y) - 2) + d\sum_i 1 - \frac{1}{d_i} \tag{4.3}$$

If f is not necessarily Galois let's compute, for later use, the genus $g(\tilde{X})$ where $\tilde{f} : \tilde{X} \to Y$ is the associated Galois cover. The local monodromy $\sigma_i \in \mathbb{S}_d$ of f at y_i is a product of disjoint d_{ij}-cycles (see (2.4)) and hence its order is the least common multiple δ_i of the d_{ij}'s $(j = 1, 2, \dots)$. The monodromy group M of f is generated by $\sigma_1, \dots, \sigma_w$ and the local monodromy of $\tilde{f}$ at y_i is $L(\sigma_i)$ where $L : M \to \mathbb{S}(M)$ is the left regular representation (1.9). Being a regular element of order δ_i, $L(\sigma_i)$ is a product of $|M|/\delta_i$ disjoint δ_i-cycles. Applying (4.3) we obtain

$$2g(\tilde{X}) - 2 = |M|(2g(Y) - 2) + |M| \sum_{i=1}^{w} (1 - \frac{1}{\delta_i}) \tag{4.4}$$

Notice that $g(\tilde{X})$ is not determined by the branching data d_{ij} only, it also dependes on $|M|$, see (4.7).

Coming back to (4.1), we now review the well-known Galois case, due to Klein. If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is Galois of degree d, it follows from (4.3) that

$$2 - \frac{2}{d} = \sum_i 1 - \frac{1}{d_i} \geq \frac{w}{2} \tag{4.5}$$

and hence $w \leq 3$.

If $w = 2$, it follows from (4.5) that $d_1 = d_2 = d$; thus $\sigma_1 = \sigma_2^{-1}$ are both d-cycles and M is cyclic of order d.

If $w = 3$, choose notation so that $2 \leq d_1 \leq d_2 \leq d_3 \leq d$. If $d_1 \geq 3$ then (4.5) is violated, hence $d_1 = 2$ (d is then even) and (4.5) becomes

$$\frac{1}{d_2} + \frac{1}{d_3} = \frac{1}{2} + \frac{2}{d} \tag{4.6}$$

If $d_2 = 2$ then $d_3 = d/2$; M is generated by $\sigma_1, \sigma_2, \sigma_3$, the first two are products of $d/2$ disjoint 2-cycles, and σ_3 is the product of 2 disjoint $d/2$-cycles, satisfying $\sigma_1\sigma_2\sigma_3 = 1$; it follows that $\sigma_2\sigma_3\sigma_2 = \sigma_3^{-1}$ and thus M is dihedral.

If $d_2 = 3$ then (4.6) implies $(6 - d_3)d = 12d_3$, so $d_3 \leq 5$.

If $d_3 = 3$ then $d = 12$ and M is the tetrahedral group.

If $d_3 = 4$ then $d = 24$ and M is the octahedral group.

If $d_3 = 5$ then $d = 60$ and M is the icosahedral group.

Remark 4.7. In general, the local monodromies do not determine the monodromy group. For example, if $x = (1234) \in \mathbb{S}_4$, the data

(a) $x, x^2, x^{-1}, x, x^2, x^{-1}$ and
(b) $x, y, x^{-1}, x, y, x^{-1}$, with $y = (12)(34)$
both determine coverings of $\mathbb{P}^1$ with the same local monodromies, but the monodromy groups are cyclic and $\mathbb{S}_4$ respectively.
The previous analysis shows that for the Galois coverings the local monodromies do determine the monodromy group.

Zariski considers coverings $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that the monodromy group is a Frobenius group. He determines which such groups occur and, more importantly, how they occur. The result is stated in (4.22).

We recall some definitions and facts from group theory before carrying out Zariski's analysis. See for example [A], [Hall] for more details.

Definitions 4.8. *A permutation $\sigma \in \mathbb{S}_d$ has class s if it permutes exactly s letters (i.e. it leaves fixed exactly $d - s$ letters); σ is semi-regular if all the (non-trivial) cycles in its cycle decomposition have the same length, and σ is regular if it is semi-regular of class d.*
A subgroup $M \subset \mathbb{S}_d$ has class s if all its elements (except the identity) have class at least s and if it contains elements of class s. For example, a regular permutation group (that is, a group acting on itself by translations) consists of regular elements and has class d; it is easy to see that, conversely, a transitive group of class d is regular.
A group is a Frobenius group if it is isomorphic to a transitive subgroup $M \subset \mathbb{S}_d$ of class $d - 1$.

As an example, it is a theorem of Galois that a transitive subgroup $M \subset \mathbb{S}_d$, with d prime, has class $d - 1$ if and only if it is solvable. In this case, M is realized as a subgroup of the affine group of a one dimensional vector space over the field with d elements (see [R] and the introduction to [Z]).

