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# SOME ASPECTS OF RADICAL THEORY

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**Abstract**

In this survey we attempt to glimpse the utility of radical theory in the development of ring theory. Various radicals, namely the Baer (prime), the Köthe (nil), the Jacobson, the Brown-McCoy, the torsion and the von Neumann regular radicals will be considered, and structure theorems concerning these radicals will be given. Distinguishing the different radicals led to interesting ring constructions which contributed substantially to the better understanding of the structure of rings. It is also an important issue when certain radicals coincide, for instance, when the Jacobson radical becomes nil or nilpotent. Radicals of related rings, such as matrix rings and polynomial rings, will be mentioned including also some very recent results. Common characteristic properties of radicals will be briefly touched. Finally, we shall illustrate by some recent results a typical way of developing radical theory: the introduction of new radicals led to new types of structure theorems and to the construction of a ring with unusual properties.

**1.** It was Wedderburn [30] in 1908 who used an ingenious technique in the study of rings and algebras. He discarded or ignored a "bad" ideal $R$ of a ring or algebra $A$ such that the factor ring $A/R$ had no longer "bad" ideals and, in addition, $A/R$ had a "nice" structure (representable by rings of linear transformations on vector spaces). In 1930 Köthe [16] considered the unique largest nil ideal

$$\mathcal{N}(A) = \sum (I \triangleleft A \mid I \text{ is a nil ring})$$

as a "bad" ideal of $A$, and determined the structure of $A/\mathcal{N}(A)$ in terms of rings of linear transformations. To each nonzero element $a \in \mathcal{N}(A)$, being nilpotent,

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there exists an exponent $n > 1$ with $a^n = 0$, whence the element $a \in \mathcal{N}(A)$ can be viewed as an $n$-th root of 0. Root in Latin is *radix*, so Köthe called the "pathological" ideal $\mathcal{N}(A)$ as the *radical* of $A$. Note that $\mathcal{N}(A)$ has also an intersection representation,

$$\mathcal{N}(A) = \cap(J \triangleleft A \,|\, A/J \text{ is a prime ring without nonzero nil ideals}).$$

Köthe posed also the problem: *does $\mathcal{N}(A)$ contain every nil left ideal of $A$?* Köthe's problem is still open, it is one of the central problems, and it seems to be the hardest one in ring theory.

This was the genesis of radical theory.

Later several other successful radicals have been introduced, based on various calculation rules. In 1943 Baer investigated the structure of rings which have no nonzero nilpotent ideals. These rings are the *semiprime rings.* The ideal

$$\beta(A) = \cap(I \triangleleft A \,|\, A/I \text{ is a prime ring})$$

of a ring $A$, is called the *Baer* or *prime radical* of $A$. A ring $A$ is semiprime if and only if $\beta(A) = 0$.

In a ring $A$ we may define an additional operation $\circ$ by

$$a \circ b = a + b - ab \qquad \forall a, b \in A.$$

An ideal $I$ of the ring is called a *quasi-regular* ideal, if $(I, \circ)$ is a group with unity element 0. The unique maximal quasi-regular ideal

$$\mathcal{J}(A) = \sum(I \triangleleft A \,|\, (I, \circ) \text{ is a group})$$

is called the *Jacobson radical* of $A$ ([15]). Important is that the Jacobson radical has also an intersection representation. A ring $A$ is called *(left) primitive*, if $A$ contains a maximal left ideal $L$ such that $xA \subseteq L$ implies $x = 0$, (or equivalently, there exists a faithful irreducible $A$-module). The Jacobson radical of $A$ is equal to

$$\mathcal{J}(A) = \cap(J \triangleleft A \,|\, A/J \text{ is a primitive ring}).$$

The Jacobson radical, introduced in 1945, is claimed to be the queen of all radicals, because it turned out to be the most successful to prove structure theorems for rings with 0 Jacobson radical.

In 1947 another radical, the *Brown–McCoy radical* has been introduced. Call a ring $A$ a *G-ring*, if each element $a \in A$ is contained in the set

$$\{ax - x + ya - y \mid x, y \in A\}$$

Then the Brown-McCoy radical $\mathcal{G}(A)$ of a ring $A$ is

$$\begin{aligned}\mathcal{G}(A) &= \sum(I \triangleleft A \mid I \text{ is a } G\text{-ring}) = \\ &= \cap(J \triangleleft A \mid A/J \text{ is a simple ring with } 1).\end{aligned}$$

All these four classical radicals share a common property:

$$\mathcal{N}(A) = \beta(A) = \mathcal{J}(A) = \mathcal{G}(A) = A,$$

whenever $A$ is a nilpotent ring. There are also other radicals of rings which do not have this property. We mention two of them.

The maximal torsion ideal

$$\tau(A) = \{a \in A \mid na = 0 \text{ for some integer } n \neq 0\}$$

may be regarded as the *torsion radical* of the ring $A$. In fact, $A/\tau(A)$ is torsionfree, so $\tau(A/\tau(A)) = 0$. Fields as well as nilpotent rings may be torsion rings.

A ring $A$ is a von *Neumann regular ring*, if $a \in aAa$ for every element $a \in A$. Every ring has a unique maximal von Neumann regular ideal $\nu(A)$, called the *von Neumann regular radical* of $A$. Again we have $\nu(A/\nu(A)) = 0$. Rings of linear transformations on vector spaces are von Neumann regular, so von Neumann regularity cannot be considered as a "bad" property of rings. On the other hand, $\gamma(A) = 0$ for every nilpotent ring $A$.

Though the torsion radical and the von Neumann regular radical look suspicious, they provide useful informations concerning the structure of rings, as we shall see later.

**2.** The factor rings $A/\mathcal{N}(A)$, $A/\beta(A)$, $A/\mathcal{J}(A)$ and $A/\mathcal{G}(A)$ have 0 Köthe, Baer, Jacobson and Brown–McCoy radical, respectively, so we expect to get structure theorems for such rings. If a radical has an intersection representation, then the rings with 0 radical have a subdirect decomposition, namely,

$$\begin{aligned}
\beta(A) = 0 \Leftrightarrow A &= \sum_{\text{subdirect}} (A/I_\lambda \mid \text{each } A/I_\lambda \text{ is a prime ring}),\\
\mathcal{N}(A) = 0 \Leftrightarrow A &= \sum_{\text{subdirect}} (A/I_\lambda \mid \text{each } A/I_\lambda \text{ is a prime ring without}\\
&\qquad \text{nonzero nil ideals}),\\
\mathcal{J}(A) = 0 \Leftrightarrow A &= \sum_{\text{subdirect}} (A/I_\lambda \mid \text{each } A/I_\lambda \text{ is a primitive ring}),\\
\mathcal{G}(A) = 0 \Leftrightarrow A &= \sum_{\text{subdirect}} (A/I_\lambda \mid \text{each } A/I_\lambda \text{ is a simple ring with } 1).
\end{aligned}$$

The primitive rings, the subdirect components of rings with 0 Jacobson radical, have a beautiful description. A ring $A$ is said to be a *dense subring of linear transformations on a vector space* $V$ , if for every finitely many linearly independent elements $x_1, \dots, x_n \in V$ and arbitrary elements $y_1, \dots, y_n \in V$ there exists a linear transformation $t \in A$ such that $tx_i = y_i$ for $i = 1, \dots, n$. *Jacobson's Density Theorem* states that a ring $A$ is primitive if and only if $A$ is isomorphic to a dense subring of linear transformations on a vector space over a division ring.