(4.9) Zariski's motivation for considering coverings with Frobenius monodromy group is the following: if $K \subset L$ is the field extension obtained by adjoining a root of $p \in K[t]$ (char K $= 0$) then the extension is Galois when all other roots belong to L, that is, all roots are rational functions of any one of them, and the extension is Frobenius when all the roots are rational functions of any two of them. Then, Frobenius is the "next" case after Galois. In the context of coverings, let $f : X \to Y$ be a covering with monodromy group $M \subset \mathbb{S}_d$. Then f is Galois when the stabilizers M_i of a letter i are trivial, and f is Frobenius when the stabilizers $M_{i,j}$ of two letters are trivial, that is, when M has class $d-1$.

(4.10) Let $M \subset \mathbb{S}_d$ be a transitive group of class $d-1$, and define

$$M_i = \{m \in M : m(i) = i\} \quad (i = 1, \dots, d)$$

$$H = \{m \in M : m \text{ has class } d\} \cup \{1\}$$

The following is a well known theorem in group theory ([A], [Hall])

Theorem 4.11 (Frobenius). *H is a subgroup of M (clearly normal).*

(4.12) Since M is transitive, $|M| = de$ where $e = |M_i|$ for any i, and since the class of M is $d-1$, $M_i \cap M_j = \{1\}$ for $i \neq j$, and thus M is a disjoint union $M = H \coprod \coprod_{i=1}^{d} M_i - \{1\}$; then $de = |H| + d(e-1)$ and it follows that $|H| = d$. Combining this with the fact that every element of H is fixed-point-free, we conclude that H is transitive and hence regular. Also, M_i acts on $\{1, 2, \dots, d\} - \{i\}$ without fixed points; this implies that e is a divisor of $d-1$. Also, we claim that M_i is semi-regular: if $s \in M_i$ contained in its decomposition into disjoint cycles an a-cycle and a b-cycle with $a < b$ then $s^a \in M_i - \{1\}$ would have more than one fixed point.

Lemma 4.13. *If $M \subset \mathbb{S}_d$ is a transitive subgroup of order de with $e > 1$, then M is of class $d-1$ if and only if* *(a) M contains a normal subgroup H of order d*

(b) $hm \neq mh$ for $h \in H - \{1\}, m \in M - H$

(c) e divides $d - 1$

Proof. If M is Frobenius, (a) and (c) follow from (4.11) and (4.12), and (b) is clear since a permutation of class d cannot commute with one of class $d - 1$. Conversely, if M_i denotes the stabilizer of i,(c) implies that $e = |M_i|$ is coprime with $d = |H|$ and hence $M_i \cap H = \{1\}$. Hence, H is of class d and, as in (4.12), it is regular. Suppose that there existed $m \in M$ fixing two letters, say i and j. Since H is regular there exists a unique $h \in H$ such that $h(i) = j$. The element mhm^{-1} belongs to H (H is normal) maps i to j, and hence equals h, which contradicts (b).

(4.14) Let X be a Riemann surface of genus p and $f : X \to \mathbb{P}^1$ a map of degree d with monodromy group $M \subset \mathbb{S}_d$. Assume that M is of class $d - 1$; we will use the notation introduced in (4.10). If $\tilde{X} \to \mathbb{P}^1$ is the associated Galois map then the presence of the normal subgroup H provides the diagram

where d and e indicate degrees and all maps, except f, are Galois.

Denote the local monodromies of f by $\sigma_1, \ldots, \sigma_\alpha \in H$ and $\tau_1, \ldots, \tau_\beta \in M - H$ where σ_i has order s_i and hence is a product of d/s_i disjoint s_i-cycles, and τ_j has order t_j and is a product of $(d-1)/t_j$ disjoint t_j-cycles. By Riemann-Hurwitz,

$$\sum_i \frac{d}{s_i}(s_i - 1) + \sum_j \frac{(d-1)}{t_j}(t_j - 1) = 2d + 2p - 2 \tag{4.16}$$

Combining (4.4) and (4.16) we obtain

$$2g(\tilde{X}) - 2 = \frac{(2pd - r)e}{d - 1} \tag{4.17}$$

where $r = \sum_{i=1}^{\alpha} d(1 - 1/s_i)$, and it follows that

(4.18) If $p = 0$ then $g(\tilde{X}) = 0$ or 1.

Lemma 4.19. *With the notation above, we have* $2g(\tilde{X}) - 2 = 2g(\tilde{X}/H) - 2 + 2ep$.

Proof. the Galois covering $\tilde{f}$ corresponds to a homomorphism of the fundamental group $\pi_1 \to M$ and $g : \tilde{X}/H \to \tilde{X}/M$ corresponds to the composition $\pi_1 \to M \to M/H$. Under this map σ_i goes to zero and τ_j goes to an element of the same order t_j (because $M_k \cap H = \{1\}$). By Riemann-Hurwitz,

$$2g(\tilde{X}/H) - 2 = -2e + e \sum_j 1 - \frac{1}{t_j}$$

and the result follows by combining with (4.16) and (4.17).