If a ring $A$ is *(left) artinian* (that is, $A$ satisfies the descending chain condition on left ideals), then

$$\beta(A) = 0 \Leftrightarrow \mathcal{N}(A) = 0 \Leftrightarrow J(A) = 0 \Leftrightarrow \mathcal{G}(A) = 0,$$

and the subdirect decomposition becomes a finite direct sum

$$A = A_1 \boxplus \dots \boxplus A_n$$

where each of $A_1, \dots, A_n$ is a simple artinian ring isomorphic to a matrix ring over a division ring, (that is, a ring of linear transformations on a finite dimensional vector space). This is the famous Wedderburn–Artin Structure Theorem.

Also the torsion radical and the von Neumann regular radical play a role, a different role, in the structure theory of rings. The theorem of F. Szász [26] tells us that *in every artinian ring $A$ the torsion radical is a direct summand.* A similar, even more explicit result is true for the von Neumann regular radical: $\nu(A)$ *is a direct summand in every artinian ring $A$* and $\nu(A)$ is a finite direct sum of matrix rings over division rings.

**3.** Beside the primary task of radical theory, secondary but not inferior problems arise in a natural way. The first question one may ask is: are the Köthe, Baer, Jacobson and Brown–McCoy radicals really different? The answer is affirmative. One can easily see that

$$\beta(A) \subseteq \mathcal{N}(A) \subseteq \mathcal{J}(A) \subseteq \mathcal{G}(A)$$

for all rings $A$.

Let $V$ be a countably infinite dimensional vector space. It is well-known that the subring

$$H = \{t \in Hom(V, V) \,|\, t(V) \text{ is finite dimensional}\}$$

of finite valued linear transformations is a simple ring without unity element, and, of course, a dense subring of linear transformations. Hence

$$0 = \mathcal{J}(H) \neq H = \mathcal{G}(H).$$

The set

$$A = \left\{ \frac{2x}{2y+1} \,|x, y \text{ are integers and g.c.d } (2x, 2y+1) = 1 \right\}$$

of all rationals of even numerator and odd denominator forms a subring of rationals. For any $a = \frac{2x}{2y+1} \in A$ the equation

$$a \circ z = a + z - az = 0$$

has a solution

$$z = \frac{a}{a-1} = \frac{2x}{2(x-y-1)+1} \in A,$$

whence $A$ is quasi-regular. Clearly $A$ has no nonzero nilpotent elements. Hence

$$0 = \mathcal{N}(A) \neq A = \mathcal{J}(A).$$

To show that $\beta(A) \neq \mathcal{N}(A)$ may happen, one has to construct, for instance, a nil ring $A$ which is also a prime ring. The construction of such rings is a very difficult and involved task. An example of such a ring was given first by Baer [7] (see also [27]), and another one by Zelmanov [31].

The definition of the Jacobson radical in terms of quasi-regularity is left and right symmetric, but the notion of primitivity is not. Bergman [10] (see also [27]) constructed a ring which is left but not right primitive. Although, as already seen, a Jacobson radical ring $A$ (that is, $\mathcal{J}(A) = A$) need not be a nil ring, the question arises: does there exist a simple idempotent Jacobson radical ring $A$, (that is, $\mathcal{J}(A) = A = A^2 \neq 0$)? An affirmative answer was given by Sąsiada [23], [24].

Levitzki posed a similar question which is still open: does there exist a simple idempotent nil ring $A$, (that is, $\mathcal{N}(A) = A = A^2 \neq 0$)?

Köthe's problem can be put also in the following form: does there exist a ring $A$ such that $A$ contains a nonzero nil left ideal and $\mathcal{N}(A) = 0$? Köthe's problem has many equivalent formulations. The interested reader is referred to Puczyłowski [19] and Rowen [21].

Constructing rings with peculiar properties exhibits how delicate or nasty (up to the reader's taste) the behaviour of rings may be. This activity contributes substantially to the better understanding of the structure of rings, and gives impetus for further researches.

**4.** One may investigate also the coincidence of certain radicals on a given class of rings.

On the class of artinian rings the Baer, Köthe, Jacobson and Brown–McCoy radicals coincide, that is,

$$\beta(A) = \mathcal{N}(A) = \mathcal{J}(A) = \mathcal{G}(A)$$

for all artinian rings $A$. Moreover, for any artinian ring $A$ the Jacobson radical $\mathcal{J}(A)$ is nilpotent, (though $\mathcal{J}(A)$ need not be artinian).

If a ring $A$ is *noetherian*, (that is, $A$ satisfies the ascending chain condition on left ideals), then its nil radical $\mathcal{N}(A)$ is nilpotent, and so $\mathcal{N}(A) = \beta(A)$. Furthermore, every nil left ideal of a noetherian ring is nilpotent.

In a commutative ring $A$ the set $N$ of all nilpotent elements form an ideal and

$$N = \beta(A) = \mathcal{N}(A).$$

In the theory of group rings a problem of central importance is to decide as whether the Jacobson radical is nil or nilpotent. Such investigations involve a lot of group theory.

At this point we turn again to Köthe's problem. Köthe's problem has a positive solution, (that is, every nil left ideal of a ring $A$ is contained in $\mathcal{N}(A)$) if and only if the sum of two nil left ideals is necessarily nil. Let $A$ be a ring such that $A$ is the sum of two subrings $B$ and $C$. Kegel [15] proved that if $B$ and $C$ are nilpotent rings then $A$ is nilpotent. Nevertheless, Salwa [22] has given recently an example of a ring $A = B + C$ such that $B$ as well as $C$ are sums of their nilpotent ideals but $\mathcal{N}(A) = 0$. Ferrero and Puczyłowski [12] developed further Kegel's result, and got that

if $B$ is nilpotent and $\mathcal{J}(C) = C$ then $\mathcal{J}(A) = A$,
if $\mathcal{N}(B) = B$ and $\mathcal{G}(C) = C$ then $\mathcal{G}(A) = A$.

They also proved in [12] that the positive solution of Köthe's problem is equivalent to the condition:

if $B$ is nilpotent and $\mathcal{N}(C) = C$ then $\mathcal{N}(A) = A$.

**5.** Given a radical (the Jacobson, Köthe, etc), it is natural to ask about the relation between the radical of a ring $A$ and the radical of the matrix ring (polynomial ring, skew polynomial ring, formal power series ring, etc) over $A$.

A ring $A$ is primitive if and only if the $n \times n$ matrix ring $A_n$ over $A$ is primitive. The Jacobson radical $\mathcal{J}(A_n)$ of the $n \times n$ matrix ring over a ring $A$ is the $n \times n$ matrix ring over $\mathcal{J}(A)$, that is, the matrix equation

$$\mathcal{J}(A_n) = (\mathcal{J}(A))_n \qquad (*)$$

holds for every ring $A$. The same is true for the Baer, Brown–McCoy, von Neumann regular and the torsion radical, that is, in (*) we may write $\beta, \mathcal{G}, \nu$ or $\tau$ in place of $\mathcal{J}$, (cf[19] and [27]). The analogous question for Köthe's nil radical is more delicate, Köthe's problem is equivalent to a special case of the equation $\mathcal{N}(A_n) = (\mathcal{N}(A))_n$. Krempa [17] proved that Köthe's problem has a positive solution if and only if $A = \mathcal{N}(A)$ implies $A_2 = \mathcal{N}(A_2)$ where $A_2$ stands for the $2 \times 2$ matrix ring over $A$.