Corollary 4.20. *If* $p = 0$ *and* $g(\tilde{X}) = 1$ *then* H *is a group of translations of the elliptic curve* $\tilde{X}$*.*

Proof. Otherwise H would contain elements with a fixed point and $g(\tilde{X}/H)$ would be zero, contradicting (4.19).

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree d with monodromy M of class $d - 1$. According to (4.18) we consider 2 cases: $g(\tilde{X}) = 0$ or 1.

Suppose $g(\tilde{X}) = 0$. Then $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is one of the Klein types.

(i) M cyclic is excluded by (4.13)(b).

(ii) since the icosahedral group is simple, it is excluded by (4.13)(a).

(iii) if $M \cong \mathbb{S}_4$ is octahedral, its only normal subroups are the Klein 4-group and the alternating group, both of which violate (4.13)(c).

(iv) if M is tetrahedral ($\cong$ alternating in 4 letters) then (4.13) is satisfied for H cyclic of order 4, and if M_1 is the group of symmetries of the tetrahedron fixing one vertex then

$$\mathbb{P}^1/M_1 \to \mathbb{P}^1/M$$

is Frobenius of degree 4. It is easy to check that the branching occurs at three points, with local monodromies
3-cycle + 1-cycle, 3-cycle + 1-cycle, 2-cycle + 2-cycle.

(v) if $M \cong D_d$ is dihedral then (4.13) is satisfied (only) with H cyclic of order d. By (4.13)(c), d is odd. If $\tau \in M - H$ is any element of order two then

$$f : \mathbb{P}^1/(\tau) \to \mathbb{P}^1/M$$

is Frobenius of order d. If $p = \{p_1, \ldots, p_d\}$ are the vertices of a regular d-gon on the ecuator of the Riemann sphere, $q = \{q_1, \ldots, q_d\}$ are the midpoints of the sides of the d-gon, and $P = \{N, S\}$ are the two poles, then it is easy to check that f is branched at the three orbits p, q and P with respective local monodromies
1-cycle + $(d-1)/2$ 2-cycles, 1-cycle + $(d-1)/2$ 2-cycles and d-cycle.

Now suppose $\tilde{X} = E$ is a curve of genus one. The group of linear equivalence classes of divisors on E of degree zero acts on E, denote by $T \subset \text{Aut } E$ the corresponding group of translations; T is a normal abelian subgroup and it is well known (see [Ha] for example) that $(\text{Aut } E)/T$ is cyclic of order 2, 4 or 6. We need to characterize finite subgroups $M \subset \text{Aut } E$ with a normal subgroup H satisfying (4.13). From (4.20), we know that H has to be a subgroup of T. Since T is abelian, (4.13)(b) implies that $M \cap T = H$, and then $M/H \hookrightarrow \text{Aut } E/T$. It follows that M/H is cyclic of order 2, 3, 4 or 6, and if $\mu \in M$ generates M modulo H then M is the semi-direct product $M = (\mu).H$. The translations of E are characterized as those automorphisms without fixed points. Let's choose a fixed point 0 of μ as an origin for a group structure on E; then μ is a group automorphism and H corresponds to a subgroup $K \subset E \cong \mathbb{S}^1 \times \mathbb{S}^1$. The condition that H is normal in M translates into the fact that K should be μ-stable and, by (4.13)(c), $|K| - 1$ should be divisible by the order of μ. We obtain a Frobenius map

$$f : E/(\mu) = \mathbb{P}^1 \to E/M = \mathbb{P}^1$$

of degree $d = |K|$ and monodromy M. Now we distinguish several cases:

(a) E is any elliptic curve; $\mu(x) = -x$.
Here $K \subset E$ is any subgroup with $d = |K|$ odd. To work out the branching pattern of f consider, more generally, the situation

(4.21) X is a curve, $M = H.G \subset \text{Aut } X$, $H \cap G = \{1\}$, $H \subset M$ is normal, $X \xrightarrow{g} X/G \xrightarrow{f} X/M$ are the natural maps and $\tilde{f} = fg$. For $x \in X$ let $y = \tilde{f}(x)$, then $g_* \tilde{f}^*(y) = g_* g^* f^*(y) = \deg g . f^*(y)$ and hence

$$f^*(y) = (\deg g)^{-1} \sum_{m \in M} g(m.x) = \sum_{Gm \in G \backslash M} g(m.x) = \sum_{h \in H} g(h.x)$$

since the natural map $H \to G \setminus M$ is bijective.
In the present case, let $\{0 = p_1, p_2, p_3, p_4\}$ be the 2-torsion points. Then, g is branched at p_i and unbranched at $h.p_i$ for $h \in H - \{1\}$. It follows that

$$f^*(\tilde{f}(p_i)) = \sum_{h \in H} g(h.p_i) = g(p_i) + 2 \sum_h g(h.p_i)$$

where the sum is over representatives of $\{h, h^{-1}\}$ for $h \in H - \{1\}$, and then f is branched over four points with local monodromies
$(d-1)/2$ 2-cyles + one 1-cycle