Let $A[x]$ denote the ring of polynomials over a ring $A$. Amitsur [4] proved that $\mathcal{N}(A) = 0$ implies $\mathcal{J}(A[x]) = 0$, or equivalently,

$$\text{if } \mathcal{J}(A[x]) = A[x] \text{ then } \mathcal{N}(A) = A.$$

The converse implication,

$$\text{if } \mathcal{N}(A) = A \text{ then } \mathcal{J}(A[x]) = A[x]) \qquad (**)$$

is equivalent to the positive solution of Köthe's problem (see Krempa [17] and Puczyłowski [19]). As Edmund Puczyłowski reported at the XV Escola de Álgebra, recent investigations approximate the implication (**) from below and from above. Agata Smoktunowicz [25] solved a long standing problem of Amitsur by constructing a ring $A$ such that

$$\mathcal{N}(A) = A \text{ but } \mathcal{N}(A[x]) \neq A[x].$$

Nevertheless,

$$\text{if } \mathcal{N}(A) = A \text{ then } \mathcal{G}(A[x]) = A[x],$$

as proved by Puczyłowski and Smoktunowicz [20]. The coincidence of radicals is sometimes reflected also on polynomial rings. Watters [28], [29] proved that for every homomorphic image $B$ of a ring $A$

$$\mathcal{J}(B) = \beta(B) \text{ if and only if } \mathcal{J}(B[x]) = \beta(B[x])$$

and

$$\mathcal{G}(B) = \beta(B) \text{ if and only if } \mathcal{G}(B[x]) = \beta(B[x]).$$

For more general results the reader is referred to Ferrero and Parmenter [12].

**6.** Many more radicals have been introduced beside the ones we have already seen. The various radicals may be defined by a property $\gamma$ of rings, or equivalently, by the class of all rings possessing property $\gamma$. Thus the class

$$\mathcal{N} = \{\text{all nil rings}\} = \{A \mid \mathcal{N}(A) = A\}$$

is a *radical class*, that of *Köthe radical rings*. Similarly, the class

$$\mathcal{J} = \{\text{all quasi-regular rings}\} = \{A \mid \mathcal{J}(A) = A\},$$
$$\mathcal{G} = \{\text{all } \mathcal{G}\text{-regular rings}\} = \{A \mid \mathcal{G}(A) = A\},$$
$$\nu = \{\text{all von Neumann regular rings}\} = \{A \mid \nu(A) = A\},$$
$$\tau = \{\text{all rings with torsion additive group}\} = \{A \mid \tau(A) = A\}$$

define *the Jacobson, Brown–McCoy, von Neumann regular and torsion radical classes*, respectively. Also the *Baer radical class* can be given as

$$\beta = \{A \mid \beta(A) = A\} = \left\{ A \;\middle|\; \begin{array}{l} \text{every nonzero homomorpic image} \\ \text{of } A \text{ has a nonzero nilpotent ideal} \end{array} \right\}.$$

We are faced with the question: given an arbitrary class of rings, (that is, a property of rings), is it a radical class? In the early fifties Amitsur [1], [2], [3] and Kurosh [18] independently observed that all radical classes $\gamma$ have the following common properties

(i) the class $\gamma$ is homomorphically closed: $f\colon A \mapsto f(A)$ and $A \in \gamma$ imply $f(A) \in \gamma$ for every homomorphism $f$,

(ii) the radical $\gamma(A)$ of the ring $A$ is

$$\gamma(A) = \sum(I \triangleleft A \mid I \in \gamma) \text{ and } \gamma(A) \in \gamma \text{ for all } A,$$

(iii) $\gamma(A/\gamma(A)) = 0$ for all $A$.

Thereafter, a class $\gamma$ of rings is called a *radical class* in the sense of Kurosh and Amitsur, if $\gamma$ fulfils conditions (i), (ii) and (iii). A Kurosh–Amitsur radical $\gamma$ is a *general radical*, and examples, such as the Köthe, Baer, Jacobson, Brown–McCoy, von Neumann regular and torsion radicals are *concrete radicals*. The introduction of a new, more abstract notion like that of Kurosh–Amitsur radicals must be justified by important statements. The most significant statement is expressed in the *ADS-Theorem*, named after Anderson, Divinsky and Suliński [5]: *for any Kurosh–Amitsur radical* $\gamma$, *if* $I \triangleleft A$ *then* $\gamma(I) \triangleleft A$. The lack of transitivity of the relation "$I$ is an ideal in $A$" makes the situation difficult in the variety of rings. The ADS–Theorem is a remedy and establishes a bridge for the missing transitivity. We mention two easy but important consequences of the ADS–Theorem.