(b) $E = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ $(i^2 = -1)$; $\mu(z) = i.z,\ z \in \mathbb{C}$.
Now $K \subset E$ is an i-stable subgroup of order d with $d-1$ divisible by 4. The only points in E with non-trivial stabilizer under (μ) are the 2-torsion points, giving three orbits $\{0\}, \{1/2, i/2\}, \{1/2 + i/2\}$ and then, by (4.21), f is branched over three points with local monodromies
$(d-1)/4$ 4-cycles + one 1-cycle, $(d-1)/4$ 4-cycles + one 1-cycle, $(d-1)/2$ 2-cycles + one 1-cycle.

(c) $E = \mathbb{C}/\mathbb{Z} + \rho\mathbb{Z}$ $(\rho^2 - \rho + 1 = 0)$; $\mu(z) = \rho.z,\ z \in \mathbb{C}$ $(\mu^6 = 1)$.
$K \subset E$ is a ρ-stable subgroup of order d congruent to 1 modulo 6. The special

orbits of (μ) are $\{0\}, \{1/2, \rho/2, 1/2+\rho/2\}$ and $\{1/3+\rho/3, 2/3+2\rho/3\}$. It follows from (4.21) that f is branched over 3 points, with local monodromies $(d-1)/6$ 6-cycles + one 1-cycle, $(d-1)/3$ 3-cycles + one 1-cycle, $(d-1)/2$ 2-cycles + one 1-cycle.

(d) $E = \mathbb{C}/\mathbb{Z} + \rho\mathbb{Z}$ $(\rho^2 - \rho + 1 = 0)$; $\mu(z) = \rho^2.z$, $z \in \mathbb{C}$ $(\mu^3 = 1)$.
$K \subset E$ is a ρ-stable subgroup of order d congruent to 1 modulo 3. The fixed points of μ are $0, 1/3+\rho/3$ and $2/3+2\rho/3$, and hence f is branched over 3 points, with local monodromy
$(d-1)/3$ 3-cycles + one 1-cycle.

Summarizing, we proved the following

Theorem 4.22. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree d such that the monodromy group is a Frobenius group. Then there exists a curve $\tilde{X}$ and subgroups $M_1 \subset M \subset Aut(\tilde{X})$ such that*

$$f : \mathbb{P}^1 = \tilde{X}/M_1 \to \mathbb{P}^1 = \tilde{X}/M$$

The possible $(\tilde{X}, M, M_1)$ are the following:

a) $\tilde{X} = \mathbb{P}^1$, $M =$ alternating group in four letters, acting as direct symmetries of a regular tetrahedron, $M_1 =$ symmetries fixing one vertex. Here $d = 4$.

b) $\tilde{X} = \mathbb{P}^1$, $M = D_d$ dihedral group, acting as symmetries of a regular d-gon, $M_1 = (\tau) \subset D_d$ subgroup generated by an element $\tau \in D_d$ of order two.
In the next cases $\tilde{X} = E$ is an elliptic curve, with $0 \in E$ neutral element for the group structure. We consider $E \subset Aut(E)$ as the subgroup of translations. Also, $\mu \in Aut(E, 0)$ denotes a group automorphism, $K \subset E$ a subgroup and $M_1 = (\mu) \subset M = (\mu).K \subset Aut(E)$.

c) Here E is any elliptic curve, $\mu(x) = -x$ and $K \subset E$ any subgroup with *cardinality d odd.*

d) E is the special elliptic curve $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$ $(i^2=-1)$ with $\mu(z)=i.z$, $z\in\mathbb{C}$. The subgroup $K\subset E$ is any i-stable subgroup of order d, with $d-1$ divisible by 4.

e) $E=\mathbb{C}/\mathbb{Z}+\rho\mathbb{Z}$ $(\rho^2-\rho+1=0)$; $\mu(z)=\rho.z$, $z\in\mathbb{C}$. Here $K\subset E$ is any ρ-stable subgroup of order d, with $d-1$ divisible by 6.

f) $E=\mathbb{C}/\mathbb{Z}+\rho\mathbb{Z}$ $(\rho^2-\rho+1=0)$; $\mu(z)=\rho^2.z$, $z\in\mathbb{C}$. Here $K\subset E$ is any ρ-stable subgroup of order d, with $d-1$ divisible by 3.

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