Let $\gamma$ be any Kurosh–Amitsur radical. If $I \triangleleft A$ and $\gamma(A) = 0$, then also $\gamma(I) = 0$. In fact, $\gamma(I) \in \gamma$ and by the ADS–Theorem $\gamma(I) \triangleleft A$. Hence $\gamma(I) \subseteq \gamma(A) = 0$.

Many radical classes, including all the six concrete radicals we have seen, are *hereditary*: if $I \triangleleft A \in \gamma$, then also $I \in \gamma$. Let $\gamma$ be a hereditary radical class and $I \triangleleft A$. Then $\gamma(I) = \gamma(A) \cap I$. In fact, since $\gamma(A) \in \gamma$ and the class $\gamma$ is hereditary, we have that $\gamma(A) \cap I \subseteq \gamma(I)$. The inclusion $\gamma(I) \subseteq \gamma(A)$ is a direct consequence of the ADS–Theorem.

There are also recipes how to build a radical class. Starting from any hereditary class $\sigma$ of rings, the class

$$\gamma = \mathcal{U}\sigma = \{A \mid A \text{ has no nonzero homomorphic image in } \sigma\}$$

is always a radical class, the largest radical class $\gamma$ such that $\gamma \cap \sigma = \{0\}$. Of particular interest are the special radicals introduced by Andrunakievich [6]. Let $\sigma$ be a hereditary class of prime rings satisfying the following condition

> if $I$ is an *essential ideal* of a ring $A$, (that is, $I \cap K \neq 0$
> for every nonzero ideal $K$ of $A$) and $I \in \sigma$, then $A \in \sigma$.

Such a class $\sigma$ is referred to as a *special class* and the radical $\gamma = \mathcal{U}\sigma$ is called a *special radical*. Note that the class of all prime rings, all prime rings with nonzero nil ideals, all primitive rings, all simple rings with unity element, respectively, are special classes, and so $\beta, \mathcal{N}, \mathcal{J}$ and $\mathcal{G}$ are all special radicals.

The importance of special radicals is featured in the following: *if $\sigma$ is a special class, then the special radical $\gamma = \mathcal{U}\sigma$ is hereditary and $\gamma(A) = 0$ if and only if $A$ is a subdirect sum of $\sigma$-rings.*

**7.** Finally, some recent results will be surveyed which will show a way how to make radical theory.

For module theoretic purposes Kasch defined the *total* of a ring $A$ by

$$Tot(A) = \{a \in A \mid aA \text{ does not contain nonzero idempotents}\}.$$

$Tot(A)$ resembles to a radical assignment. The shortcoming is that $Tot(A)$ is not closed under addition, so it cannot be an ideal of $A$. To overcome this difficulty, in [9] we considered the class

$$\mu = \{\text{all rings } A \mid Tot(A) = 0\},$$

and proved that $\mu$ is a hereditary class. Thus $\mathcal{K} = \mathcal{U}\mu$ is a radical class what we called the *Kasch radical.* The Kasch radical $\mathcal{K}$ has several decent properties, for instance $\mathcal{K}$ *is hereditary*, $\mathcal{J}(A) \subseteq \mathcal{K}(A)$ for every ring $A$, but $\mathcal{K}$ is not comparable with the Brown–McCoy radical $\mathcal{G}$, that is, there exist rings $A$ and $B$ such that $\mathcal{K}(A) \not\subseteq \mathcal{G}(A)$ and $\mathcal{G}(B) \not\subseteq \mathcal{K}(B)$. Furthermore, $\mathcal{K}$ *is a left and right hereditary radical*, that is, if $A \in \mathcal{K}$ and $I$ is a one-sided ideal of $A$ then also $I \in \mathcal{K}$, (the Baer, Köthe, Jacobson and torsion radicals are left and right hereditary but the Brown–McCoy and the von Neumann regular radicals are not). $\mathcal{K}$ is also *a left and right strong radical*, that is, if $I \in \mathcal{K}$ for a one-sided ideal $I$ of $A$ then $I \subseteq \mathcal{K}(A)$, (the Baer, Jacobson and torsion radicals are left and right strong but the Brown–McCoy and the von Neumann regular radicals are not; as whether the nil radical $\mathcal{N}$ is left or right strong, is exactly Köthe's problem). We have shown in [9] that *the Kasch radical is not a special radical* in the following way. We proved that the class

$$\mu_p = \{\text{all prime rings } A \mid Tot(A) = 0\}$$

is a special class, so $\mathcal{K}_p = \mathcal{U}\mu_p$ is a special radical which is also left and right hereditary and left and right strong. To decide that $\mathcal{K}$ and $\mathcal{K}_p$ are different radicals, Beidar [9] gave an ingenious and involved construction of a ring $A$ such that $Tot(A) = 0$ but $Tot(A/P) \neq 0$ for all prime factor rings $A/P$ of $A$. Then $\mathcal{K}(A) = 0$ but $\mathcal{K}_p(A) = A$. This example shows also that $\mathcal{K}$ is not a special radical.

The study of rings with zero total yields interesting structure theorems. Beidar [8] determined the structure of rings $A$ such that $Tot(A) = 0$ and the degrees of nilpotency of elements in $A$ are bounded. In particular, $A$ is a prime ring, $Tot(A) = 0$ and the degrees of nilpotency of elements in $A$ are bounded if and only if $A$ is isomorphic to a matrix ring over a division ring.

Another recent develpoment in radical theory is due to Ferrero and Puczyłowski [13]. Specializing a module theoretic notion to rings, the *singular ideal* $Z(A)$ of a ring $A$ is defined as the set of all elements $a \in A$ such that the right annihilator $r_A(a)$ has zero intersection with every nonzero right ideal of $A$. A ring $A$ is called *singular* if $Z(A) = A$, and *non-singular* if $Z(A) = 0$. The class

$$\mathcal{S} = \left\{ A \,\middle|\, \begin{array}{l} \text{every nonzero homomorphic image of } A \\ \text{has a nonzero ideal which is a singular ring} \end{array} \right\} = \left\{ A \,\middle|\, \begin{array}{l} A \text{ cannot be homomorphically mapped onto} \\ \text{a nonzero semiprime non-singular ring} \end{array} \right\}$$

is then a radical class, called the *singular radical*. For every ring $A$, $\mathcal{S}(A)$ is not far from being a singular ring whereas $A/\mathcal{S}(A)$ is close to being a non-singular ring. The radical $\mathcal{S}$ is left and right hereditary, left and right strong, and contains all nilpotent rings.

As whether $\mathcal{S}$ is a special radical, is an open question.

## References

**[1]** Amitsur, S. A., *A general theory of radicals I, Radicals in complete lattices*, Amer. J. Math., 74 (1952), 774–786.

**[2]** Amitsur, S. A., *A general theory of radicals II, Radicals in rings and bicategories*, Amer. J. Math., 76 (1954), 100–125.

**[3]** Amitsur, S. A., *A general theory of radicals III, Applications,* Amer. J. Math., 76 (1954), 126–136.

**[4]** Amitsur, S.A., *Radicals of polynomial rings,* Canad. J. Math., 8 (1956), 355–361.

**[5]** Anderson, T., Divinsky, N. and Suliński, A., *Hereditary radicals in associative and alternative rings,* Canad. J. Math., 17 (1965), 594–603.

**[6]** Andrunakievich, V. A., *Radicals of associative rings I,* Mat. Sb., 44(86) (1958), 179–212 (Russian), English translation: Amer. Math. Soc. Transl., (2) 52 (1966), 95–128.

**[7]** Baer, R., *Radical ideals,* Amer. J. Math., 65 (1943), 537–568.

**[8]** Beidar, K. I., *On rings with zero total,* Beiträge Alg. und Geom., 38 (1997), 233–239.

**[9]** Beidar, K. I. and Wiegandt, R., *Radicals induced by the total of rings,* Beiträge Alg. und Geom., 38 (1997), 150–159.

**[10]** Bergman, G., *A ring primitive on the right but not on the left,* Proc. Amer. Math. Soc., 15 (1964), 473–475.

**[11** Ferrero, M. and Parmenter, M. M., *A note on Jacobson rings and polynomial rings,* Proc. Amer. Math. Soc., 105 (1989), 281–286.

**[12]** Ferrero, M. and Puczyłowski, E. R., *On rings which are sums of two subrings,* Arch. Math., 53 (1989), 4–10.

**[13]** Ferrero, M. and Puczyłowski, E. R., *The singular ideal and radicals,* J. Austral. Math. Soc. (Series A), 64 (1998), 195–209.

**[14]** Jacobson, N., *The radical and semisimplicity of arbitrary rings,* Amer. J. Math., 67 (1945), 300–320.

**[15]** Kegel, O. H., *Zur Nilpotenz gewisser assoziativer Ringe*, Math. Ann., 149 (1962/63), 258–260.

**[16]** Köthe, G., *Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist*, Math. Z., 32 (1930), 161–186.

**[17]** Krempa, J., *Logical connection among some open problems in noncommutative rings*, Fund. Math., 76 (1972), 121–130.

**[18]** Kurosh, A. G., *Radicals of rings and algebras*, Mat. Sb., 33 (1953), 13–26 (Russian); English translation: Coll. Soc. Math. Soc. J. Bolyai 6, Rings, Modules and Radicals, Keszthely 1971, North–Holland 1973, pp.297–314.

**[19]** Puczyłowski, E. R., *On Koethe's problem*, Inst. Mat., UFRGS, Série A, No 36, Porto Alegre, 1993.

**[20]** Puczyłowski, E. R. and Smoktunowicz, A., *On minimal ideals and the Brown – McCoy radical of polynomial rings*, Comm. in Algebra, 26 (1998), 2473–2482.

**[21]** Rowen, L. H., *Ring theory* I, Academic Press, 1988.

**[22]** Salwa, A., *Rings that are sums of two locally nilpotent subrings*, Comm. in Algebra, 24 (1996), 3921–3931.

**[23]** Sąsiada, E., *Solution of a problem of existence of a simple radical ring*, Bull. Acad. Polon. Sci., 9 (1961), 257.

**[24]** Sąsiada, E. and Cohn, P. M., *An example of a simple radical ring*, J. Alg., 5 (1967), 373–377.

**[25]** Agata Smoktunowicz, *Polynomial rings over nil rings need not be nil*, Preprint, 1998.

**[26]** Szász, F., *Über artinsche Ringe*, Bull. Acad. Polon. Sci., 11 (1963), 351–354.

**[27]** Szász, F. A., *Radicals of rings*, Disqu. Math. Hungar. 11, Akadémiai Kiadó, Budapest 1981.

**[28]** Watters, J. F., *Polynomial extensions of Jacobson rings*, J. Alg., 36 (1975), 302–308.

**[29]** Watters, J. F., *The Brown–McCoy radical and Jacobson rings*, Bull. Acad. Polon. Sci., 24 (1976), 91–99.

**[30]** Wedderburn, J. H. M., *On hypercomplex numbers*, Proc. London Math. Soc., (2) (1908), 77–118.

**[31]** Zel'manov, E. I., *An example of a finitely generated primary ring*, Sibirsk. Mat. Zh., 20 (1979), 423, (Russian).

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